

Skeletal stochastic differential equations for continuous-state branching process

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DEFINITION OF ψ -CSBP.

A CSBP (X, \mathbb{P}_x) is a non-negative valued strong Markov process with probabilities $(\mathbb{P}_x, x \geq 0)$ such that for any $x, y \geq 0$, $\mathbb{P}_{x+y} = \mathbb{P}_x * \mathbb{P}_y$.

In particular

$$\mathbb{E}_x(e^{-\theta X_t}) = e^{-xu_t(\theta)}, \quad x, \theta, t \geq 0,$$

where $u_t(\theta)$ uniquely solves the evolution equation

$$u_t(\theta) + \int_0^t \psi(u_s(\theta)) ds = \theta, \quad t \geq 0.$$

Here, we assume that the so-called branching mechanism ψ takes the form

$$\psi(\theta) = -\alpha\theta + \beta\theta^2 + \int_{(0, \infty)} (e^{-\theta x} - 1 + \theta x) \Pi(dx), \quad \theta \geq 0,$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ and Π is a measure concentrated on $(0, \infty)$ which satisfies $\int_{(0, \infty)} (x \wedge x^2) \Pi(dx) < \infty$.

PROPERTIES.

We assume that the process is **conservative**, i.e.

$$\int_{0+}^{\infty} \frac{1}{|\psi(\xi)|} d\xi = \infty.$$

It is easily verified that

$$\mathbb{E}_x[X_t] = xe^{-\psi'(0+)t}, \quad t, x \geq 0.$$

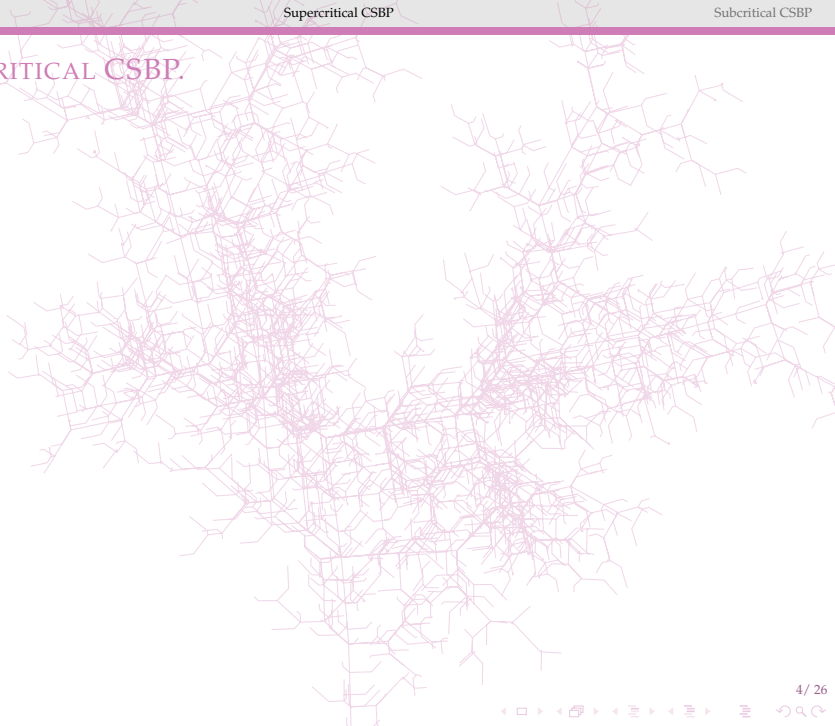
We say that the CSBP is **supercritical**, **critical** or **subcritical** accordingly as $-\psi'(0+) = \alpha$ is strictly positive, equal to zero or strictly negative.

For a **supercritical** ψ -CSBP the **probability of extinction** is

$$\mathbb{P}_x(\lim_{t \uparrow \infty} X_t = 0) = e^{-\lambda^* x},$$

where λ^* is the unique root on $(0, \infty)$ of the equation $\psi(\theta) = 0$.

