Self-similar Markov processes

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- Diffusions → Brownian motion ✓
- Cts-time Markov processes with jumps → Lévy processes ✗
- Self-similar Markov processes ↑

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Lévy processes

- Stick to one-dimension
- A Lévy process is an $\mathbb{R}$-valued random trajectory $\{X_t : t \geq 0\}$ issued from the origin with paths that are right-continuous and left limits and which has stationary and independent increments.
- More formally stationary and independent increments means:
  - for $0 \leq s \leq t < \infty$, $X_t - X_s = X_{t-s}$
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- It can be shown that this means the entire process is characterised by its position at time 1

$$\mathbb{E}[e^{i\theta X_t}] = e^{-\Psi(\theta)t}$$

for some appropriate function Ψ (the characteristic exponent).

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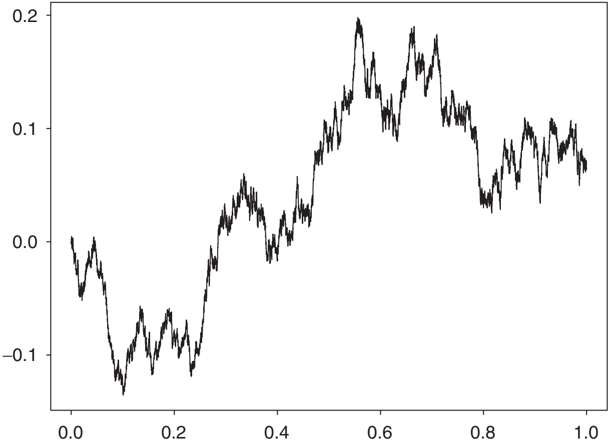
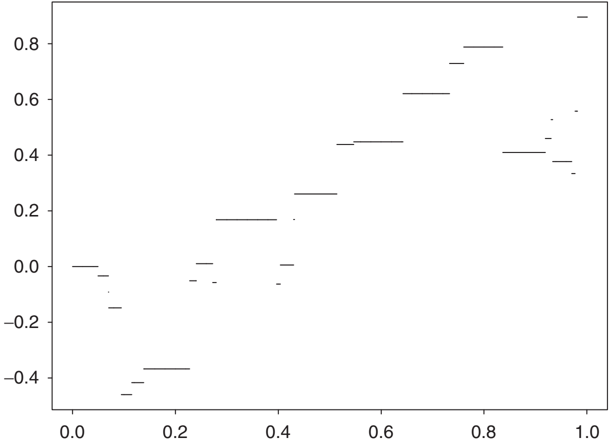
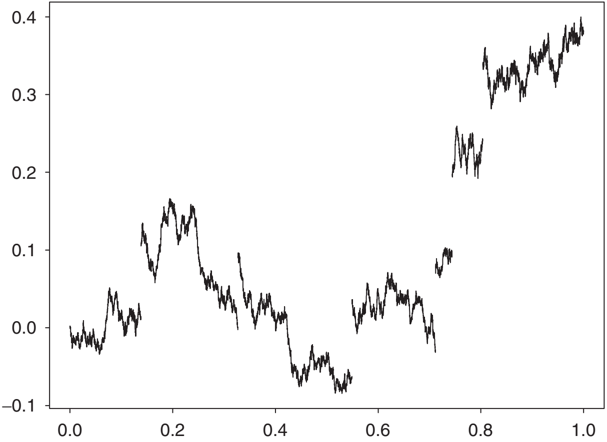


