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A note on branching Lévy processes

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Abstract

We show for the branching Lévy process that it is possible to construct two classes of multiplicative martingales using stopping lines and solutions to one of two source equations. The first class, similar to those martingales of Chauvin (1991, Ann. Probab. 30, 1195–1205) and Neveu (1988, Seminar on Stochastic Processes 1987, Progress in Probability and Statistics, vol. 15, Birkhaüser, Boston, pp. 223–241) have a source equation which provides travelling wave solutions to a generalized version of the K-P-P equation. For the second class of martingales, similar to those of Biggins and Kyprianou (1997, Ann. Probab. 25, 337–360), the source equation is a functional equation. We show further that under reasonably broad circumstances, these equations share the same solutions and hence the two types of martingales are one and the same. This conclusion also tells us something more about the nature of the solutions to the first of our two equations.

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1. Introduction

The branching Lévy process is defined as follows. An initial ancestor begins its existence at the origin of both the real line and time. It lives for Λ units of time where Λ is exponentially distributed with mean $1/\beta$. During its lifetime, it moves according to a Lévy process, $X(t)$. At the instant of its death, the initial ancestor scatters a random number of children in space relative to its death point according to the point process $\{\zeta_i\colon i=1,\ldots,\Sigma(\mathbb{R})\}$. Attached to each of these children is an independent copy of the triple (Λ, X, Σ) so that they live, move and reproduce independently of one another and in a manner that is stochastically identical to the initial ancestor, and so on. We will assume in this presentation that the process is non-explosive (see for example Athreya

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and Ney (1972, Chapter III) for necessary and sufficient conditions) and supercritical ($E\Sigma(\mathbb{R}) = m > 1$).

This process has been considered recently in the context of martingale convergence by Biggins (1992) and Biggins and Kyprianou (1996). There are also other examples of where this process has been studied in a more specific context. When X is a Brownian motion and $\zeta_i = 0$ for all $i = 1, \ldots, \Sigma(\mathbb{R})$, then we have branching Brownian motion, see for example Athreya and Ney (1972). When there is no movement, so that $X \equiv 0$ then we have a process whose spatial growth was studied by Uchiyama (1982). If in this case we were to make further the restriction that the life length distribution Λ is equal to 1 with probability one, then we have returned to the branching random walk; although the Markovian nature has been lost.

In this report we will bring the results of Neveu (1988), Chauvin (1991) and Biggins and Kyprianou (1997), all of whom explored certain types of multiplicative martingales structured from spatial branching processes, into the context of the branching Lévy process. By doing so we will demonstrate the existence of a class of functions that may be expressed simultaneously as the unique solution of two apparently very different types of equations.

The results of Neveu that were followed by Chauvin's refinements deal with the case of branching Brownian motion. (We should also add here that independently and in parallel Lalley and Sellke (1987) produced results similar to Neveu's). Biggins and Kyprianou then developed the theory of these martingales for the branching random walk. In their case, multiplicative martingales were used as an essential tool to help prove the existence of Seneta–Heyde norming constants for a class of additive martingales. In both types of processes the multiplicative martingales were constructed from the branching process using two main ingredients; stopping lines and a function $\Phi : \mathbb{R} \to [0, 1]$ satisfying a particular source equation. Loosely speaking, stopping lines may be considered as special subsets of the realized branching tree that are measurable in some sense analogous with the concept of stopping times and further have the predominant feature that no two individuals in a stopping line can be a descendent or ancestor to one another; a more precise definition will follow however in the next section. The source equation for the function Φ is different for each of the two processes. In the case of branching Brownian motion with $f(s) = E(s^{\Sigma(\mathbb{R})})$, we have that Φ is a travelling wave solution to the K-P-P equation $\partial u/\partial t = (1/2)\partial^2 u/\partial x^2 + \beta[f(u) - u]$ with boundary conditions. That is to say that Φ satisfies

$$\tfrac{1}{2}\Phi'' + \lambda\Phi' + \beta[f(\Phi) - \Phi] = 0 \text{ such that } \Phi(\infty) = 1 \text{ and } \Phi(-\infty) = 0, \tag{1}$$

where λ is the wave speed. In the branching random walk, this equation is replaced by the functional equation (also known as a smoothing transform)

$$\Phi(x) = E\left[\prod_{i=1}^{\Sigma(\mathbb{R})} \Phi(\mathrm{e}^{-\theta(\zeta_i + L(\theta))})\right] \tag{2}$$

where $L(\theta) = \theta^{-1} \log E(\sum_{i=1}^{\Sigma(\mathbb{R})} \exp\{-\theta\zeta_i\})$.

The multiplicative martingale is constructed by first defining a special sequence of stopping lines (whose index also acts as the index of the martingale). Then for each index, we multiply over every individual in the stopping line a copy of the function Φ, with its argument weighted in a special way by the spatial position of that individual. More explicit details of this weighting are dealt with in the third and fourth sections.

Apart from inducing a deeper understanding of growth through branching, the study of these martingales has provided insight into the nature of solutions to ordinary differential equations and functional equations of the type (1) and (2), respectively. Solutions to the travelling wave solution equation (1) have applications, for example, in genetic theory (see Kolmogorov et al., 1937; Fisher, 1937), and also in statistical physics (see Derrida and Spohn, 1988). Examples of applications of the functional equation (2) include Kahane and Peyrière (1976) who studied this equation in relation to turbulence models, Chauvin and Rouault (1997) who studied its solutions in relation to overlaps in the branching random walk and the Boltzmann-Gibbs measure and Liu (1998) who considered the functional equation in the context of smoothing transforms.