SUPERCritical CSBP.



PROLIFIC SKELETON I.

The **supercritical** ψ -CSBP is equal in law to the total mass process obtained by the following construction.

- ▶ Initiate **Po**(λ^*x) **independent Galton-Watson processes** with branching generator

$$q \left(\sum_{k \geq 0} p_k r^k - r \right) = \frac{1}{\lambda^*} \psi(\lambda^*(1-r)), \quad r \in [0, 1],$$

where $q = \psi'(\lambda^*)$, $p_0 = p_1 = 0$ and for $k \geq 2$

$$p_k = \frac{1}{\lambda^* \psi'(\lambda^*)} \left\{ \beta(\lambda^*)^2 \mathbf{1}_{\{k=2\}} + (\lambda^*)^k \int_{(0, \infty)} \frac{r^k}{k!} e^{-\lambda^* r} \Pi(dr) \right\}.$$

- ▶ Along the edges **immigrate CSBPs** at rate

$$2\beta d\mathbb{Q}^* + \int_0^\infty y e^{-\lambda^* y} \Pi(dy) d\mathbb{P}_y^*,$$

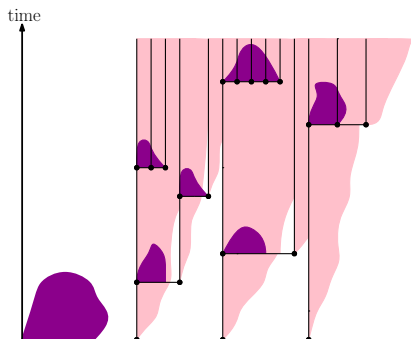
where \mathbb{P}_x^* , $x \geq 0$ is the law of the CSBP **with branching mechanism** $\psi^*(\lambda) = \psi(\lambda + \lambda^*)$ and \mathbb{Q}^* is the associated excursion measure.

PROLIFIC SKELETON II.

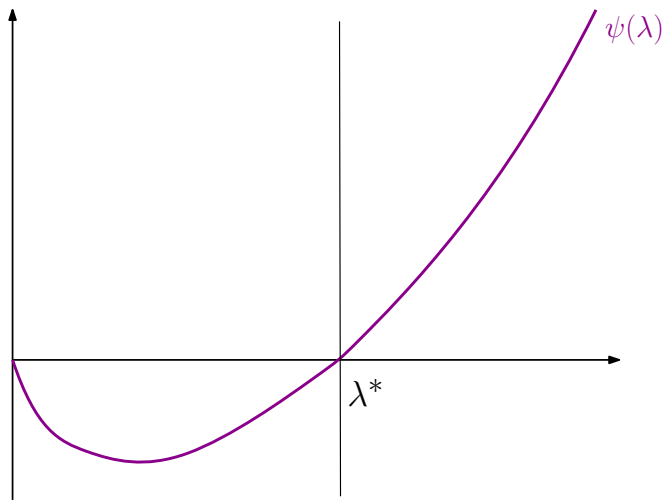
- ▶ Given that an individual dies and branches into $k \geq 2$ offspring, an independent ψ^* -CSBP is immigrated with initial mass r with probability

$$\eta_k(dr) = \frac{1}{p_k \lambda^* \psi'(\lambda^*)} \left\{ \beta (\lambda^*)^2 \delta_0(dr) \mathbf{1}_{\{k=2\}} + (\lambda^*)^k \frac{r^k}{k!} e^{-\lambda^* r} \Pi(dr) \right\}.$$