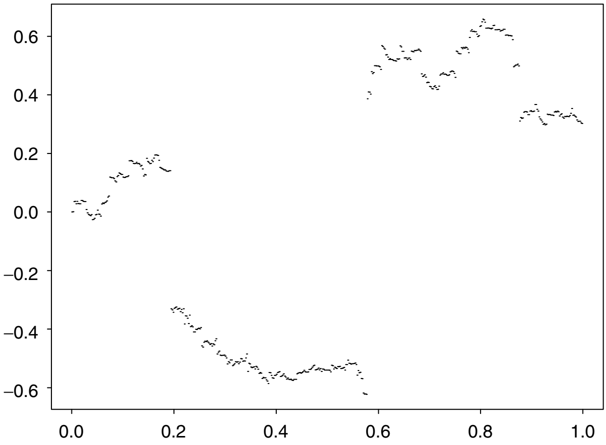
Compound Poisson process



Brownian motion + compound Poisson process



Unbounded variation paths



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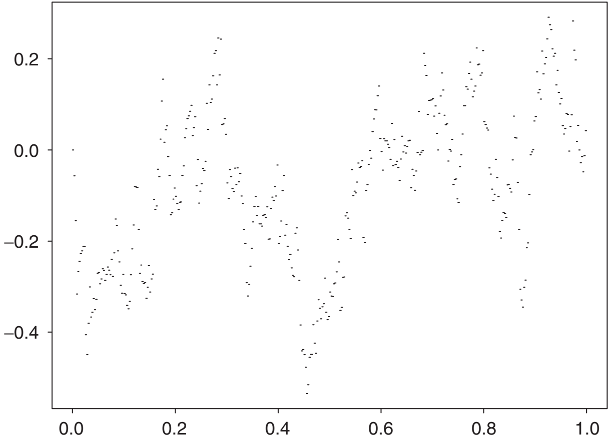

Space exploration: some successes and dissatisfaction

- Fundamentally we want to understand how Lévy processes explore space.
- 25 years of research has been very successful in giving an (relatively) complete theoretical description .....
- ......with the caveat that the database of tractable examples for the aforesaid theory is uncomfortably small (relative to Markov chains and diffusions).

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Example 1:

$$\mathbb{P}(\text{Process first exceeds level } x \text{ by an amount y}) = \int_{[0,x)} U(\mathrm{d}z)\bar{\nu}(z-x+y)$$

where

$$\Psi(\theta) = \kappa^+(-\mathrm{i}\theta)\kappa^-(\mathrm{i}\theta), \qquad \theta \in \mathbb{R},$$

$$\kappa^+(\lambda) = q + \delta\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x})\nu(\mathrm{d}x), \qquad \lambda \geq 0,$$

$$\bar{\nu}(x) = \nu(x,\infty) \qquad \text{and} \qquad \int_{[0,\infty)} \mathrm{e}^{-\lambda x} U(\mathrm{d}x) = \frac{1}{\kappa^+(\lambda)}$$

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Example 2:

Under appropriate assumptions,

$$\mathbb{P}(\text{Process ever hits a point } x) = \frac{u(x)}{u(0)}, \qquad x \in \mathbb{R},$$

where

$$\int_{\mathbb{R}} e^{i\theta x} u(x) dx = \frac{1}{\Psi(\theta)}, \qquad \theta \in \mathbb{R}.$$

Self-similar Markov processes on $\mathbb{R}$

α-ssMp

$\mathbb{R}$-valued Markov process,
equipped with initial measures P_x, $x \in \mathbb{R}\backslash\{0\}$,
with 0 an absorbing state,
satisfying the scaling property

$$\left(cX_{c^{-\alpha}t}\right)_{t\geq 0}\Big|_{\mathrm{P}_x} \stackrel{d}{=} X|_{\mathrm{P}_{cx}}, \qquad x, c > 0$$

Space-time changes and modulation

It turns out that everyn $\mathbb{R}$-valued ssMp can be characterised using polar coordinates in $\mathbb{C}$ (think $re^{i\theta}$) as follows:

$$X_t = |x| \exp\left\{\xi_{\varphi(|x|^{-\alpha}t)} + i\pi(J_{\varphi(|x|^{-\alpha}t)} + 1)\right\}, \qquad t \geq 0, x \neq 0,$$

where (ξ, J) is a so-called Markov modulated Lévy process and

$$\varphi(t) = \inf\left\{s > 0 : \int_0^s e^{\alpha\xi_u} du > t\right\}.$$

(ξ, J):

- $J = \{J_t : t \geq 0\}$ is a Markov chain on $\{1, 2\}$ with intensity matrix Q.
- When $J_t = i$, ξ moves as a Lévy process of type i. "$d\xi_t = d\xi_t^{(i)}$"
- When J makes a jump at time t, e.g. $1 \to 2$, then ξ experiences an additional jump $\Delta\xi_t$ which is an i.i.d. copy of some pre specified r.v. $U_{1,2}$.

