

Self-similar Markov processes

Andreas E. Kyprianou¹

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- Diffusions \rightarrow Brownian motion ✓
- Cts-time Markov processes with jumps \rightarrow Lévy processes ✓
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Lévy processes

- Stick to one-dimension
- A Lévy process is an \mathbb{R} -valued random trajectory $\{X_t : t \geq 0\}$ issued from the origin with paths that are right-continuous and left limits and which has stationary and independent increments.
- More formally stationary and independent increments means:
 - for $0 \leq s \leq t < \infty$, $X_t - X_s = X_{t-s}$
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- It can be shown that this means the entire process is characterised by its position at time 1

$$\mathbb{E}[e^{i\theta X_t}] = e^{-\Psi(\theta)t}$$

for some appropriate function Ψ (the characteristic exponent).

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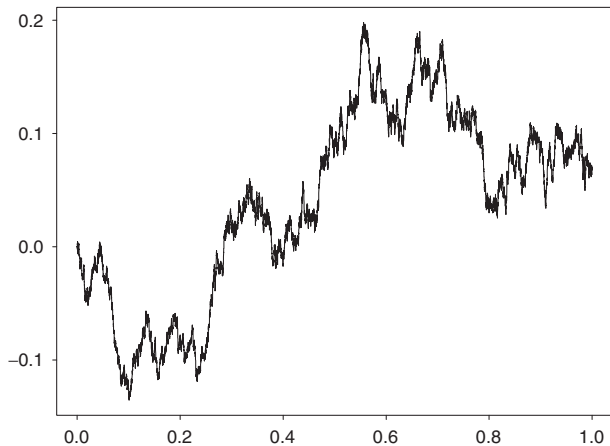
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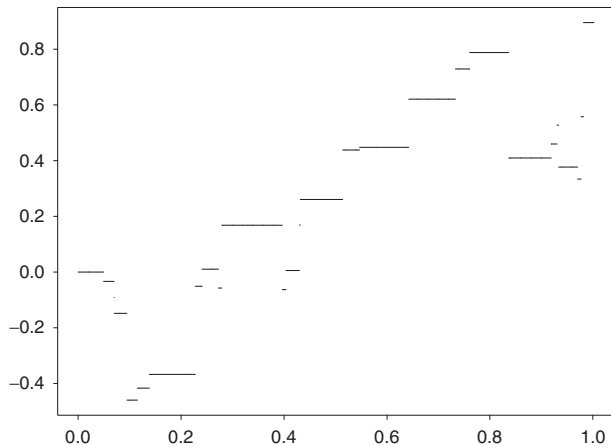
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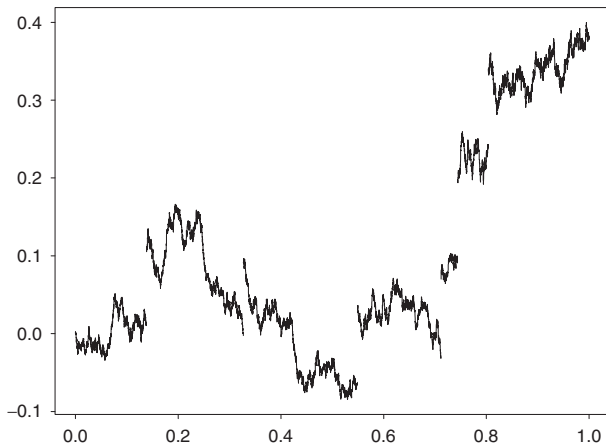
Brownian motion



