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Some calculations for Israeli options

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Abstract. Recently Kifer (2000) introduced the concept of an Israeli (or Game) option. That is a general American-type option with the added possibility that the writer may terminate the contract early inducing a payment exceeding the holder's claim had they exercised at that moment. Kifer shows that pricing and hedging of these options reduces to evaluating a saddle point problem associated with Dynkin games. In this short text we give two examples of perpetual Israeli options where the solutions are explicit.

Key words: Stochastic games, option pricing, fluctuation theory, American options, Russian options, Israeli options

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1 Israeli options

Consider the Black-Scholes market. That is, a market with a risky asset S and a riskless bond, B. The bond evolves according to the dynamic

$$dB_t = rB_t dt \text{ where } r, t \geq 0.$$

The value of the risky asset is written as the process $S = \{S_t : t \geq 0\}$ where

$$S_t = s \exp\{\sigma W_t + \mu t\}$$

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where $s > 0$ is the initial value of S and $W = \{W_t : t \geq 0\}$ is a Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, P)$ satisfying the usual conditions.

Let $0 < T \leq \infty$. Suppose that $X = \{X_t : t \in [0,T]\}$ and $Y = \{Y_t : t \in [0,T]\}$ be two continuous stochastic processes defined on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ such that with probability one $Y_t \geq X_t$ for all $t \in [0,T]$. The Israeli option, introduced by Kifer (2000), is a contract between a writer and holder at time $t = 0$ such that both have the right to exercise at any $\mathbb{F}$-stopping time before the expiry date T. If the holder exercises, then (s)he may claim the value of X at the exercise date and if the writer exercises, (s)he is obliged to pay to the holder the value of Y at the time of exercise. If neither have exercised at time T and $T < \infty$ then the writer pays the holder the value X_T. If both decide to claim at the same time then the lesser of the two claims is paid. (Note that the assumption that X and Y are continuous processes is not the most generic case but will suffice for the following discussion). In short, if the holder will exercise with strategy σ and the writer with strategy τ we can conclude that at any moment during the life of the contract, the holder can expect to receive $Z_{\sigma,\tau}$ where

$$Z_{s,t} = X_s \mathbf{1}_{(s\leq t)} + Y_t \mathbf{1}_{(t<s)}.$$

Suppose now that $\mathbb{P}_s$ is the risk-neutral measure for S under the assumption that $S_0 = s$. [Note that standard Black-Scholes theory dictates that this measure exists and is uniquely defined via a Girsanov change of measure]. We shall denote $\mathbb{E}_s$ to be expectation under $\mathbb{P}_s$. The following Theorem is Kifer's pricing result.

Theorem 1 (Kifer) *Suppose that for all* $s > 0$

$$\mathbb{E}_s \left(\sup_{0\leq t\leq T} e^{-rt} Y_t \right) < \infty$$

and if $T = \infty$ *that* $\mathbb{P}_s\left(\lim_{t\uparrow\infty} e^{-rt} Y_t = 0\right) = 1$. *Let* $\mathcal{T}_{t,T}$ *be the class of* $\mathbb{F}$*-stopping times valued in* $[t,T]$. *The value of the Israeli option under the Black-Scholes framework is given by* $V = \{V_t : t \in [0,T]\}$ *where*

$$V_t = \textit{ess-inf}_{\tau\in\mathcal{T}_{t,T}} \textit{ess-sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}_s \left(e^{-r(\sigma\wedge\tau - t)} Z_{\sigma,\tau} \middle| \mathcal{F}_t \right) \tag{1}$$

$$= \textit{ess-sup}_{\sigma\in\mathcal{T}_{t,T}} \textit{ess-inf}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}_s \left(e^{-r(\sigma\wedge\tau - t)} Z_{\sigma,\tau} \middle| \mathcal{F}_t \right) \tag{2}$$

Further the optimal stopping strategies for the holder and writer respectively are

$$\sigma^* = \inf\{t \in [0,T] : V_t \leq X_t\} \wedge T \textit{ and } \tau^* = \inf\{t \geq 0 : V_t \geq Y_t\} \wedge T$$

(with the usual definition $\inf \emptyset = \infty$.) The formulae given in this theorem reflect the fact that the essence of this option contract is based on the older theory of Dynkin games or stochastic games; see Friedman (1976) or Dynkin (1969) for example. In this paper we shall perform calculations showing that for certain familiar choices of X and Y exact strategies σ^* and τ^* can be obtained giving an explicit expression for the process V when $T = \infty$ (perpetual options). The two cases we shall consider are as follows.

Israeli δ-penalty put options. In this case, the holder may claim as a normal American put,

$$X_t = (K - S_t)^+.$$

The writer on the other hand will be assumed to payout the holders claim plus a constant,

$$Y_t = (K - S_t)^+ + \delta \text{ for } \delta > 0.$$

Israeli δ-penalty Russian options. The holder may exercise to take a normal Russian claim,

$$X_t = e^{-\alpha t} \max \left\{ m, \sup_{u \in [0,t]} S_u \right\} \text{ for } \alpha > 0, m > s$$

and the writer is punished by an amount $e^{-\alpha t} \delta S_t$ for annulling the contract early,

$$Y_t = e^{-\alpha t} \left(\max \left\{ m, \sup_{u \in [0,t]} S_u \right\} + \delta S_t \right) \text{ for } \delta > 0.$$

Our method of analysis is straightforward. Relying on the results for American put and Russian options (cf., McKean 1965; Shepp and Shiryayev 1995; Graversen and Peškir 1998; Kyprianou and Pistorius 2000; Avram et al. 2002a) we guess the form of the optimal stopping strategies using heuristic arguments based on fluctuation theory and then show, using martingale techniques, that the suggested solutions solve the associated saddle point problem suggested by Kifer's pricing Theorem. For both Israeli δ-penalty put and Russian claim structures, when solving the saddle point problem (V, σ^*, τ^*), it will be the case that $\{e^{-rt} V_t : t \leq \sigma^* \wedge \tau^*\}$ is a uniformly integrable $\mathbb{P}$-martingale whose terminal value is equal to $e^{-r(\sigma^* \wedge \tau^*)} Z_{\sigma^*, \tau^*}$. This is sufficient to construct a hedge and hence the value process of the Israeli option is indeed equal to V. In the following two sections we deal with the Isreali δ-penalty put and Russian options respectively. We conclude the paper with some remarks about Canadization and the finite expiry case.