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(ξ, J): Markov modulated Lévy processes can also be characterised by a "characteristic exponetnt".

$$\mathbb{E}_i[e^{\mathrm{i}\theta X_t}; J_t = j] = (\exp\{-\boldsymbol{\Psi}(\theta)t\})_{i,j}$$

where

$$\boldsymbol{\Psi}(\theta) = \begin{pmatrix} \Psi_1(\theta) & 0 \\ 0 & \Psi_2(\theta) \end{pmatrix} - Q \circ \begin{pmatrix} 1 & \mathbb{E}(e^{\mathrm{i}\theta U_{1,2}}) \\ \mathbb{E}(e^{\mathrm{i}\theta U_{2,1}}) & 1 \end{pmatrix}$$

Space-time changes and modulation

If the Markov chain has an absorbing state, then the ssMp is in effect a "positive self-similar Markov process" (pssMp)

$$X_t = |x| \exp\left\{\xi_{\varphi(|x|^{-\alpha} t)}\right\}, \qquad t \geq 0, x \neq 0,$$

where ξ is a Lévy process.

Positive feedback

- There is one class of Lévy processes which has always been considered to be "the next best thing after Brownian motion": the (α, ρ)-stable process.
- $\Psi(\theta) = |\theta|^\alpha \left(e^{\pi i \alpha(\frac{1}{2}-\rho)} \mathbf{1}_{\{\theta>0\}} + e^{-\pi i \alpha(\frac{1}{2}-\rho)} \mathbf{1}_{\{\theta<0\}} \right), \qquad \theta \in \mathbb{R},$
  Only allowed to take $\alpha \in (0,2]$, $\rho \in [0,1]$.
- In fact stable processes are also self-similar Markov processes:

$$\mathbb{E}[e^{i\theta X_t}] = e^{-\Psi(\theta)t} \text{ and } \mathbb{E}[e^{i\theta c X_{c^{-\alpha}t}}] = e^{-\Psi(\theta)t} \text{ for all } c > 0.$$

- So α is the index of self-similarity, but $\rho \in [0,1]$ is a symmetry index. When $\rho = 1/2$, $\Psi(\theta) = |\theta|^\alpha$ so $-X =^d X$.
- When $\rho \in (0,1)$ (keep away from complete asymmetry!) and $\alpha \in (0,2)$ then the underlying Markov modulated Lévy process has exponent

$$\boldsymbol{\Psi}(\theta) = \begin{pmatrix} -\frac{\Gamma(\alpha - i\theta)\Gamma(1+i\theta)}{\Gamma(\alpha\hat{\rho} - i\theta)\Gamma(1-\alpha\hat{\rho}+i\theta)} & \frac{\Gamma(\alpha-i\theta)\Gamma(1+i\theta)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \\ \frac{\Gamma(\alpha-i\theta)\Gamma(1+i\theta)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & -\frac{\Gamma(\alpha-i\theta))\Gamma(1+i\theta)}{\Gamma(\alpha\rho-i\theta)\Gamma(1-\alpha\rho+i\theta)} \end{pmatrix}.$$

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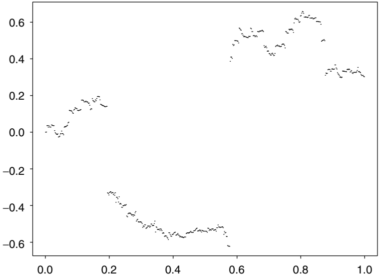

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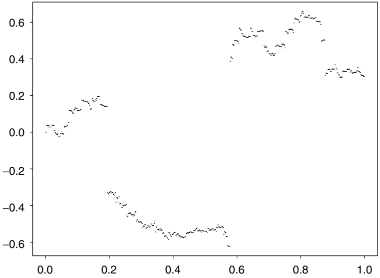