We begin in the next section by refreshing our knowledge on *stopping* lines. We discuss how the Markovian nature of the process and the definitions and properties of stopping lines rigorously outlined by Chauvin (1991) for branching Brownian motion still apply in this case. In the subsequent two sections we will develop separately multiplicative martingales using appropriate generalizations of the source equations (1) and (2). Section 3 is based on ideas that appear mainly in Chauvin (1991) and Section 4 on work appearing in Biggins and Kyprianou (1997). We then demonstrate in the final section that under reasonably broad conditions the two classes of martingales are in fact one and the same because their source equations share the same solutions. This link allows us to exchange information about solutions to the two source equations and thus we obtain new results concerning the existence and uniqueness of solutions to a generalized version of (1).

2. A refresher on stopping lines

For the construction and definition of stopping lines we shall use the Ulam–Harris labelling system. Define the space of all possible nodes

$$\mathscr{I} = \mho \cup \bigcup_{n=1}^{\infty} \mathbb{N}^n$$

where $\mho$ is the label we use for the initial ancestor. A node is labelled $u=(i_1,i_2,\dots,i_n)$, meaning that u is the i_nth child of the i_{n-1}th child of … of the i_1th child of $\mho$. The branching process is considered to be a random tree of nodes, $\mathscr{T}$, rooted at $\mho$, where any realization of $\mathscr{T}$ is a subset of $\mathscr{I}$. For each $u\in\mathscr{T}$ we refer to the generation in which u resides to be $|u|$. There is a partial ordering of individuals on the branching tree in that we can write $u<v$ if u is a strict ancestor of v. We also refer to an individual as uv if that individual is a descendant of u and has label v in the subtree rooted at u.

The birth time of an individual $u\in\mathscr{T}$ is written as σ_u ($\sigma_{\mho}=0$) and thus the death time of the same individual will be $\sigma_u+\Lambda_u$. The birth position is labelled p_u ($p_{\mho}=0$)

and thus the birth position of the ith child of u can be expressed as

$$p_{ui} = p_u + X_u(\Lambda_u) + \zeta_i^{(u)}$$

where X_u is the independent copy of X attached to u and $\zeta_i^{(u)}$ is the ith point of the point process Σ_u, which is itself an independent copy of Σ. The position of u at age s, where $s \leqslant \Lambda_u$ is $\Xi_u(s) := p_u + X_u(s)$.

We are now ready to define a stopping line. Let $\mathscr{B}_u(s)$ be the sigma algebra containing information about the biography of individual u up to and including age s together with the biographies of all of u's ancestry. In particular we have that σ_u, p_u, and $\Xi_u(s')$ $(0 \leqslant s' \leqslant s)$, are all $\mathscr{B}_u(s)$-measurable. Suppose that $\tau : \mathscr{I} \to [0, \Lambda]^{\mathscr{I}}$ is an ensemble of maps such that $\tau_u \in [0, \Lambda_u]$ for each $u \in \mathscr{T}$. The set of individuals $\mathscr{L}(\tau) \subseteq \mathscr{T}$ is a stopping line if

(i) for each $u, v \in \mathscr{L}(\tau)$ such that $u \neq v$, $u \nless v$ and $u \ngtr v$,
(ii) $\tau_u < \Lambda_u$ and
(iii) τ_u is a stopping time for $\mathscr{B}_u(\cdot)$.

A more intuitive construction would be to imagine selecting and pruning branches of the tree $\mathscr{T}$ according to a decision rule τ in which, moving from root upwards, our decision to prune is based purely on what we have seen in moving from the root to that point in the tree. (It is natural in this context that pruned branches can never correspond to individuals who are ancestral/descendent to one another). The branches and points at which we would prune, comprise the stopping line $\mathscr{L}(\tau)$.

In what follows we will often omit the dependency of $\mathscr{L}$ on τ unless we wish to emphasize its importance.

Jagers (1989) offers a less stringent definition for stopping lines using the nodes of a branching tree only. Suppose ℓ is a subset of $\mathscr{I}$ with property (i), let $\mathscr{F}_\ell$ be the sigma algebra containing information about the full life histories of individuals that are neither in nor a descendent of ℓ. Jagers defined a stopping line $\mathscr{L}$ to be a random set of individuals with property (i) for which $\{\forall u \in \ell,\ \exists v \in \mathscr{L} : v \leqslant u\} \in \mathscr{F}_\ell$ for all possible ℓ. The equivalent of this definition for the branching Lévy processes would allow for a greater correlation between the mappings τ_u than in the structure of τ outlined above; an individual's presence in a stopping line may well be dependent on the presence of another individual in the stopping line rather than just the history of its ancestry to date.

For use later on we define

$$\mathscr{D}_{\mathscr{L}} := \{u \in \mathscr{T} : \exists v < u,\ v \in \mathscr{L}\},$$

the set of individuals who have a strict ancestor on the line $\mathscr{L}$ and

$$\mathscr{A}_{\mathscr{L}}(n) := \{u \in \mathscr{T} : |u| = n,\ \nexists v \leqslant u,\ v \in \mathscr{L}\},$$

the set of individuals in the nth generation who have no ancestor (including themselves) on the stopping line. A stopping line $\mathscr{L}$ is called dissecting if $\sup\{n : \mathscr{A}_{\mathscr{L}}(n) \neq \emptyset\} < \infty$ almost surely. Intuitively this implies that any line of descent emanating from the initial ancestor will either die out or meet the stopping line (whichever happens first) with probability one. For two stopping lines, $\mathscr{L}_1$ and $\mathscr{L}_2$ we say that $\mathscr{L}_2$ dominates $\mathscr{L}_1$ if for each $u \in \mathscr{L}_2$ there exists a $v \in \mathscr{L}_1$ such that $v \leqslant u$. A sequence of stopping lines