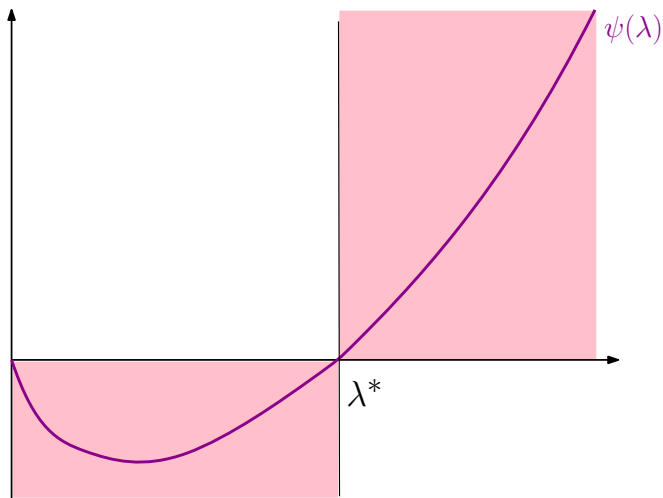
- ▶ Finally an **independent ψ^* -CSBP** is issued at time zero with initial mass x .



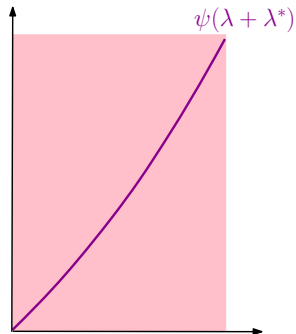
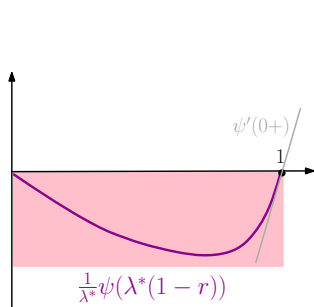
BRANCHING MECHANISM.



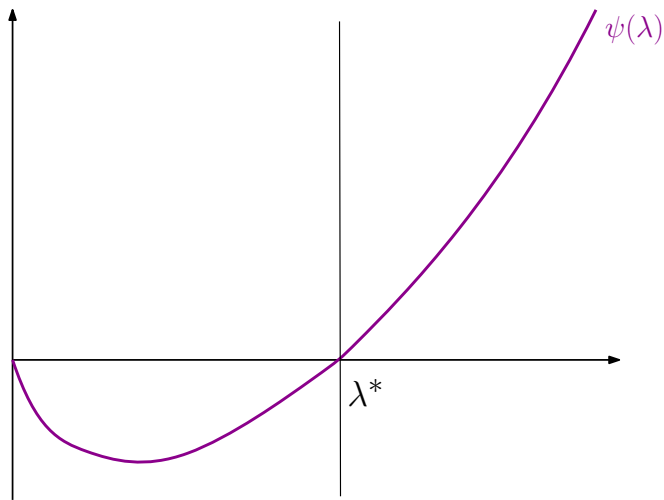
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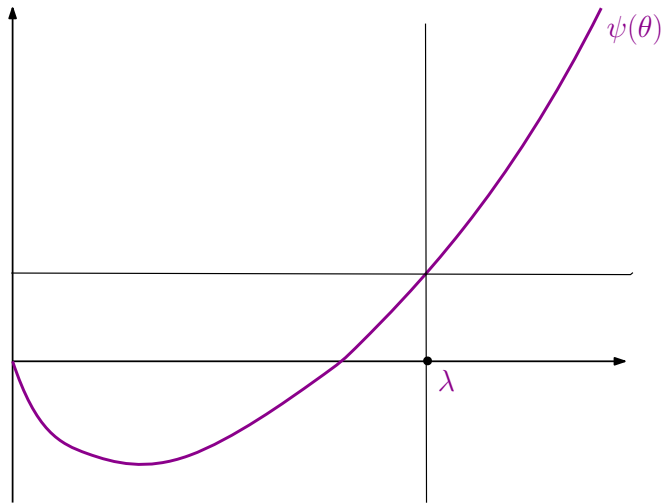
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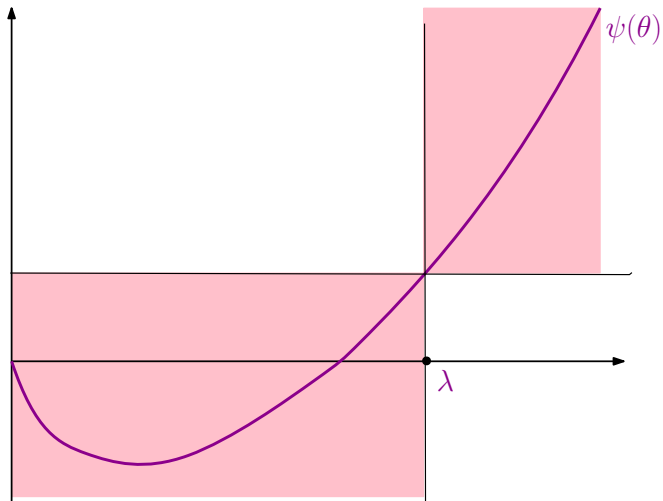
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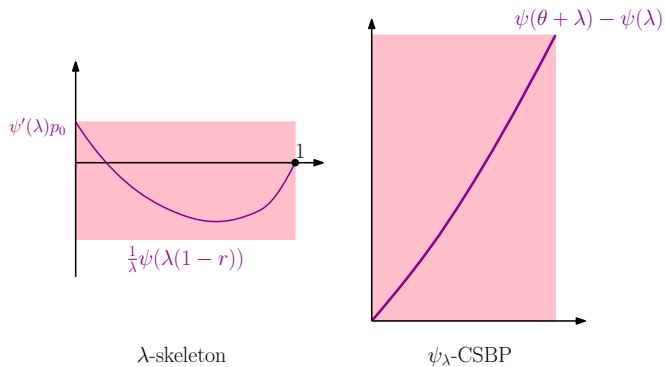
BRANCHING MECHANISM.