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Positive feedback

- $\alpha \in (0,1)$

$$\begin{aligned}&\mathbb{P}(\text{Stable process first enters } [0,1] \text{ in d}y)\\&\quad = \mathbb{P}(\xi \text{ first enters } (-\infty,0] \text{ in d}(\log y))\\&\quad = \frac{\sin(\pi\alpha\hat\rho)}{\pi} x^{\alpha\rho} y^{-\alpha\rho}(x-1)^{\alpha\hat\rho}(1-y)^{-\alpha\hat\rho}(x-y)^{-1}\text{d}y\end{aligned}$$

- $\alpha \in (1,2)$

$$\begin{aligned}&\mathbb{P}(\text{Stable process hits 1 before 0 when starting from } x > 0)\\&= \mathbb{P}(\xi \text{ ever hits 0 when starting from } \log x)\\&= \frac{\sin(\pi\rho\alpha) - |x-1|^{\alpha-1}[\mathbf{1}_{(x>1)}\sin(\pi\hat\rho\alpha) + \mathbf{1}_{(0<x<1)}\sin(\pi\rho\alpha)] + x^{\alpha-1}\sin(\pi\hat\rho\alpha)}{(\sin(\pi\rho\alpha)+\sin(\pi\hat\rho\alpha))}.\end{aligned}$$

Positive feedback

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$$\begin{aligned}&\mathbb{P}(\text{Stable process first enters } [0,1] \text{ in } \mathrm{d}y)\\ &\quad = \mathbb{P}(\xi \text{ first enters } (-\infty,0] \text{ in } \mathrm{d}(\log y))\\ &\quad = \frac{\sin(\pi\alpha\hat\rho)}{\pi} x^{\alpha\rho} y^{-\alpha\rho}(x-1)^{\alpha\hat\rho}(1-y)^{-\alpha\hat\rho}(x-y)^{-1}\mathrm{d}y\end{aligned}$$

- $\alpha \in (1,2)$

$$\begin{aligned}&\mathbb{P}(\text{Stable process hits 1 before 0 when starting from x} > 0)\\ &\quad = \mathbb{P}(\xi \text{ ever hits 0 when starting from } \log x)\\ &\quad = \frac{\sin(\pi\rho\alpha) - |x-1|^{\alpha-1}[\mathbf{1}_{(x>1)}\sin(\pi\hat\rho\alpha) + \mathbf{1}_{(0<x<1)}\sin(\pi\rho\alpha)] + x^{\alpha-1}\sin(\pi\hat\rho\alpha)}{(\sin(\pi\rho\alpha)+\sin(\pi\hat\rho\alpha))}.\end{aligned}$$

A bigger picture

- It's possible to extend the notion of both Lévy processes and ssMp to higher dimensions
- For example, a d-dimensional isotropic stable Lévy process is also a ssMp:

$$\mathbf{E}[e^{i\theta\cdot X_t}] = \exp\{-|\theta|^\alpha t\}, \qquad t \geq 0, \theta \in \mathbb{R}^d,$$

  necessarily $\alpha \in (0, 2]$.
- The radial distance of such a process from the origin, $|X_t|$, $t \geq 0$, is a pssMp. Its underlying Lévy process has characteristic exponent

$$\Psi(\theta) = \frac{\Gamma(\frac{1}{2}(-i\theta + \alpha))}{\Gamma(-\frac{1}{2}i\theta)} \frac{\Gamma(\frac{1}{2}(i\theta + d))}{\Gamma(\frac{1}{2}(i\theta + d - \alpha))}, \qquad \theta \in \mathbb{R}.$$

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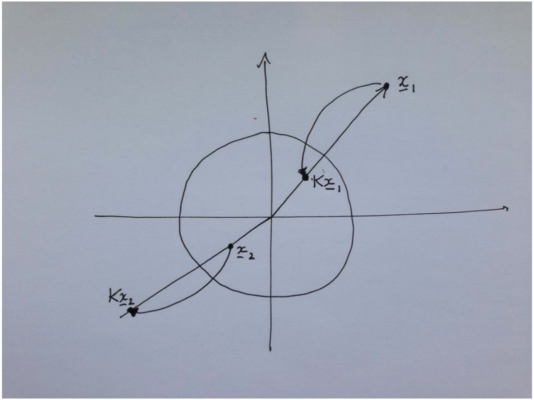
Bogdan-Zak transform