2 Perpetual Israeli δ-penalty puts

For reflection, let us consider the case of the perpetual American put option with the same parameter K. In this case it is known that the option value is given by the process $\{v^A(S_t) : t \geq 0\}$ where

$$v^A(s) = \sup_{\sigma \in \mathcal{T}_{0,\infty}} \mathbb{E}_s \left(e^{-r\sigma} (K - S_\sigma)^+ \right)$$

which may otherwise be expressed as

$$v^A(s) = \begin{cases} (K - s) & s \in (0, s^*] \\ (K - s^*) (s^*/s)^{2r/\sigma^2} & s \in (s^*, \infty) \end{cases}$$

with

$$s^* = \frac{K}{(1 + \sigma^2/2r)}.$$

Further the optimal stopping strategy is $\sigma_{s^*} = \inf\{t \geq 0 : S_t \leq s^*\}$.

The logic behind this solution is as follows. The holder is interested in stopping to claim when S is as small as possible. On the other hand, if (s)he waits too long for this to happen, then (s)he will be punished through the exponential discounting. The compromise is to stop at some boundary close to zero. Suppose now that the holder has not yet exercised, then the remaining time to expiry in this option is still infinite suggesting the solution is time invariant, that is to say the boundary is a fixed level.

Now let us turn our attention to the Israeli δ-penalty put. In this case, the holder is still interested in stopping as close to zero as possible but without waiting too long. From the writer's perspective, there is a chance to exercise when the value of the asset S is small enough to make $(K - S_\tau)^+ = 0$ in which case, they are only left with the burden of a payment of the form $\delta e^{-r\tau}$. The later this can happen the better. If the initial value of the risky asset is below K then it would seem rational to cancel the contract as soon as S hits K. On the other hand, if the initial value of the risky asset is above K then it would seem rational to wait until the last moment that $S_t \geq K$ in order to prolong the payment. Again, the perpetual nature of the option suggests a time invariant approach to the writers strategy. The conclusion would seem to be a hitting problem of the set $(0, k^*) \cup \{K\}$ for some choice of $k^* < K$. This will turn out to be the case providing the value of δ is not too large. Beyond a certain value of δ it would not seem efficient for the writer to exercise at all. We shall show in this case that the solution is, as one would expect, the same as the American put.

Theorem 2 *Let* $\gamma = \left(r/\sigma^2 + 1/2\right)$ *and define*

$$\delta^* = v^A(K) = \frac{K}{2\gamma}\left(\frac{2\gamma - 1}{2\gamma}\right)^{(2\gamma-1)}.$$

(i) If $\delta \geq \delta^*$ *then the perpetual Israeli* δ*-penalty put option is nothing more than an American put option, that is, the writer will never exercise.*

(ii) If $\delta < \delta^*$ *then the perpetual Israeli* δ*-Put option has value process* $V_t = I^P(S_t)$ *where* $I^P(s)$ *is given by*

$$\begin{array}{ll} K - s & s \in (0, k^*] \\ (K - k^*)\left(\frac{s}{k^*}\right)^{-(\gamma-1)} \frac{(s/K)^\gamma - (s/K)^{-\gamma}}{(k^*/K)^\gamma - (k^*/K)^{-\gamma}} & \\ +\delta\left(\frac{s}{K}\right)^{-(\gamma-1)} \frac{(s/k^*)^{-\gamma} - (s/k^*)^\gamma}{(k^*/K)^\gamma - (k^*/K)^{-\gamma}} & s \in (k^*, K) \\ \delta\left(\frac{s}{K}\right)^{-(2\gamma-1)} & s \in [K, \infty) \end{array}$$

and the optimal stopping strategies for the holder and writer respectively are

$$\sigma^* = \inf\{t \geq 0 : S_t \leq k^*\} \text{ and } \tau^* = \inf\{t \geq 0 : S_t = K\}$$

where k^*/K is the solution in $(0,1)$ to the equation

$$y^{2\gamma} + 2\gamma - 1 = 2\gamma\left(1 + \frac{\delta}{K}\right)y$$

Proof (i) Suppose that $\delta > \delta^*$. Taking the value function $v^A(s)$ recall the well established facts that

$$\left\{e^{-rt}v^A(S_t) : t \geq 0\right\} \text{ and } \left\{e^{-r(t\wedge\sigma_{s^*})}v^A(S_{t\wedge\sigma_{s^*}}) : t \geq 0\right\}$$

are a supermartingale and martingale respectively where $\sigma_{s^*} = \inf\{t \geq 0 : S_t = s^*\}$. Since $\delta > \delta^*$ it follows that