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BRANCHING MECHANISM.



λ -SKELETON I.

Let $\lambda \geq \lambda^*$.

Define the Esscher transformed branching mechanism $\psi_\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ for $\theta \geq -\lambda$ and $\lambda \geq \lambda^*$ by $\psi_\lambda(\theta) = \psi(\theta + \lambda) - \psi(\lambda)$.

The **supercritical** ψ -CSBP is equal in law to the total mass process obtained by the following construction.

- Initiate **Po(λx) independent Galton-Watson processes** with branching generator

$$q \left(\sum_{k \geq 0} p_k r^k - r \right) = \frac{1}{\lambda} \psi(\lambda(1-r)), \quad r \in [0, 1],$$

where $q = \psi'(\lambda)$, $p_0 = \psi(\lambda) / \lambda \psi'(\lambda)$, $p_1 = 0$ and for $k \geq 2$

$$p_k = \frac{1}{\lambda \psi'(\lambda)} \left\{ \beta \lambda^2 \mathbf{1}_{\{k=2\}} + \int_{(0, \infty)} \frac{(\lambda r)^k}{k!} e^{-\lambda r} \Pi(dr) \right\}.$$

λ -SKELETON II.

- ▶ Along the edges **immigrate CSBPs** at rate

$$2\beta d\mathbb{Q}^{(\lambda)} + \int_0^\infty ye^{-\lambda y}\Pi(dy)d\mathbb{P}_y^{(\lambda)},$$

where $\mathbb{P}_x^{(\lambda)}$, $x \geq 0$ is the law of the CSBP **with branching mechanism ψ_λ** and $\mathbb{Q}^{(\lambda)}$ is the associated excursion measure.

- ▶ Given that an individual dies and branches into $k \in \mathbb{N}_0 \setminus \{1\}$ offspring, an independent ψ_λ -CSBP is immigrated with initial mass r with probability

$$\eta_k(dr) = \frac{1}{p_k \lambda \psi'(\lambda)} \left\{ \psi(\lambda) \mathbf{1}_{\{k=0\}} \delta_0(dr) + \beta \lambda^2 \mathbf{1}_{\{k=2\}} \delta_0(dr) + \mathbf{1}_{\{k \geq 2\}} \frac{(\lambda r)^k}{k!} e^{-\lambda r} \Pi(dr) \right\},$$

- ▶ Finally an **independent ψ_λ -CSBP** is issued at time zero **with initial mass x** .