$\{\mathscr{L}_t\}_{t>0}$ is said to be increasing to infinity if $\mathscr{L}_t$ dominates $\mathscr{L}_s$ for all $0 \leqslant s \leqslant t$ and $\lim_{t\uparrow\infty} \inf\{|u|: u \in \mathscr{L}_t\} = \infty$ almost surely. We should note briefly here that because the movement of individuals is not necessarily continuous, not all dissecting stopping lines in branching Brownian motion have a dissecting equivalent in the branching Lévy process. Consider for example those individuals who are first in their line of descent to cross the space–time line $y = x - at$. This is a stopping line constructed from the mapping τ such that

$$\tau_u = \eta_u \widetilde{\tau}_u + (1 - \eta_u)\Lambda_u$$

where $\eta_u = I(\tau_v = \Lambda_v, \ \forall v < u)$ and $\widetilde{\tau}_u = \inf\{s \in [0, \Lambda_u)\colon \Xi_u(s) = x - a(\sigma_u + s)\}$ if it exists and Λ_u otherwise. (Note that although this is a rather complicated expression for τ, conceptually the idea of first crossings is quite simple.)

There is a natural sigma algebra associated with the stopping line $\mathscr{L}$

$$\mathscr{G}_{\mathscr{L}} = \bigvee_{u \notin \mathscr{D}_{\mathscr{L}}} \mathscr{B}_u(\tau_u).$$

This sigma algebra gives information about the biographies of all individuals on the line $\mathscr{L}$, up to the age at which they meet the stopping line, the biographies of the lives of all the ancestry of all these individuals and the biographies of the lives of all individuals who are part of a line of descent that never meets the stopping line. Using this sigma algebra Chauvin (1991, 1986) and Neveu (1988) have developed a *Strong Markov branching property* on stopping lines for the case of branching Brownian motion. We claim the same result here for the case of branching Lévy processes. No proof is offered however since the Markovian nature of the process ensures that the justifications for this claim do not differ from those of its original proof in branching Brownian motion.

Theorem 1. *Let $\mathscr{L}$ $(=\mathscr{L}(\tau))$ be a stopping line. Given $\mathscr{G}_{\mathscr{L}}$, the trees emanating from each $u \in \mathscr{L}$ at age τ_u are independent stochastic copies of $\mathscr{T}$. This is expressed by the following identity*:

$$E\left(\prod_{u\in\mathscr{L}} f_u \circ T_{u,\tau_u}(x + \cdot) \,\middle|\, \mathscr{G}_{\mathscr{L}}\right) = \prod_{u\in\mathscr{L}} E(f_u(x + \Xi_u(\tau_u) + \cdot) \,|\, \mathscr{G}_{\mathscr{L}}),$$

where T_{u,τ_u} is the shift operator that renders $u \in \mathscr{T}$ at age τ_u the initial ancestor.

Chauvin (1991, 1986) also established the following lemma that illuminates further the Markovian structure present in sequences of stopping lines. In the same way as above, we claim that the lemma, presented below, is equally valid for branching Lévy processes.

Lemma 2. *Suppose $\mathscr{L}_1$ $(=\mathscr{L}_1(\tau))$ and $\mathscr{L}_2$ are two stopping lines such that $\mathscr{L}_2$ dominates $\mathscr{L}_1$. Then $\mathscr{L}_2$ can be partitioned exhaustively and uniquely into mutually exclusive subsets*

$$\mathscr{L}_2^{(v)} := \{u \in \mathscr{L}_2 \colon u \geqslant v \in \mathscr{L}_1\}$$

such that conditional on $\mathscr{G}_{\mathscr{L}_1}$, $\mathscr{L}_2^{(v)}$ is a stopping line on the tree $\mathscr{T} \circ T_{v,\tau_v}$.

The key element to note of the proof of this lemma is that by gluing independent copies of $\mathscr{T}$ onto the points $\{\Xi_v(\tau_v)\}_{v\in\mathscr{L}_1}$ it becomes clear that conditions (i)–(iii) are satisfied locally on these trees. This is because criteria for an individual to be in $\mathscr{L}_2^{(v)}$ requires now only its ancestral history as far back as individual v at age τ_v. We claim here that this insight also shows that the equivalent lemma is not necessarily true for the stopping lines outlined by Jagers. In his definition one can accommodate for sufficiently strong dependencies between $\{\tau_u: u \in \mathscr{L}_2\}$ conditional on $\mathscr{G}_{\mathscr{L}_1}$ to the extent that the $\mathscr{L}_2^{(v)}$ are correlated and thus cannot necessarily be locally stopping lines on the trees emanating from each $v \in \mathscr{L}_1$.