$$(K-s)^+ \leq v^A(s) < (K-s)^+ + \delta \tag{3}$$

(a sketch may help) and hence

$$\begin{aligned}
&v^A(s)\\
&= \inf_{\tau\in\mathcal{T}_{0,\infty}} E_s\left(e^{-r(\tau\wedge\sigma_{s^*})}v^A(S_{\tau\wedge\sigma_{s^*}})\right)\\
&\leq \inf_{\tau\in\mathcal{T}_{0,\infty}} E_s\left(e^{-r(\tau\wedge\sigma_{s^*})}\left[(K-S_{\sigma_{s^*}})^+\mathbf{1}_{(\sigma_{s^*}\leq\tau)} + \{(K-S_\tau)^+ + \delta\}\mathbf{1}_{(\sigma_{s^*}>\tau)}\right]\right)\\
&\leq \sup_{\sigma\in\mathcal{T}_{0,\infty}}\inf_{\tau\in\mathcal{T}_{0,\infty}} E_s\left(e^{-r(\tau\wedge\sigma)}\left[(K-S_\sigma)^+\mathbf{1}_{(\sigma\leq\tau)} + \{(K-S_\tau)^+ + \delta\}\mathbf{1}_{(\sigma>\tau)}\right]\right)\\
&\leq \sup_{\sigma\in\mathcal{T}_{0,\infty}} E_s\left(e^{-r\sigma}(K-S_\sigma)^+\right)\\
&= v^A(s).
\end{aligned}$$

The first equality follows from the martingale property and Doob's Optional Stopping Theorem. The first inequality follows from (3) and the third inequality uses the fact that the infimum can be no greater than the expectation evaluated at $\tau = \infty$. Note also that the order of the supremum and infimum in the second inequality above can also be reversed by starting from the right hand side and reasoning in a similar manner towards the left hand side. It follows with the help of the strong Markov property that both (1) and (2) can be written as $v^A(S_t)$ and a saddle point occurs at $\sigma^* = \sigma_{s^*}$ and $\tau^* = \infty$.

(ii) Let us now suppose then that $\delta \leq \delta^*$. We thus need to conclude that both (1) and (2) are equal to $I^P(S_t)$ and further this is achieved by stopping at $\sigma^* \wedge \tau^*$ with σ^* and τ^* as defined in the statement of the Theorem. To this end, define for general $k \leq K$

$$v(s) = \mathbb{E}_s\left(e^{-r(\sigma_k\wedge\tau_K)}Z_{\sigma_k,\tau_K}\right)$$

where

$$\sigma_k = \inf\{t \geq 0 : S_t \leq k\} \text{ and } \tau_K = \inf\{t \geq 0 : S_t = K\}.$$

We can write

$$v(s) = \begin{cases} K - s & s \in (0,k] \\ (K-k)\mathbb{E}_s\left(e^{-r\sigma_k}\mathbf{1}_{(\sigma_k\leq\tau_K)}\right) + \mathbb{E}_s\left(\delta e^{-r\tau_K}\mathbf{1}_{(\sigma_k>\tau_K)}\right) & s \in (k,K) \\ \delta\mathbb{E}_s\left(e^{-r\tau_K}\right) & s \in [K,\infty) \end{cases}.$$

The expectations in the previous expression are the classic objects of study from the two sided exit problem of Brownian motion; see for example Borodin and Salminen (1996) or Karatzas and Shreve (1988). Filling in we have $v(s)$ is equal to

$$\begin{array}{cc} K-s & s\in(0,k] \\ (K-k)\left(\frac{s}{k}\right)^{-(\gamma-1)}\frac{(s/K)^{\gamma}-(s/K)^{-\gamma}}{(k/K)^{\gamma}-(k/K)^{-\gamma}}+\delta\left(\frac{s}{K}\right)^{-(\gamma-1)}\frac{(s/k)^{-\gamma}-(s/k)^{\gamma}}{(k/K)^{\gamma}-(k/K)^{-\gamma}} & s\in(k,K) \\ \delta\left(\frac{s}{K}\right)^{-(2\gamma-1)} & s\in[K,\infty) \end{array}$$

where $\gamma = (r/\sigma^2 + 1/2)$. Note that there is continuity at $s = k$ and $s = K$.

The remainder of the proof will centre around martingale properties associated with the function v which we shall now discuss. It is immediate from the two sided exit problem that when $s \in (k, K)$

$$\left\{e^{-r(t\wedge\tau_K\wedge\sigma_k)}v\left(S_{t\wedge\tau_K\wedge\sigma_k}\right) : t \geq 0\right\}$$

is a $\mathbb{P}_s$-martingale, alternatively that $(\mathcal{L} - r)\, v\,(s) = 0$ on (k, K) where $\mathcal{L}$ is the infinitesimal generator of the process $(S, \mathbb{P})$. Similarly, from the one sided exit problem, it follows that when $s \geq K$

$$\left\{e^{-r(t\wedge\tau_K)}v\left(S_{t\wedge\tau_K}\right) : t \geq 0\right\}$$

is a $\mathbb{P}_s$-martingale from which it follows that $(\mathcal{L} - r)\, v\,(s) = 0$ on (K, ∞). Finally we can add to these variational equalities that by a trivial computation $(\mathcal{L} - r)\, v\,(s) \leq 0$ on $(0, k)$.

For $s < K$ we want to deduce that $\left\{e^{-r(t\wedge\tau_K)}v\left(S_{t\wedge\tau_K}\right) : t \geq 0\right\}$ is a $\mathbb{P}_s$-supermartingale by applying the Itô formula. The minimum requirement of smoothness on v we can allow without involving local time in the computation is that k is chosen to be a special value k^* such that $v'(k^*) = -1$. That is, there is continuity in v' at k^*. A rather tedious calculation reveals that this condition on k^* amounts to finding a solution in $(0, K)$ to the equation

$$\left(\frac{k^*}{K}\right)^{2\gamma} + 2\gamma - 1 = 2\gamma\left(1 + \frac{\delta}{K}\right)\left(\frac{k^*}{K}\right). \tag{4}$$