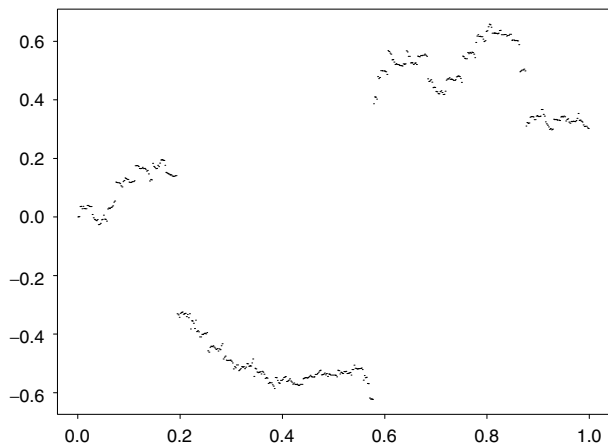
Compound Poisson process



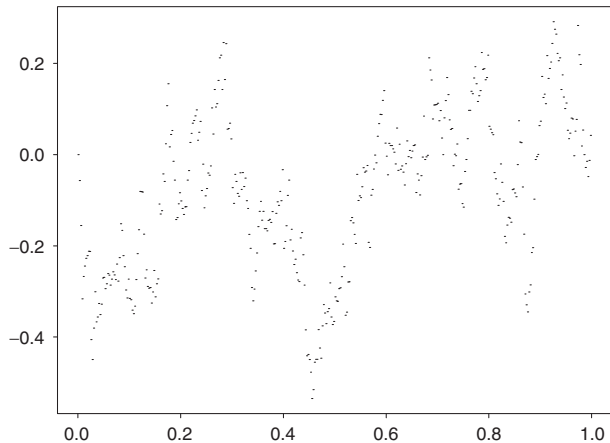
Brownian motion + compound Poisson process



Unbounded variation paths



Bounded variation paths



Space exploration: some successes and dissatisfaction

- Fundamentally we want to understand how Lévy processes explore space.
- 25 years of research has been very successful in giving an (relatively) complete theoretical description
-with the caveat that the database of tractable examples for the aforesaid theory is uncomfortably small (relative to Markov chains and diffusions).

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Space exploration: some successes and dissatisfaction

Example 1:

$$\mathbb{P}(\text{Process first exceeds level } x \text{ by an amount } y) = \int_{[0,x)} U(dz) \bar{\nu}(z-x+y)$$

where

$$\Psi(\theta) = \kappa^+(-i\theta)\kappa^-(i\theta), \quad \theta \in \mathbb{R},$$

$$\kappa^+(\lambda) = q + \delta\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x}) \nu(dx), \quad \lambda \geq 0,$$

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Example 2:

Under appropriate assumptions,

$$\mathbb{P}(\text{Process ever hits a point } x) = \frac{u(x)}{u(0)}, \quad x \in \mathbb{R},$$

where

$$\int_{\mathbb{R}} e^{i\theta x} u(x) dx = \frac{1}{\Psi(\theta)}, \quad \theta \in \mathbb{R}.$$

Self-similar Markov processes on \mathbb{R}

α -ssMp

\mathbb{R} -valued Markov process,
equipped with initial measures P_x , $x \in \mathbb{R} \setminus \{0\}$,
with 0 an absorbing state,
satisfying the scaling property

$$(cX_{c^{-\alpha}t})_{t \geq 0} \Big|_{P_x} \stackrel{d}{=} X \Big|_{P_{cx}}, \quad x, c > 0$$

Space-time changes and modulation

It turns out that every \mathbb{R} -valued ssMp can be characterised using polar coordinates in \mathbb{C} (think $re^{i\theta}$) as follows:

$$X_t = |x| \exp \left\{ \xi_{\varphi(|x|^{-\alpha}t)} + i\pi(J_{\varphi(|x|^{-\alpha}t)} + 1) \right\}, \quad t \geq 0, x \neq 0,$$

where (ξ, J) is a so-called Markov modulated Lévy process and

$$\varphi(t) = \inf \left\{ s > 0 : \int_0^s e^{\alpha \xi_u} du > t \right\}.$$

(ξ, J) :

- $J = \{J_t : t \geq 0\}$ is a Markov chain on $\{1, 2\}$ with intensity matrix Q .
- When $J_t = i$, ξ moves as a Lévy process of type i . “ $d\xi_t = d\xi_t^{(i)}$ ”
- When J makes a jump at time t , e.g. $1 \rightarrow 2$, then ξ experiences an additional jump $\Delta\xi_t$ which is an i.i.d. copy of some pre specified r.v. $U_{1,2}$.

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(ξ, J) : Markov modulated Lévy processes can also be characterised by a “characteristic exponent”.

$$\mathbb{E}_i[e^{i\theta X_t}; J_t = j] = (\exp\{-\Psi(\theta)t\})_{i,j}$$

where

$$\Psi(\theta) = \begin{pmatrix} \Psi_1(\theta) & 0 \\ 0 & \Psi_2(\theta) \end{pmatrix} - Q \circ \begin{pmatrix} 1 & \mathbb{E}(e^{i\theta U_{1,2}}) \\ \mathbb{E}(e^{i\theta U_{2,1}}) & 1 \end{pmatrix}$$

Space-time changes and modulation

If the Markov chain has an absorbing state, then the ssMp is in effect a “positive self-similar Markov process” (pssMp)

$$X_t = |x| \exp \{ \xi_{\varphi(|x|^{-\alpha} t)} \}, \quad t \geq 0, x \neq 0,$$

where ξ is a Lévy process.