3. Martingales from travelling waves

Let $\mathscr{C}$ be the set of measurable functions $\Phi: \mathbb{R} \to [0,1]$ with the property that Φ is twice differentiable and continuous with its derivatives such that $\Phi(\infty) = 1$ and $\Phi(-\infty)=q$, where q is the probability of extinction. Let $\mathscr{Q}$ be the infinitesimal generator of X with domain $\mathscr{D}(\mathscr{Q})$. It is known that $\mathscr{C} \subseteq \mathscr{D}(\mathscr{Q})$ (see, for example, Bertoin, 1996). We assume for the remainder of this section that $\Phi \in \mathscr{C}$ is a travelling wave solution with wave speed λ to the generalized K-P-P diffusion equation $\partial\pi(x,t)/\partial t = \mathscr{Q}\pi(x,t) + \beta\left[E\prod_{|u|=1}\pi(x+\zeta_u,t) - \pi(x,t)\right]$. That is to say that Φ satisfies

$$\mathscr{Q}\Phi + \lambda\Phi' + \beta\left[E\prod_{|u|=1}\Phi(\cdot + \zeta_u) - \Phi\right] = 0. \tag{3}$$

One notices that when the Lévy process is a straightforward Brownian motion, then Eq. (3) becomes (1) and thus gives travelling wave solutions to the K-P-P equation. There is of course the issue over whether such a solution exists. This will be dealt with later on in Section 5. For the time being we will assume that at least one solution does exist. The main result of this section follows.

Theorem 3. *Let $\{\mathscr{L}_t\}_{t\geqslant 0}$ be an increasing sequence of dissecting stopping lines tending to infinity. The multiplicative structure*

$$M[\lambda,\mathscr{L}_t](x) := \prod_{u\in\mathscr{L}_t}\Phi(x + \Xi_u(\tau_u(t)) + \lambda(\sigma_u + \tau_u(t))) \tag{4}$$

is a $\mathscr{G}_{\mathscr{L}_t}$-martingale, converging almost surely and in expectation to the same variable irrespective of the choice of $\{\mathscr{L}_t\}_{t\geqslant 0}$.

To prove this theorem we need the following lemma.

Lemma 4. *For each $t > 0$,*

$$E\left[I(\Lambda > t)\Phi(x + X(t) + \lambda t) + I(\Lambda \leqslant t)\prod_{|u|=1}\Phi(x + \zeta_u + X(\Lambda) + \lambda\Lambda)\right] = \Phi(x).$$

Proof. Breaking the expectation so that we first average over Σ and then Λ, we are left to evaluate the expectation of

$$I_{t,x} := \mathrm{e}^{-\beta t}\Phi(x + X(t) + \lambda t) + \int_0^t E\left[\prod_{|u|=1} \Phi(x + \zeta_u + X(s) + \lambda s)\right] \beta \mathrm{e}^{-\beta s}\,\mathrm{d}s.$$

Recalling that Φ is a travelling wave solution to the generalized K-P-P equation, we have, defining $\Gamma(t,x) = \exp(-\beta t)\Phi(x + \lambda t)$, that

$$I_{t,x} = \Gamma(t, x + X(t)) - \int_0^t \left(\frac{\partial}{\partial t} + \mathscr{Q}\right) \Gamma(t, x + X(s))\,\mathrm{d}s. \tag{5}$$

The proof can now be completed easily by observing from Stroock (1975) that X is characterized by its operator $\mathscr{Q}$ via the Martingale problem. That is to say that $I_{t,x}$ is a martingale with respect to the natural filtration $\mathscr{F}(t) = \sigma(X(s){:}\ s \leqslant t)$ and thus $EI_{t,x} = EI_{0,x} = \Phi(x)$. □

In proving an equivalent result for branching Brownian motion, Chauvin (1991, Lemma 3.2) referred to a computation involving the Itô formula. Indeed in this case, working via the Itô formula, it is also possible to obtain the result. The necessary steps are to substitute the two-dimensional Itô formula for the Lévy process, (see, for example, Protter, 1991) and the specific form of the operator $\mathscr{Q}$ (see, for example, Bertoin, 1996) in the first and second terms of (5), respectively. Taking expectations, the result follows by a tedious process of expansion and cancellation of terms. Note, however, in the case of branching Brownian motion, the computation will be considerably simpler as there are no jump or drift terms to deal with.

Proof of Theorem 3. Let $\mathscr{L}$ be any dissecting stopping line. Define the approximation to $M[\lambda, \mathscr{L}](x)$,

$$M^{(n)}[\lambda, \mathscr{L}](x) = \prod_{\substack{|u| \leqslant n \\ u \in \mathscr{L}}} \Phi(x + \Xi_u(\tau_u) + \lambda(\sigma_u + \tau_u))$$
$$\times \prod_{u \in \mathscr{A}_{\mathscr{L}}(n)} \prod_{|v|=1} \Phi(x + \zeta_v + \Xi_u(\Lambda_u) + \lambda(\sigma_u + \Lambda_u)).$$

Let $\mathscr{F}_n$ be the biography of all individuals in the nth generation and their ancestry. We begin by showing that $M^{(n)}[\lambda, \mathscr{L}](x)$ is an $\mathscr{F}_n$-martingale. Since $\mathscr{L}$ is dissecting we then have that $M^{(n)}[\lambda, \mathscr{L}](x)$ converges almost surely and in expectation to $M[\lambda, \mathscr{L}](x)$ as $n \uparrow \infty$ and hence $\Phi(x) = EM^{(0)}[\lambda, \mathscr{L}](x) = EM[\lambda, \mathscr{L}](x)$. So, to show the martingale property, consider the following decomposition:

$$E[M^{(n+1)}[\lambda, \mathscr{L}](x) \mid \mathscr{F}_n]$$
$$= \prod_{\substack{|u| \leqslant n \\ u \in \mathscr{L}}} \Phi(x + \Xi_u(\tau_u) + \lambda(\sigma_u + \tau_u))$$

$$\times E_{\mathscr{F}_n} \prod_{\substack{|u|=n+1\\ u\notin\mathscr{D}_{\mathscr{L}}}} \Bigg\{ I(\Lambda_u > \tau_u)\Phi(x + p_u + X_u(\tau_u) + \lambda(\sigma_u + \tau_u)) + I(\Lambda_u \leqslant \tau_u)\prod_{|v|=1}\Phi(x + \zeta_v + p_u + X_u(\Lambda_u) + \lambda(\sigma_u + \Lambda_u))\Bigg\}.$$