Note that if $\delta = \delta^*$ then the solution is easily seen on inspection to be $k^* = s^* = K\,(2\gamma - 1)\,/2\gamma$. Further, as δ decreases the solution k^* increases until $\delta = 0$ where the solution becomes $k^* = K$. It can be further checked that with $k = k^*$, it is also true that v is a convex function on $(0, \infty)$ such that

$$(K - s)^+ \leq v\,(s) \leq (K - s)^+ + \delta. \tag{5}$$

Since now $v(s) \in C^1\ (0, K) \cup C^2\ [(0, K)\backslash\{k^*\}]$ and $(\mathcal{L} - r)\, v\,(s) \leq 0$ on $(0, K)\backslash\{k^*\}$ we can apply the Itô formula to the process

$$\left\{e^{-r(t\wedge\tau_K)}v\left(S_{t\wedge\tau_K}\right) : t \geq 0\right\}$$

and deduce that it is a $\mathbb{P}_s$-supermartingale.

It also follows from Itô's rule for convex functions that on $t \leq \sigma_{k^*}$

$$d\left[e^{-rt}v\left(S_t\right)\right]=e^{-rt}\left(\mathcal{L}-r\right)v\left(S_t\right)dt \\ +e^{-rt}\left(v'\left(K^+\right)-v'\left(K^-\right)\right)L_t^K+dM_t$$

where L^K is the local time at K of S, $v'(K^+)$ and $v'(K^-)$ are the right and left first derivatives of v at K and M_t is a pure martingale term (cf. Karatzas and Shreve Problem 3.6.24). Since $v'\left(K^+\right)-v'\left(K^-\right) \geq 0$ (because of convexity) and $\left(\mathcal{L}-r\right)v\left(s\right)=0$ on $(k^*,\infty)\backslash\{K\}$ it follows that

$$\left\{e^{-r(t\wedge\sigma_{k^*})}v\left(S_{t\wedge\sigma_{k^*}}\right):t\geq 0\right\}$$

is a $\mathbb{P}_s$-submartingale.

With all the previous observations concerning martingales we now have

$$\begin{aligned}
&v\left(s\right)\\
&\leq \inf_{\tau\in\mathcal{T}_{0,\infty}} E_s\left(e^{-r(\tau\wedge\sigma_{k^*})}v\left(S_{\tau\wedge\sigma_{k^*}}\right)\right)\\
&\leq \inf_{\tau\in\mathcal{T}_{0,\infty}} E_s\left(e^{-r(\tau\wedge\sigma_{s^*})}\left[\left(K-S_{\sigma_{k^*}}\right)^+\mathbf{1}_{(\sigma_{k^*}<\tau)}+\{\left(K-S_\tau\right)^+ +\delta\}\mathbf{1}_{(\tau\leq\sigma_{k^*})}\right]\right)\\
&\leq \sup_{\sigma\in\mathcal{T}_{0,\infty}}\inf_{\tau\in\mathcal{T}_{0,\infty}} E_s\left(e^{-r(\tau\wedge\sigma)}\left[\left(K-S_\sigma\right)^+\mathbf{1}_{(\sigma<\tau)}+\{\left(K-S_\tau\right)^+ +\delta\}\mathbf{1}_{(\tau\leq\sigma)}\right]\right)\\
&\leq \sup_{\sigma\in\mathcal{T}_{0,\infty}} E_s\left(e^{-r(\tau_K\wedge\sigma)}\left[\left(K-S_\sigma\right)^+\mathbf{1}_{(\sigma<\tau_K)}+\{\left(K-S_{\tau_K}\right)^+ +\delta\}\mathbf{1}_{(\tau_K\leq\sigma)}\right]\right)\\
&\leq \sup_{\sigma\in\mathcal{T}_{0,\infty}} E_s\left(e^{-r(\tau_K\wedge\sigma)}v\left(S_{\tau_K\wedge\sigma}\right)\right)\\
&\leq v\left(s\right).
\end{aligned}$$

We have used the submartingale property in the first inequality and (5) in the second. For the fourth inequality we have used the fact that the infimum of the expectation over τ is no greater that the expectation evaluated at τ_K. The fifth inequality uses (5) again and the sixth uses the supermartingale property. Again the order of the supremum and infimum can be exchanged by starting from the right hand side and working the inequalities in reverse. Again with the help of the strong Markov Property we have thus established the saddle point in (1) and (2), hence $V_t = I^P(S_t) = v(S_t)$ with $k = k^*$ given by (4). □

Remark 3 There is an intuitive way to see the results that have appeared in Theorem 2. Consider in the same diagram the graph of $(K-s)^+$, $v^A\left(s\right)$ and $(K-s)^+ + \delta$. Given that the writer now has the possibility of removing the rights of the holder, one should expect to see that $I^P\left(s\right)$ is bounded above by the smaller of $v^A\left(s\right)$ and $(K-s)^+ + \delta$. On the other hand, it is also clear that if the writer is to exercise at all, then they should do it when $s \geq K$. With this in mind, when $\delta \geq \delta^*$ a possibility that would make sense is that the writer never exercises and hence the option is nothing more than an American put as the graph of $v^A(s)$ fits between those of $(K-s)^+$ and $(K-s)^+ + \delta$. When $\delta < \delta^*$ and the graphs of $v^A\left(s\right)$ and

$(K-s)^{+}+\delta$ cross over one another, things change. The shape of $I^{P}(s)$ could be imagined to be the result of the following deformation of $v^{A}(s)$. Slowly decrease δ from a large value, so that the curve $(K-s)^{+}+\delta$ pushes down on $v^{A}(s)$ at the contact point $s=K$ reshaping it. A non-smooth ‘angle’ will form at $s=K$ and the smooth join to the line $(K-s)$ will also be dragged towards K.