Positive feedback

- There is one class of Lévy processes which has always been considered to be “the next best thing after Brownian motion”: the (α, ρ) -stable process.

- $\Psi(\theta) = |\theta|^\alpha \left(e^{\pi i \alpha (\frac{1}{2} - \rho)} \mathbf{1}_{\{\theta > 0\}} + e^{-\pi i \alpha (\frac{1}{2} - \rho)} \mathbf{1}_{\{\theta < 0\}} \right), \quad \theta \in \mathbb{R},$
Only allowed to take $\alpha \in (0, 2], \rho \in [0, 1]$.

- In fact stable processes are also self-similar Markov processes:

$$\mathbb{E}[e^{i\theta X_t}] = e^{-\Psi(\theta)t} \quad \text{and} \quad \mathbb{E}[e^{i\theta c X_{c^{-\alpha}t}}] = e^{-\Psi(\theta)t} \quad \text{for all } c > 0.$$

- So α is the index of self-similarity, but $\rho \in [0, 1]$ is a symmetry index. When $\rho = 1/2$, $\Psi(\theta) = |\theta|^\alpha$ so $-X = \overset{d}{=} X$.
- When $\rho \in (0, 1)$ (keep away from complete asymmetry!) and $\alpha \in (0, 2)$ then the underlying Markov modulated Lévy process has exponent

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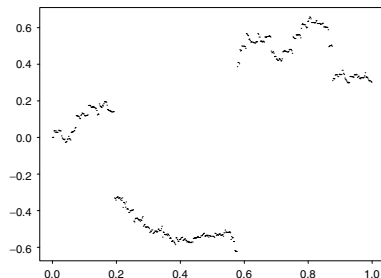
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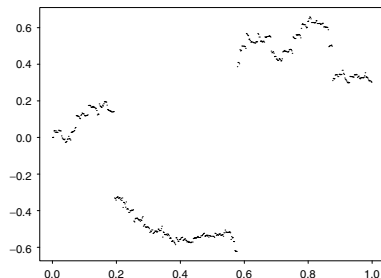


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- $\alpha \in (0, 1)$

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- $\alpha \in (1, 2)$

$\mathbb{P}(\text{Stable process hits 1 before 0 when starting from } x > 0)$

$= \mathbb{P}(\xi \text{ ever hits 0 when starting from } \log x)$

$$= \frac{\sin(\pi\rho\alpha) - |x-1|^{\alpha-1} [\mathbf{1}_{(x>1)} \sin(\pi\hat{\rho}\alpha) + \mathbf{1}_{(0<x<1)} \sin(\pi\rho\alpha)] + x^{\alpha-1} \sin(\pi\hat{\rho}\alpha)}{(\sin(\pi\rho\alpha) + \sin(\pi\hat{\rho}\alpha))}$$

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$$\begin{aligned} &= \mathbb{P}(\xi \text{ ever hits 0 when starting from } \log x) \\ &= \frac{\sin(\pi\rho\alpha) - |x-1|^{\alpha-1} [\mathbf{1}_{(x>1)} \sin(\pi\hat{\rho}\alpha) + \mathbf{1}_{(0<x<1)} \sin(\pi\rho\alpha)] + x^{\alpha-1} \sin(\pi\hat{\rho}\alpha)}{(\sin(\pi\rho\alpha) + \sin(\pi\hat{\rho}\alpha))}. \end{aligned}$$

A bigger picture

- It's possible to extend the notion of both Lévy processes and ssMp to higher dimensions
- For example, a d -dimensional isotropic stable Lévy process is also a ssMp:

$$\mathbf{E}[e^{i\theta \cdot X_t}] = \exp\{-|\theta|^\alpha t\}, \quad t \geq 0, \theta \in \mathbb{R}^d,$$

necessarily $\alpha \in (0, 2]$.