Let $\mathscr{Q}(\rho)$ be the stopping line with $\rho_u = 0$ if $u \notin \mathscr{D}_{\mathscr{L}}$ and $|u| = n+1$, $\rho_u = \tau_u$ if $u \in \mathscr{L}$ and $|u| \leqslant n$ and $\rho_u = \Lambda_u$, otherwise. Noting that p_u and σ_u are all $\mathscr{F}_n$-measurable, Lemma 2 applied to the stopping lines $\mathscr{L}$ and $\mathscr{Q}$, Theorem 1 and the previous lemma, we conclude that the expectation on the right-hand side is equal to

$$\prod_{\substack{|u|=n+1\\ u\notin\mathscr{D}_{\mathscr{L}}}} \Phi(x + p_u + \lambda\sigma_u) = \prod_{u\in\mathscr{A}_{\mathscr{L}}(n)}\prod_{|v|=1}\Phi(x + p_{uv} + \lambda\sigma_{uv})$$

$$= \prod_{u\in\mathscr{A}_{\mathscr{L}}(n)}\prod_{|v|=1}\Phi(x + \Xi_u(\Lambda_u) + \zeta_v + \lambda(\sigma_u + \Lambda_u))$$

hence the result follows. For $0 \leqslant s \leqslant t$, members of $\mathscr{L}_t$ may be divided exhaustively into groups of descendents, labelled $\mathscr{L}^{(u)}_{s,t}$, from each member of $u \in \mathscr{L}_s$. Hence Lemma 2 and Theorem 1 combined give the following identity:

$$E(M[\lambda, \mathscr{L}_t](x) \,|\, \mathscr{G}_{\mathscr{L}_s}) = \prod_{u\in\mathscr{L}_s} E(M_u[\lambda, \mathscr{L}^{(u)}_{s,t}](x + \Xi_u(\tau_u(s)) + \lambda(\sigma_u + \tau_u(s))) \,|\, \mathscr{G}_{\mathscr{L}_s}) \tag{6}$$

where given $\mathscr{G}_{\mathscr{L}_s}$, $M_u[\lambda,\cdot](\cdot)$ is an independent copy of $M[\lambda,\cdot](\cdot)$. Finally our previous conclusion that $EM[\lambda,\mathscr{L}](x) = \Phi(x)$ completes the proof that $M[\lambda,\mathscr{L}_t](x)$ is a martingale.

To prove that the martingale limit is sequence independent, we consider $\{\mathscr{L}_t\}_{t\geqslant 0}$ and $\{\mathscr{K}_t\}_{t\geqslant 0}$ satisfying the conditions of the theorem. The respective associated multiplicative martingales have limits (by the *bounded martingale convergence theorem*) which we shall label $M_{\mathscr{L}}[\lambda](x)$ and $M_{\mathscr{K}}[\lambda](x)$. For each $t > 0$ it is possible to find a stopping time $T(\mathscr{L}_t)$ with respect to $\{\mathscr{G}_{\mathscr{L}_t}\}_{t\geqslant 0}$ such that $\mathscr{K}_{T(\mathscr{L}_t)+s}$ dominates $\mathscr{L}_t$ for all $s > 0$. Likewise we can find a stopping time $T(\mathscr{K}_t)$ fulfilling a symetrical role to $T(\mathscr{L}_t)$. Consequently, $M[\lambda,\mathscr{L}_t](x) = E(M[\lambda,\mathscr{K}_{T(\mathscr{L}_t)+s}](x) \,|\, \mathscr{G}_{\mathscr{L}_t})$ and $M[\lambda,\mathscr{K}_t](x) = E(M[\lambda,\mathscr{L}_{T(\mathscr{K}_t)+s}](x) \,|\, \mathscr{G}_{\mathscr{K}_t})$. Taking limits as $s \uparrow \infty$ the *bounded convergence theorem* implies

$$M[\lambda,\mathscr{L}_t](x) = E(M_{\mathscr{K}}[\lambda](x) \,|\, \mathscr{G}_{\mathscr{L}_t}) \quad\text{and}\quad M[\lambda,\mathscr{K}_t](x) = E(M_{\mathscr{L}}[\lambda](x) \,|\, \mathscr{G}_{\mathscr{K}_t}),$$

the consequence of which is that $M[\lambda,\mathscr{L}_t](x)$ and $M[\lambda,\mathscr{K}_t](x)$ have the same limit as $t \uparrow \infty$. □

4. Martingales from the functional equation

Before proceeding to our construction of our second type of multiplicative martingales, we will consider the functional equation from which we will build them.

By defining the map τ such that

$$\tau_u = \eta_u(t-\sigma_u) + (1-\eta_u)\Lambda_u$$

where $\eta_u = I(\sigma_u \leqslant t < \sigma_u + \Lambda_u)$, it can be checked that the corresponding stopping line, which we shall label $\mathscr{N}_t$, is dissecting and consists of individuals alive at time t. In this section, we will assume that Φ is a solution to the equation

$$\Phi(x) = E\left[\prod_{u\in\mathscr{N}_t} \Phi(x + \Xi_u(t-\sigma_u) + \lambda t)\right] \quad \text{for all } t>0 \tag{7}$$

and some $\lambda \in \mathbb{R}$. By making the transform $\Phi(x) = \Psi(\exp(-\theta x))$ one can see that this functional equation is a continuous version of the functional equation (2). The existence of solutions to this functional equation again is an issue that needs to be dealt with; however again, we leave that for the remaining section. Thus follows the main result of this section.