SDE.

The process (X, \mathbb{P}_x) , $x > 0$, can be represented as the unique strong solution to the stochastic differential equation (SDE)

$$X_t = x + \alpha \int_0^t X_{s-} ds + \sqrt{2\beta} \int_0^t \int_0^{X_{s-}} W(ds, du) + \int_0^t \int_0^\infty \int_0^{X_{s-}} r \tilde{N}(ds, dr, d\nu),$$
(1)

for $x > 0, t \geq 0$, where

- ▶ $W(ds, du)$ is a **white noise** process on $(0, \infty)^2$ based on the Lebesgue measure $ds \otimes du$,
- ▶ $N(ds, dr, d\nu)$ is a **Poisson point process** on $(0, \infty)^3$ with intensity $ds \otimes \Pi(dr) \otimes d\nu$, and $\tilde{N}(ds, dr, d\nu)$ the compensated measure of $N(ds, dr, d\nu)$.

THINNING OF THE SDE I.

We can introduce an **additional mark** to atoms of N , resulting in an ‘extended’ Poisson random measure, $\mathcal{N}(ds, dr, d\nu, dk)$ on $(0, \infty)^3 \times \mathbb{N}_0$ with intensity

$$ds \otimes \Pi(dr) \otimes d\nu \otimes \frac{(\lambda r)^k}{k!} e^{-\lambda r} \#(dk).$$

Define three random measures by

$$N^0(ds, dr, d\nu) = \mathcal{N}(ds, dr, d\nu, \{k = 0\}),$$

$$N^1(ds, dr, d\nu) = \mathcal{N}(ds, dr, d\nu, \{k = 1\})$$

and

$$N^2(ds, dr, d\nu) = \mathcal{N}(ds, dr, d\nu, \{k \geq 2\}).$$

We have that N^0 , N^1 and N^2 are independent Poisson point processes on $(0, \infty)^3$ with respective intensities $ds \otimes e^{-\lambda r} \Pi(dr) \otimes d\nu$, $ds \otimes (\lambda r) e^{-\lambda r} \Pi(dr) \otimes d\nu$ and $ds \otimes \sum_{k=2}^{\infty} (\lambda r)^k e^{-\lambda r} \Pi(dr) / k! \otimes d\nu$.

THINNING OF THE SDE II.

$$\begin{aligned}
X_t &= x + \alpha \int_0^t X_{s-} ds + \sqrt{2\beta} \int_0^t \int_0^{X_{s-}} W(ds, du) + \int_0^t \int_0^\infty \int_0^{X_{s-}} r\tilde{N}^0(ds, dr, d\nu) \\
&\quad + \int_0^t \int_0^\infty \int_0^{X_{s-}} rN^1(ds, dr, d\nu) + \int_0^t \int_0^\infty \int_0^{X_{s-}} rN^2(ds, dr, d\nu) \\
&\quad - \int_0^t \int_0^\infty X_{s-} \sum_{n=1}^{\infty} \frac{(\lambda r)^n}{n!} e^{-\lambda r} r \Pi(dr) ds \\
&= x - \psi'(\lambda) \int_0^t X_s ds + \sqrt{2\beta} \int_0^t \int_0^{X_{s-}} W(ds, du) + \int_0^t \int_0^\infty \int_0^{X_{s-}} r\tilde{N}^0(ds, dr, d\nu) \\
&\quad + \int_0^t \int_0^\infty \int_0^{X_{s-}} rN^1(ds, dr, d\nu) + 2\beta\lambda \int_0^t X_{s-} ds \\
&\quad + \int_0^t \int_0^\infty \int_0^{X_{s-}} rN^2(ds, dr, d\nu),
\end{aligned}$$

(In the last equality we have used that $-\int_{(0,\infty)}(1 - e^{-\lambda r})r\Pi(dr) = -\alpha + 2\beta\lambda - \psi'(\lambda)$).