3 Perpetual Israeli δ-penalty Russian

We begin again by considering the older relative of this option, the perpetual Russian option; a full account of which may be found in the articles of its inventors, Shepp and Shiryayev (1993, 1995). Recall the Russian option has value given by

$$\text{ess-sup}_{\sigma\in\mathcal{T}_{t,\infty}}\mathbb{E}_{s}\left(e^{-r(\sigma-t)}e^{-\alpha\sigma}\max\left\{m,\overline{S}_{\sigma}\right\}\middle|\mathcal{F}_{t}\right)$$

where $\overline{S}_{t}=\sup_{u\in[0,t]}S_{u}$ and $m>s>0$. By using a second change of measure (over and above moving to the risk neutral measure)

$$\left.\frac{d\widetilde{\mathbb{P}}_{s}}{d\mathbb{P}_{s}}\right|_{\mathcal{F}_{t}}=\frac{e^{-rt}S_{t}}{s}$$

and defining $\widetilde{\mathbb{P}}_{m/s}(\cdot)=\widetilde{\mathbb{P}}_{s}\left(\cdot|\overline{S}_{0}=m\right)$, Shepp and Shiryayev (1995) have shown using this change of measure (with the help of the Markov Property) that the value of the option can be more neatly written as

$$\left\{e^{-\alpha t}S_{t}v^{R}\left(\Psi_{t}\right):t\geq 0\right\}$$

where $\Psi=\{\Psi_{t}=\overline{S}_{t}/S_{t}:t\geq 0\}$ and

$$v^{R}(\psi)=\sup_{\sigma\in\mathcal{T}_{0,\infty}}\widetilde{\mathbb{E}}_{\psi}\left(e^{-\alpha\sigma}\Psi_{\sigma}\right).$$

Further, with $\gamma=\left(r/\sigma^{2}+1/2\right)$ as before and $\eta:=\sqrt{2\alpha/\sigma^{2}+\gamma^{2}}$, $v^{R}(\psi)$ can be written as

$$\begin{cases}(\psi_{*}/2\eta)\left[(\gamma+\eta-1)(\psi/\psi_{*})^{\gamma-\eta}+(1-\gamma+\eta)(\psi/\psi_{*})^{\gamma+\eta}\right] & \psi\in[1,\psi_{*}]\\ \psi & \psi\in(\psi_{*},\infty)\end{cases},$$

where

$$\psi_{*}=\left(\frac{\gamma+\eta}{\eta-\gamma}\cdot\frac{\eta-\gamma+1}{\gamma+\eta-1}\right)^{1/2\eta}.$$

Finally, the optimal stopping strategy is given by $\sigma_{\psi_{*}}=\inf\{t\geq 0:\Psi_{t}\geq\psi_{*}\}$.

The logic behind this result when $\alpha>0$ is as follows. The holder is interested in the supremum of the value of the risky asset reaching a high level. However waiting too long for this to happen will again will count against the holder because of the exponential weighting in the payout. If S experiences an excursion from $\overline{S}$ which is large, then the holder will wait a long time for the supremum to increase before the excursion is completed and thus will be penalized. Taking time invariance into account the holder of the Russian option thus behaves optimally by exercising once S gets too far from the previous maximum.

Let us assume temporarily that $\alpha = 0$. When moving to the perpetual Israeli δ-penalty Russian option, it would seem that the holder's intentions should not change if they are to act reasonably. On the other hand, the writer would like to protect themselves against large values of $\overline{S}$. To do this it would seem logical to exercise once the value of S gets too high. Indeed with in an initial value of $\overline{S}$ being m, prudence would suggest it is better to call the contract off once the value of the risky asset hits m. In both cases, the perpetual nature of the option preserves the time invariance of their stopping strategies. With the obvious restriction that δ is not too large we shall show that this is indeed the case. When δ takes large values, it would again not seem rational for the writer to exercise at all, in which case we have returned to the case of the Russian option.

Theorem 4 *Define*

$$\delta_* = v^R(1) - 1 = \frac{(\eta+\gamma-1)\,\psi_*^{\eta-\gamma+1} + (1+\eta-\gamma)\psi_*^{-\gamma-\eta+1} - 2\eta}{2\eta}.$$

(i) Let $\delta \geq \delta_*$ and $\alpha > 0$ then the perpetual Israeli δ-penalty Russian option is nothing more than the perpetual Russian option. That is, the writer's strategy will be to never exercise.

(ii) Let $\delta < \delta_*$ and $\alpha \geq 0$. Define k_* as the solution in $[1,\infty)$ to

$$(\gamma+\eta-1)\,y^{\eta-\gamma+1} + (\eta-\gamma+1)y^{-(\eta+\gamma-1)} = 2\eta(1+\delta).$$

If

$$2\eta k_*^{-\gamma+1} - (1+\delta)\left[(\eta-\gamma)k_*^{\eta} + (\eta+\gamma)k_*^{-\eta}\right] \geq 0 \tag{6}$$

then $V_t = e^{-\alpha t} S_t I^R(\Psi_t)$ where

$$I^R(\psi) = \begin{cases} k_*\left(\frac{\psi}{k_*}\right)^{\gamma} \frac{\psi^{\eta}-\psi^{-\eta}}{k_*^{\eta}-k_*^{-\eta}} + (1+\delta)\,\psi^{\gamma}\frac{(\psi/k_*)^{-\eta}-(\psi/k_*)^{\eta}}{k_*^{\eta}-k_*^{-\eta}} & 1 \leq \psi < k_* \\ \psi & \psi \geq k_*. \end{cases}$$

Further the optimal stopping strategies for the holder and writer respectively are

$$\sigma^* = \inf\{t \geq 0 : \Psi_t \geq k_*\} \text{ and } \tau^* = \inf\{t \geq 0 : \Psi_t = 1\}.$$

Remark 5 Like the proof of the Israeli δ-penalty put option, the method of proof in the second part of the above theorem is to show that $\{e^{-\alpha t} I^R(\Psi_t) : t \geq 0\}$ is a martingale, supermartingale and submartingale when stopped at $\sigma^* \wedge \tau^*$, τ^* and σ^* respectively. The strange technical condition (6) guarantees that the submartingale status can be affirmed. Curiously its presence is strictly necessary as it is possible to make choices of α, σ, r such that the inequality is violated. It is little work to verify that this condition holds when for example $\alpha = 0$. This is quite a natural situation as to some extent the parameter α is a superfluous distraction here. In principle, its presence is merely for the purpose of guaranteeing that the optimal stopping problem associated with the Russian option has a solution (cf., Shepp and Shiryayev 1995).