Theorem 5. *Theorem* 3 *still holds when* (7) *is the source equation for* $M[\lambda, \mathscr{L}_t](x)$.

The proof of this theorem is motivated by arguments that appear in Biggins and Kyprianou (1997).

Proof of Theorem 5. Using the decomposition (6) specifically for the stopping line $\mathscr{N}_t$ we have, noting that $\sigma_u + \tau_u(t) = t$,

$$M[\lambda, \mathscr{N}_{t+s}](x) = \prod_{u\in\mathscr{N}_t} M_u[\lambda, \mathscr{N}_s^{(u)}](x + \Xi_u(t-\sigma_u) + \lambda t)$$

where given $\mathscr{G}_{\mathscr{N}_t}$, $\mathscr{N}_s^{(u)}$ is an independent copy of $\mathscr{N}_s$. Since Φ satisfies (7) it follows with the help of Theorem 1 that $M[\lambda, \mathscr{N}_t](x)$ is a martingale. Note further that this martingale converges almost surely and in expectation to some variable we will label $M[\lambda](x)$ which has mean $\Phi(x)$.

Consider now the following decomposition of $M[\lambda, \mathscr{N}_t](x)$ according to the intersection of $\mathscr{N}_t$ with any other dissecting stopping line $\mathscr{L}$.

$$M[\lambda, \mathscr{N}_t](x) = \prod_{u\in\mathscr{A}_{\mathscr{L}}(t)} \Phi(x + \Xi_u(t-\sigma_u) + \lambda t)$$

$$\times \prod_{\substack{u\in\mathscr{L}\\ \sigma_u+\tau_u\leqslant t}} M_u[\lambda, \mathscr{N}^{(u)}_{t-(\sigma_u+\tau_u)}](x + \Xi_u(\tau_u) + \lambda(\sigma_u+\tau_u)).$$

Here, $\mathscr{A}_{\mathscr{L}}(t)$ takes a similar definition to $\mathscr{A}_{\mathscr{L}}(n)$; the set of individuals alive at time t that have no ancestors (including themselves) in $\mathscr{L}$. Since $\mathscr{L}$ is a dissecting stopping line it has an almost surely finite number of members, each of whom has an almost surely finite time of death. Therefore it follows that $\mathscr{A}_{\mathscr{L}}(t)$ converges to the empty set as $t\uparrow\infty$. Combining this conclusion with the *bounded convergence theorem*, the fact that $M[\lambda, \mathscr{N}_t](x)$ is a martingale and Theorem 1, we have that

$$E(M[\lambda](x)\,|\,\mathscr{G}_{\mathscr{L}}) = \lim_{t\uparrow\infty} E(M[\lambda, \mathscr{N}_t](x)\,|\,\mathscr{G}_{\mathscr{L}})$$

$$= \lim_{t\uparrow\infty} \prod_{u\in\mathscr{A}_{\mathscr{L}}(t)} \Phi(x+\Xi_u(t-\sigma_u)+\lambda t)$$

$$\times \prod_{\substack{u\in\mathscr{L}\\ \sigma_{u+}\tau_u\leqslant t}} E(M_u[\lambda, \mathscr{N}^{(u)}_{t-(\sigma_{u+}\tau_u)}](x+\Xi_u(\tau_u)+\lambda(\sigma_{u+}\tau_u))\,|\,\mathscr{G}_{\mathscr{L}})$$

$$= \prod_{u\in\mathscr{L}} \Phi(x+\Xi_u(\tau_u)+\lambda(\sigma_{u+}\tau_u))$$

$$= M[\lambda, \mathscr{L}](x). \tag{8}$$

It now follows directly from (8) with the assistance of the ladder property of conditional expectation that $M[\lambda, \mathscr{L}_t](x)$ is a martingale. Taking unconditional expectations of (8) reveals that the martingale expectation is $\Phi(x)$. Further, the proof that the limit of the martingale is independent of the choice of sequence of stopping lines follows as in the proof of Theorem 3. □

5. Common solutions to the source equations

Before we can make any conclusions about the commonality of solutions to the source equations, we must be sure that at least one solution to either (3) or (7) exists. We will prove so for the latter as it follows as a simple consequence of a Seneta–Heyde norming result established in Biggins and Kyprianou (1996).

Define $Z^{(t)}$ to be the point process describing the positions of individuals alive at time t. Let $\lambda(\phi)=\phi^{-1}\log E[\int \exp\{-\phi p\}Z^{(1)}(\mathrm{d}p)]$ and $\mathscr{R}$ be the set of all measurable functions $\Psi: \mathbb{R}\mapsto[0,1]$, such that $x^{-1}[1-\Psi(x)]$ is monotone decreasing. For computational purposes it should be noted that $\lambda(\phi)$ can be written $\phi^{-1}[\beta b(\phi)-\beta+a(\phi)]$ where $b(\phi)=E\int\exp(-\phi u)\Sigma(\mathrm{d}u)$ and $a(\phi)$ is the Lévy–Khintchine exponent of the Laplace transform of X (see Biggins, 1992).