- The radial distance of such a process from the origin, $|X_t|$, $t \geq 0$, is a pssMp. Its underlying Lévy process has characteristic exponent

$$\Psi(\theta) = \frac{\Gamma(\frac{1}{2}(-i\theta + \alpha))}{\Gamma(-\frac{1}{2}i\theta)} \frac{\Gamma(\frac{1}{2}(i\theta + d))}{\Gamma(\frac{1}{2}(i\theta + d - \alpha))}, \quad \theta \in \mathbb{R}.$$

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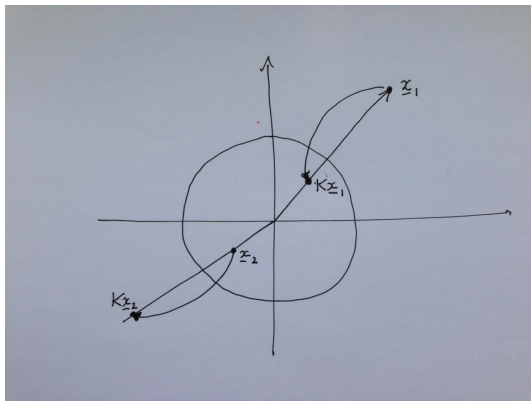
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Bogdan-Zak transform

The Kelvin transform is an inversion of \mathbb{R}^d through the unit sphere:

$$Kx = \frac{x}{|x|^2}, \quad x \in \mathbb{R}^d.$$



Bogdan-Zak transform

Bogdan-Zak transform

Suppose that X is a d -dimensional isotropic stable process with $d \geq 2$. Define

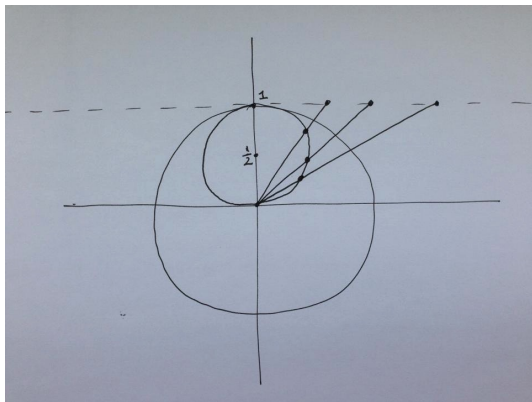
$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} du > t\}, \quad t \geq 0.$$

Then, for all $x \in \mathbb{R}^d \setminus \{0\}$, $\{KX_{\eta(t)} : t \geq 0\}$ under \mathbb{P}_x is equal in law to (X, \mathbb{P}_{Kx}^h) , where

$$\frac{d\mathbb{P}_x^h}{d\mathbb{P}_x} \Big|_{\sigma(X_s : s \leq t)} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \quad t \geq 0,$$

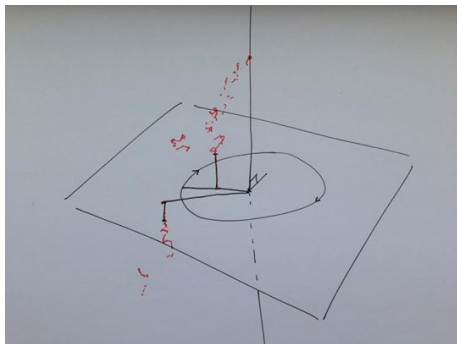
Bogdan-Zak transform

The Kelvin transform maps the ball $\{x \in \mathbb{R}^d : |x - 1/2| \leq 1/2\}$ to a half-space.



Bogdan-Zak transform

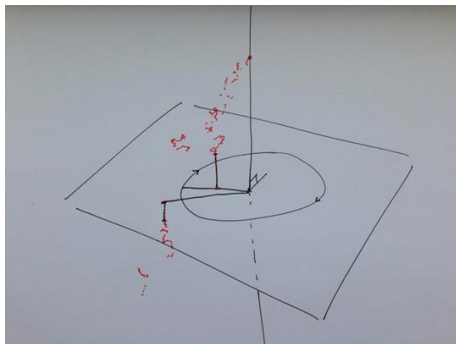
- How does an isotropic stable process first enter/exit a ball? → how does an isotropic stable process cross a hyperplane → use rotational symmetry to the orthogonal projection



- Where does an isotropic stable process first hit the surface of a sphere? → where does an isotropic stable process hit a hyperplane → use rotational symmetry to the orthogonal projection

Bogdan-Zak transform

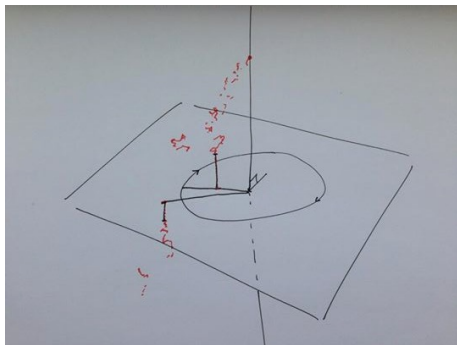
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