Proof of Theorem 4 First note that the expressions (1) and (2) can be simplified in a similar way to the Russian option. Indeed we can use the measure $\widetilde{\mathbb{P}}_{m/s}$ together with the Markov property to deduce that they are given by

$$e^{-\alpha t}S_t \times \inf_{\tau\in\mathcal{T}_{0,\infty}} \sup_{\sigma\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi \left(e^{-\alpha\sigma}\Psi_\sigma \mathbf{1}_{(\tau\geq\sigma)} + e^{-\alpha\tau}\left(\Psi_\tau + \delta\right)\mathbf{1}_{(\tau<\sigma)}\right)$$

and

$$e^{-\alpha t}S_t \times \sup_{\sigma\in\mathcal{T}_{0,\infty}} \inf_{\tau\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi \left(e^{-\alpha\sigma}\Psi_\sigma \mathbf{1}_{(\tau\geq\sigma)} + e^{-\alpha\tau}\left(\Psi_\tau + \delta\right)\mathbf{1}_{(\tau<\sigma)}\right)$$

respectively.

(i) Suppose now that $\delta \geq \delta_*$ and $\alpha > 0$. Note that when this happens, we have that

$$\psi \leq v^R(\psi) < \delta + \psi \tag{7}$$

(a quick sketch may help). Recall from well established facts concerning the Russian option

$$\left\{e^{-\alpha(t\wedge\sigma_{\psi_*})}v^R\left(\Psi_{t\wedge\sigma_{\psi_*}}\right) : t \geq 0\right\} \text{ and } \left\{e^{-\alpha t}v^R\left(\Psi_t\right) : t \geq 0\right\}$$

are a $\widetilde{\mathbb{P}}_{m/s}$-martingale and a $\widetilde{\mathbb{P}}_{m/s}$-supermartingale respectively. With these two pieces of information we can deduce that $v^R(\psi)$ is a saddle point value as follows:

$$\begin{aligned}
v^R(\psi) &\geq \sup_{\sigma\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi\left(e^{-\alpha\sigma}v^R\left(\Psi_\sigma\right)\right)\\
&\geq \sup_{\sigma\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi\left(e^{-\alpha\sigma}\Psi_\sigma\right)\\
&\geq \inf_{\tau\in\mathcal{T}_{0,\infty}} \sup_{\sigma\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi\left(e^{-\alpha\sigma}\Psi_\sigma\mathbf{1}_{(\sigma\leq\tau)} + e^{-\alpha\tau}\left(\Psi_\tau+\delta\right)\mathbf{1}_{(\sigma>\tau)}\right)\\
&\geq \inf_{\tau\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi\left(e^{-\alpha\sigma_{\psi_*}}\Psi_{\sigma_{\psi_*}}\mathbf{1}_{(\sigma_{\psi_*}\leq\tau)} + e^{-\alpha\tau}\left(\Psi_\tau+\delta\right)\mathbf{1}_{(\sigma_{\psi_*}>\tau)}\right)\\
&\geq \inf_{\tau\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi\left(e^{-\alpha(\sigma_{\psi_*}\wedge\tau)}v^R\left(\Psi_{\sigma_{\psi_*}\wedge\tau}\right)\right)\\
&= v^R(\psi).
\end{aligned}$$

The first inequality follows by the supermartingale property associated with v^R, the second by (7). The third inequality is a lower bound on the second which one can consider to have the same form of expectation as in the third inequality except with $\tau = \infty$. Note that under $\widetilde{\mathbb{P}}_\psi$ the state 1 is positive recurrent for the process Ψ (the drift of the underlying Brownian motion has become $\gamma\sigma > 0$) and hence from arguments similar to those given in Chapt. VII.2 of Bertoin (1996) one can deduce that in the Poisson point process of excursions (indexed by local time at the maximum) of $\sigma^{-1}\log\Psi_t$ there is an almost surely finite number of excursions whose maximum height, h_t, exceed $c+\xi t$ at local time t where $c, \xi > 0$ are arbitrary constants. It follows that when $\alpha > 0$,

$$\limsup_{t\uparrow\infty} e^{-\alpha t}\Psi_t \leq \limsup_{t\uparrow\infty} e^{-\alpha L^{-1}_{L_t}}e^{\sigma h_{L_t}} = \limsup_{u\uparrow\infty} e^{-\alpha L^{-1}_u}e^{\sigma h_u} = 0$$

$\widetilde{\mathbb{P}}_\psi$-almost surely where L is the local time of Ψ at 1 (recall that the inverse local time L^{-1} is a subordinator hence it grows asymptotically no slower than linearly). The fifth inequality again uses (7) together with the definition of σ_{ψ_*} and the final equality is a consequence of the martingale property associated with v^R. Reversing the arguments we can reverse the order of the infimum and supremum and we find that $e^{-\alpha t} S_t v^R(\Psi_t)$ is the required value of the saddle point problem.