Theorem 6. *When* $\theta\in\operatorname{int}\{\phi: \lambda(\phi)<\infty\}$, *assumed to be a non-empty set, and* $\lambda'(\theta)<0$, *there exists a unique solution (up to a multiplicative constant) in* $\mathscr{R}$ *to the functional equation*

$$\Psi(x) = E\left[\prod_{u\in\mathscr{N}_t}\Psi(x\mathrm{e}^{-\theta(\Xi_u(t-\sigma_u)+\lambda(\theta)t)})\right] \quad \text{for every } t>0 \tag{9}$$

Proof. The following proof is an adaptation of a method used in Biggins and Kyprianou (1996) to demonstrate the existence of a sequence of Seneta–Heyde norming constants for the martingale $W^{(t)}(\theta):=\int\exp\{-\phi x\}Z^{(t)}(\mathrm{d}x)$. By considering a skeleton of time steps on the lattice $\{n\delta\}_{n\geqslant0}$ ($\delta>0$), there exists a branching random walk embedded within the branching Lévy process. (This statement can be justified more rigorously by simply applying Theorem 1 and Lemma 2 to the stopping lines $\{\mathscr{N}_{n\delta}\}_{n\geqslant0}$). Under the conditions stated, it can trivially be checked that the Seneta–Heyde norming result of Biggins and Kyprianou (1997) holds on the embedded process (for each $\delta>0$) so that there exist a sequence of constants $\{C_n^{(\delta)}(\theta)\}$ such that

$\lim_{t\uparrow\infty} C_n^{(\delta)}(\theta)\ W^{(n\delta)}(\theta) = \Delta_\delta(\theta)$ where its Laplace transform is the unique solution in $\mathscr{R}$ to the functional equation (9) with $t=\delta$. With the reasoning in Section 6 of Biggins and Kyprianou (1996) we can use this result on the embedded process to show that indeed there exist constants $C_t^{(\delta)}(\theta){:=}C_{[t]}^{(\delta)}(\theta)$ such that $\lim_{t\uparrow\infty} C_t^{(\delta)}(\theta)\ W^{(t)}(\theta) = \Delta_\delta(\theta)$ for each $\delta > 0$. Since any two sequences of norming constants must be asymptotically equivalent the limits $\Delta_\delta(\theta)$ must be equal up to a constant of proportionality (dependent on δ) and thus the result follows. □

We can make Theorem 6 more suitable to the context of our multiplicative martingales, by making the transformation $\Phi(x) = \Psi(\exp(-\theta x))$. We have then, under the conditions of the above theorem, a unique solution in the class of functions $\mathscr{R}_\theta{:=}\{\Phi(x) = \Psi(\mathrm{e}^{-\theta x})\colon \Psi \in \mathscr{R}\}$ to Eq. (7). It is possible however to say even more as a direct consequence of taking expectations of Eq. (8).

Corollary 7. *Under the conditions of Theorem* 6 *there exists a unique solution* (*up to an additive constant*) *in* $\mathscr{R}_\theta$ *that satisfies the functional equation*

$$\Phi(x) = E\left[\prod_{u\in\mathscr{L}} \Phi(x + \Xi_u(\tau_u) + \lambda(\theta)(\sigma_u + \tau_u))\right] \tag{10}$$

for any dissecting stopping line $\mathscr{L}$.

We now state the main conclusion of this section.

Theorem 8. *The function* $\Phi \in \mathscr{C}$ *is a solution to the functional equation* (10) *for any dissecting stopping line* $\mathscr{L}$ *if and only if it is a solution to the integro-differential equation* (3).

Quite clearly, under the conditions of this theorem, both types of martingales become one and the same. The next Corollary follows immediately from this theorem and the proof of Theorem 6.

Corollary 9. *When* $\theta \in \mathrm{int}\{\phi\colon \lambda(\phi) < \infty\}$, *assumed to be a non-empty set, and* $\lambda'(\theta) < 0$, *there exists a positive variable* $\Delta(\theta)$ *such that*

$$\Phi(x) = E[\exp(-\mathrm{e}^{-\theta x}\Delta(\theta))]$$

is the unique solution (*up to an additive constant*) *in* $\mathscr{R}_\theta$ *to* (3) *for* $\lambda = \lambda(\theta)$ *with boundary condition* $\Phi(\infty) = 1$ *and* $\Phi(-\infty) = q$.

Note that when X is a basic Brownian motion and Σ is concentrated at the origin then this result is consistent with Theorem 3.4 of Chauvin (1991) although the conditions here are different. Chauvin has a class broader than $\mathscr{R}_\theta$, namely $\mathscr{C}$ (but with $\Phi(-\infty) = 0$) and also she accommodates for the case that $\lambda'(\theta) = 0$ (note that in branching Brownian motion $\lambda(\theta)$ parameterizes the wave speeds and a short computation shows $\lambda'(\theta) = 0$ corresponds to the minimal wave speed $\sqrt{2\beta(m-1)}$). The result presented here identifies specifically Δ to be that of the limiting variable in the Seneta–Heyde norming problem discussed in Biggins and Kyprianou (1996).

It is not unreasonable to conjecture that Corollary 9 can be strengthened by replacing the set $\mathscr{R}_\theta$ by $\mathscr{C}$. The way forward lies with a better understanding of asymptotics of common solutions to the source equations. Either in the light of Uchiyama (1978) and Bramson (1983) or Kyprianou (1998) the slow variation of $e^{x\theta}[1-\Phi(x)]$ as $x\uparrow\infty$ for $\Phi\in\mathscr{C}$ should be established from (3) or (7), respectively. As was demonstrated by both Chauvin (1991) and Biggins and Kyprianou (1997) this leads to the relation $-\log M[\lambda](x)=-e^{-\theta x}\log M[\lambda](1)$. Hence we have that Φ is the Laplace transform of $-\log M[\lambda](1)$ and falls also in the class of functions $\mathscr{R}_\theta$ for which we know we have uniqueness.