The Kelvin transform is an inversion of $\mathbb{R}^d$ through the unit sphere:

$$Kx = \frac{x}{|x|^2}, \qquad x \in \mathbb{R}^d.$$



Bogdan-Zak transform

Bogdan-Zak transform

Suppose that X is a d-dimensional isotropic stable process with $d \geq 2$. Define

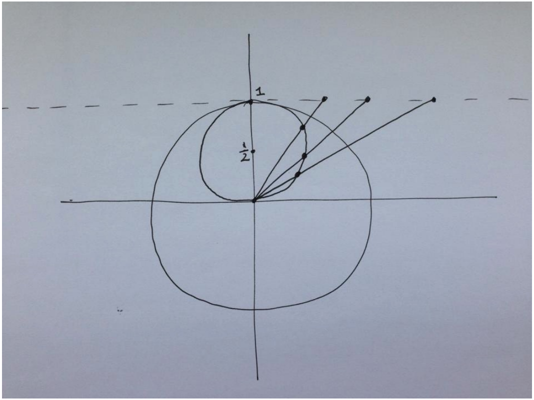
$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \mathrm{d}u > t\}, \qquad t \geq 0.$$

Then, for all $x \in \mathbb{R}^d \backslash \{0\}$, $\{KX_{\eta(t)} : t \geq 0\}$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}^h_{Kx})$, where

$$\left.\frac{\mathrm{d}\mathbb{P}^h_x}{\mathrm{d}\mathbb{P}_x}\right|_{\sigma(X_s : s \leq t)} = \frac{|X_t|^{\alpha - d}}{|x|^{\alpha - d}}, \qquad t \geq 0,$$

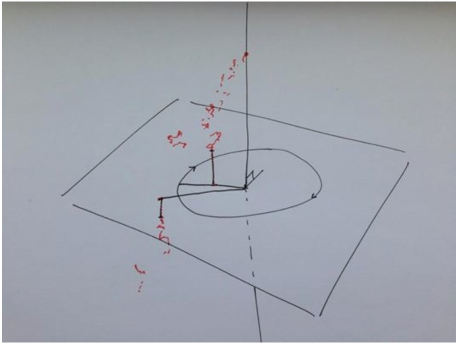
Bogdan-Zak transform

The Kelvin transform maps the ball $\{x \in \mathbb{R}^d : |x - 1/2| \leq 1/2\}$ to a half-space.



Bogdan-Zak transform

- How does an isotropic stable process first enter/exit a ball? $\rightarrow$ how does an isotropic stable process cross a hyperplane $\rightarrow$ use rotational symmetry to the orthogonal projection



- Where does an isotropic stable process first hit the surface of a sphere? $\rightarrow$ where does an isotropic stable process hit a hyperplane $\rightarrow$ use rotational symmetry to the orthogonal projection

Bogdan-Zak transform

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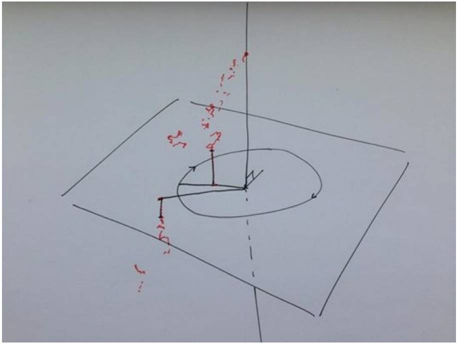

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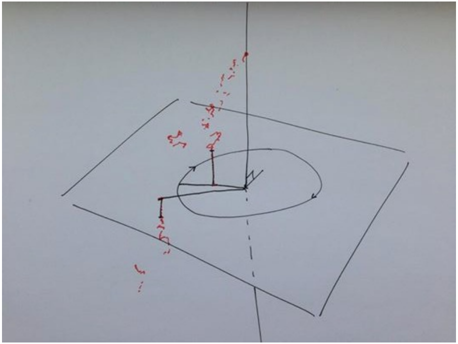

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Thank
You
Merci
Gracias
TerimaKasih
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