(ii) Let us assume that $\delta < \delta_*$ and $\alpha \geq 0$ then we want to show that $e^{-\alpha t}\ S_t\ I^R\ (\Psi_t)$ solves the saddle point problem given by (1) and (2). To this end let us define for $k > 1$

$$\tau_1 = \inf\{t \geq 0 : \Psi_t = 1\} \text{ and } \sigma_k = \inf\{t \geq 0 : \Psi_t \geq k\}$$

and for $1 \leq \psi < \infty$ the function

$$\begin{aligned} v(\psi) &= \widetilde{\mathbb{E}}_\psi \left(e^{-\alpha\sigma_k} \Psi_{\sigma_k} \mathbf{1}_{(\tau_1 > \sigma_k)} + e^{-\alpha\tau_1} \left(\Psi_{\tau_1} + \delta\right) \mathbf{1}_{(\tau_1 < \sigma_k)} \right) \\ &= \begin{cases} \widetilde{\mathbb{E}}_\psi \left(k e^{-\alpha\sigma_k} \mathbf{1}_{(\tau_1 > \sigma_k)} + e^{-\alpha\tau_1} \left(1 + \delta\right) \mathbf{1}_{(\tau_1 < \sigma_k)} \right) & 1 \leq \psi < k \\ \psi & \psi \geq k \end{cases} . \end{aligned}$$

Note that by construction (that is to say by virtue of the fact that v is the linear sum of solutions to a two sided exit problem for Ψ) we have that $\left(\widetilde{\mathcal{L}} - \alpha\right) v\left(\psi\right) = 0$ for $\psi \in (1, k)$ and $\left(\widetilde{\mathcal{L}} - \alpha\right) v\left(\psi\right) \leq 0$ for $\psi \in (k, \infty)$ where $\widetilde{\mathcal{L}}$ is the infinitesimal generator of $\left(\Psi, \widetilde{\mathbb{P}}\right)$. [To see this recall that $e^{-t\wedge\tau_1\wedge\sigma_k} v(\Psi_{t\wedge\tau_1\wedge\sigma_k})$ is a $\widetilde{\mathbb{P}}_\psi$-martingale and apply the Itô formula]. We will show that for an appropriate choice of k, $v(\psi) = I^R\left(\psi\right)$. The expectations in the right hand side of the above equation can be evaluated using again fluctuation theory. Note that

$$\sigma^{-1} \log\left(\overline{S}_t / S_t\right) = \left(\overline{\beta}_t - \beta_t\right) \tag{8}$$

where under $\widetilde{\mathbb{P}}_{m/s}$, β is a Brownian motion with drift $\sigma\gamma$ where γ was defined in the previous section as $\left(r/\sigma^2 + 1/2\right)$. Using this information, we can use the usual two sided exit problem for Brownian motion to deduce that in fact

$$v(\psi) = \begin{cases} k \left(\frac{\psi}{k}\right)^\gamma \frac{\psi^\eta - \psi^{-\eta}}{k^\eta - k^{-\eta}} + (1+\delta)\, \psi^\gamma \frac{(\psi/k)^{-\eta} - (\psi/k)^\eta}{k^\eta - k^{-\eta}} & 1 \leq \psi < k \\ \psi & \psi \geq k \end{cases} .$$

Note that $v(\psi)$ is continuous at k and $v(1) = 1 + \delta$. We would again like to apply Itô's formula to $v(\Psi_{t\wedge\tau_1})$ in which case we will need at least continuity in v' at k in order to avoid involving local time. Again a series of tedious calculations reveals that by requiring that $k = k_*$ where k_* is the solution to

$$(\gamma + \eta - 1)\, k_*^{\eta - \gamma + 1} + (\eta - \gamma + 1) k_*^{-(\eta + \gamma - 1)} = 2\eta(1 + \delta), \tag{9}$$

then $v'(k_*) = 1$ in which case v is a convex function on $(1, \infty)$ satisfying $v'(1) \geq 0$ when condition (6) holds and further

$$\psi \leq v\left(\psi\right) \leq \psi + \delta. \tag{10}$$

Note that when $\delta = \delta_*$ the solution to (9) is $k_* = \psi_*$ and as δ decreases then so does the value of k_* until finally at $\delta = 0$, $k_* = 1$.

With all the afore mentioned properties of $v(\psi)$ in mind for the choice $k = k_*$, applications of Itô's formula thus yield that

$$\left\{e^{-\alpha(t\wedge\tau_1\wedge\sigma_{k_*})}v\left(\Psi_{t\wedge\tau_1\wedge\sigma_{k_*}}\right) : t \geq 0\right\} \text{ and } \left\{e^{-\alpha(t\wedge\tau_1)}v\left(\Psi_{t\wedge\tau_1}\right) : t \geq 0\right\}$$

are a $\widetilde{\mathbb{P}}_\psi$-martingale and a $\widetilde{\mathbb{P}}_\psi$-supermartingale respectively. Further, Itô calculus (for semi-martingales) reveals that on $t \leq \sigma_{k_*}$, the non-martingale part of $d\left[e^{-\alpha t}v\left(\Psi_t\right)\right]$ takes the form

$$e^{-\alpha t}\left(\widetilde{\mathcal{L}} - \alpha\right)v\left(\Psi_t\right)dt + e^{-\alpha t}S_t^{-1}v'\left(\Psi_t\right)d\overline{S}_t = e^{-\alpha t}S_t^{-1}v'\left(\Psi_t\right)d\overline{S}_t$$