In order that we may prove Theorem 8 (and thus close this paper) we must first prove the following lemma.

Lemma 10. *Let $\Phi\in\mathscr{C}$, then*

$$\pi(x,t)=E\left[\prod_{u\in\mathscr{N}_t}\Phi(x+\Xi_u(t-\sigma_u))\right]$$

is in $\mathscr{D}(\mathscr{Q})$, differentiable with respect to t and is a solution to the generalized K-P-P equation

$$\frac{\partial\pi(x,t)}{\partial t}=\mathscr{Q}\pi(x,t)+\beta\left[E\prod_{|u|=1}\pi(x+\zeta_u,t)-\pi(x,t)\right]$$

with initial data $\pi(x,0)=\Phi(x)$.

This lemma resembles very much results found in McKean (1975) and Bramson (1978). Indeed, when X is a Brownian motion, Σ is concentrated at the origin and $\Phi(x)=I\ (x\geqslant 0)$, the results here agree with the aforementioned results. Not surprisingly the proof we offer here follows a similar logic to that of Bramson (1978, Appendix A).

Proof of Lemma 10. We begin by decomposing π according to whether the initial ancestor has deceased by time t, so that after some straightforward computation and rearrangement

$$\pi(x,t)=e^{-\beta t}E[\Phi(x+X(t))]$$
$$+\int_0^t\beta e^{-\beta(t-z)}E\left(\prod_{v\in\mathscr{N}_t}\Phi(x+\Xi_v(t-\sigma_v))\,\middle|\,\Lambda=t-z\right)dz. \quad (11)$$

Define $g(x,t)=E[\Phi(x+X(t))]$. Noting that $g(x,t)\in\mathscr{D}(\mathscr{Q})$, one can construct an argument using Dynkin's formula (similar to Theorem 8.1 of Øksendal (1995) for example) to show that $\mathscr{Q}g(\cdot,t)=\partial g(\cdot,t)/\partial t$. Further, defining $h(x,t,z)$ as the second expectation in (11) it is not difficult to see that since the product is over an almost surely finite number of terms, $h(\cdot,t,z)$ is differentiable in the sense that it is a member of $\mathscr{D}(\mathscr{Q})$ and thus the same argument used above shows that also $\mathscr{Q}h(\cdot,t,z)=\partial h(\cdot,t,z)/\partial t$.

Let us now call the two parts in the sum of Eq. (11) S_1 and S_2. A straightforward calculation shows

$$\frac{\partial S_1}{\partial t} = \mathrm{e}^{-\beta t}\mathscr{Q}E[\Phi(x+X(t))] - \beta\mathrm{e}^{-\beta t}E[\Phi(x+X(t))], \tag{12}$$

$$\begin{aligned}\frac{\partial S_2}{\partial t} &= \beta E\left(\prod_{|u|=1}\pi(x+\zeta_u,t)\right)\\ &\quad -\int_0^t \beta^2\mathrm{e}^{-\beta(t-z)}E\left(\prod_{v\in\mathcal{N}_t}\Phi(x+\Xi_v(t-\sigma_v))\middle|\Lambda=t-z\right)\mathrm{d}z\\ &\quad +\int_0^t \beta\mathrm{e}^{-\beta(t-z)}\mathscr{Q}E\left(\prod_{v\in\mathcal{N}_t}\Phi(x+\Xi_v(t-\sigma_v))\middle|\Lambda=t-z\right)\mathrm{d}z\end{aligned} \tag{13}$$

thus showing that π is differentiable with respect to t. In the final term of (13) we can move $\mathscr{Q}$ to the left of the integral by fundamental theorems of calculus (c.f Bertoin (1996), $\mathscr{Q}$ may be expressed as linear combination of differential and integral operators). Collecting and tidying terms from (12) and (13) using the decomposition (11) the result follows. □

Proof of Theorem 8. Suppose that $\Phi\in\mathscr{C}$ is a travelling wave solution satisfying (3) then from the first half of the proof of Theorem 3, for any dissecting stopping line $\mathscr{L}$, $EM[\lambda,\mathscr{L}](x)=\Phi(x)$ and hence Φ is also a solution to the functional equation (10) for all dissecting stopping lines. Suppose now that $\Phi\in\mathscr{C}$ is a solution to the functional equation (10) for all dissecting stopping lines. By the previous lemma we thus have that $\Phi(x-\lambda(\theta)t)=\pi(x,t)$ solves the generalized K-P-P equation and thus Φ is a travelling wave solution to (3). □

Theorem 8 shows a parallel between the two source equations from the point of view of building multiplicative martingales. It also shows us however the parallel between the known results of the right most particle problem for branching Brownian motion and the equivalent anticipated (but not yet proven) results for the right most particle problem for the branching random walk. In branching Brownian motion, it has been shown by Bramson (1983) that there exists a sequence of centering constants c_t such that for the displacement of the right most individual alive at time t, B_t, we have that B_t-c_t converges in distribution. The limiting distribution function is also a travelling wave solution to the K-P-P equation. For the branching random walk, it is expected (see, for example, Kyprianou (1998) or Dekking and Host (1991)) that in the analogous case, the limiting distribution does not satisfy an ordinary differential equation, but the functional equation (2). The results here show that the two cases would then be true contemporaries.

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