(cf., Shepp and Shiryayev 1995). Since $\overline{S}_t$ only increases when $\Psi_t = 1$ it follows that we could replace $v'\left(\Psi_t\right)$ by $v'(1) \geq 0$ in the above calculation. The consequence of this is that

$$\left\{e^{-\alpha(t\wedge\sigma_{k_*})}v\left(\Psi_{t\wedge\sigma_{k_*}}\right) : t \geq 0\right\}$$

is a $\widetilde{\mathbb{P}}_\psi$-submartingale. The proof of the theorem is now completed in a familiar way, making use of martingale properties, the inequalities in (10) and Doob's Optional Stopping Theorem. That is

$$\begin{aligned}
v(\psi) &\geq \sup_{\sigma\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi\left(e^{-\alpha(\tau_1\wedge\sigma)}v\left(\Psi_{\tau_1\wedge\sigma}\right)\right) \\
&\geq \sup_{\sigma\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi\left(e^{-\alpha\sigma}\Psi_\sigma\mathbf{1}_{(\tau_1>\sigma)} + e^{-\alpha\tau_1}\left(\Psi_{\tau_1}+\delta\right)\mathbf{1}_{(\tau_1<\sigma)}\right) \\
&\geq \inf_{\tau\in\mathcal{T}_{0,\infty}}\sup_{\sigma\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi\left(e^{-\alpha\sigma}\Psi_\sigma\mathbf{1}_{(\tau>\sigma)} + e^{-\alpha\tau}\left(\Psi_\tau+\delta\right)\mathbf{1}_{(\tau<\sigma)}\right) \\
&\geq \inf_{\tau\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi\left(e^{-\alpha\sigma_{k_*}}\Psi_{\sigma_{k_*}}\mathbf{1}_{(\tau>\sigma)} + e^{-\alpha\tau}\left(\Psi_\tau+\delta\right)\mathbf{1}_{(\tau<\sigma_{k_*})}\right) \\
&\geq \inf_{\tau\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi\left(e^{-\alpha\sigma_{k_*}}\Psi_{\sigma_{k_*}}\mathbf{1}_{(\tau>\sigma)} + e^{-\alpha\tau}\left(\Psi_\tau+\delta\right)\mathbf{1}_{(\tau<\sigma_{k_*})}\right) \\
&\geq \inf_{\tau\in\mathcal{T}_{0,\infty}} \widetilde{\mathbb{E}}_\psi\left(e^{-\alpha(\tau\wedge\sigma_{k_*})}v\left(\Psi_{\tau\wedge\sigma_{k_*}}\right)\right) \\
&\geq v(\psi)
\end{aligned}$$

and a similar sequence of inequalities going the other way which establishes the required saddle point. □

4 Conclusion

Israeli options generalize the concept of American options in that they give the writer the opportunity to cancel the contract. Seeing this as lesser rights from the point of view of the holder, a given Israeli option should be no more expensive than an associated American option. Based on this fact, Kifer (2000) has argued that they serve as an interesting derivative in the financial markets and offers a generic pricing formula.

We have shown here that for two familiar claim structures, the put and Russian, within a perpetual context, Kifer's pricing formula reduces to explicit expressions and the optimal stopping times of the holder and writer reduce to intuitively appealing strategies. Further, both cases are essentially barrier options in disguise.

One can also consider the solution to these problems in the context of a free boundary problem. For example, the Israeli δ-penalty put is the unique solution to

$$\begin{aligned} (\mathcal{L} - r)\, I^P(s) &= 0 \text{ on } (k^*, K) \cup (K, \infty) \\ I^P(s) &= (K - s) \text{ on } (0, k^*) \\ dI^P(k^*)/ds &= -1 \\ I^P(K) &= \delta \wedge v^A(K) \\ \lim_{s\uparrow\infty} I^P(s) &= 0 \end{aligned}$$

where k^* is to be determined. Following the terminology of Carr (1988), Canadizing an Israeli option would mean replacing a finite expiry date T by an independent exponential random variable with some rate $\lambda > 0$. If one were to proceed with the Canadized version of the Israeli δ-penalty put, then taking a free boundary perspective, one could solve the following problem for the value function $I^{CP}(s)$

$$\begin{aligned} (\mathcal{L} - r - \lambda)\, v(s) &= -\lambda(K - s)^+ \text{ on } (c^*, K) \cup (K, \infty) \\ v(s) &= (K - s) \text{ on } (0, c^*) \\ dv(c^*)/ds &= -1 \\ v(K) &= \delta \wedge v^{CA}(K) \\ \lim_{s\uparrow\infty} v\,(s) &= 0 \end{aligned}$$

where $v^{CA}(K)$ is the value of the Canadized American put and c^* is to be determined. Alternatively one could address the problem using fluctuation theory and martingales by assuming the solution takes the form

$$\mathbb{E}_s \left(e^{-(r+\lambda)(\tau_K \wedge \sigma_{c^*})} \left[(K - S_{\sigma_{c^*} \wedge \tau_K})^+ + \delta \mathbf{1}_{(\sigma_{c^*} > \tau_K)} \right] \right)$$
$$+ \lambda \mathbb{E}_s \left(\int_0^{\tau_K \wedge \sigma_{c^*}} e^{-(r+\lambda)u} \, (K - S_u)^+ \, du \right)$$

(which can be written out explicitly as a function of s using standard excursion theory for the first term and the resolvent of Brownian motion for the second term). Similar remarks can be made for Candized Israeli δ-penalty Russian options.

If one were to consider the two examples we have dealt with in this paper but for finite expiry, the optimal stopping times for writer and holder are time dependent and yet more difficult to characterize than for American put and Russian options. However in forthcoming work we hope to offer a characterization of such finite expiry Israeli options.

On a final note, it is worth remarking that given the exact analytical expressions obtained in Avram et al. (2002a,b), one may consider re-employing the methods presented here to deal with the same options under spectrally negative and phase-type models. However in these cases, the possibility of jumping over boundaries (or even two boundaries) may present some interesting consequences.

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