THINNING OF THE SDE II.

$$\begin{aligned}
X_t &= x + \alpha \int_0^t X_{s-} ds + \sqrt{2\beta} \int_0^t \int_0^{X_{s-}} W(ds, du) + \int_0^t \int_0^\infty \int_0^{X_{s-}} r\tilde{N}^0(ds, dr, d\nu) \\
&\quad + \int_0^t \int_0^\infty \int_0^{X_{s-}} rN^1(ds, dr, d\nu) + \int_0^t \int_0^\infty \int_0^{X_{s-}} rN^2(ds, dr, d\nu) \\
&\quad - \int_0^t \int_0^\infty X_{s-} \sum_{n=1}^{\infty} \frac{(\lambda r)^n}{n!} e^{-\lambda r} r \Pi(dr) ds \\
&= x - \psi'(\lambda) \int_0^t X_s ds + \sqrt{2\beta} \int_0^t \int_0^{X_{s-}} W(ds, du) + \int_0^t \int_0^\infty \int_0^{X_{s-}} r\tilde{N}^0(ds, dr, d\nu) \\
&\quad + \int_0^t \int_0^\infty \int_0^{X_{s-}} rN^1(ds, dr, d\nu) + 2\beta\lambda \int_0^t X_{s-} ds \\
&\quad + \int_0^t \int_0^\infty \int_0^{X_{s-}} rN^2(ds, dr, d\nu),
\end{aligned}$$

(In the last equality we have used that $-\int_{(0,\infty)}(1 - e^{-\lambda r})r\Pi(dr) = -\alpha + 2\beta\lambda - \psi'(\lambda)$).

Theorem

Suppose that ψ corresponds to a supercritical branching mechanism (i.e. $\alpha > 0$) and $\lambda \geq \lambda^*$. Consider the coupled system of SDEs

$$\begin{aligned}
 \begin{pmatrix} \Lambda_t \\ Z_t \end{pmatrix} &= \begin{pmatrix} \Lambda_0 \\ Z_0 \end{pmatrix} - \psi'(\lambda) \int_0^t \begin{pmatrix} \Lambda_{s-} \\ 0 \end{pmatrix} ds + \sqrt{2\beta} \int_0^t \int_0^{\Lambda_{s-}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} W(ds, du) \\
 &+ \int_0^t \int_0^\infty \int_0^{\Lambda_{s-}} \begin{pmatrix} r \\ 0 \end{pmatrix} \tilde{N}^0(ds, dr, d\nu) \\
 &+ \int_0^t \int_0^\infty \int_1^{Z_{s-}} \begin{pmatrix} r \\ 0 \end{pmatrix} N^1(ds, dr, dj) \\
 &+ \int_0^t \int_0^\infty \int_0^\infty \int_1^{Z_{s-}} \begin{pmatrix} r \\ k-1 \end{pmatrix} N^2(ds, dr, dk, dj) \\
 &+ 2\beta \int_0^t \begin{pmatrix} Z_{s-} \\ 0 \end{pmatrix} ds, \quad t \geq 0,
 \end{aligned} \tag{2}$$

with $\Lambda_0 \geq 0$ given and fixed. Under the assumption that Z_0 is an independent random variable which is Poisson distributed with intensity $\lambda\Lambda_0$ the system (2) has a unique strong solution such that:

- (i) For $t \geq 0$, $Z_t | \mathcal{F}_t^\Lambda$ is Poisson distributed with intensity $\lambda\Lambda_t$, where $\mathcal{F}_t^\Lambda := \sigma(\Lambda_s : s \leq t)$;
- (ii) Conditional on $(\mathcal{F}_t^\Lambda, t \geq 0)$, the process $(\Lambda_t, t \geq 0)$ is a weak solution to (1).

(DRIVING SOURCES OF RANDOMNESS I.)

Let $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\sharp(d\ell) = \sum_{i \in \mathbb{N}_0} \delta_i(d\ell)$, $\ell \geq 0$.

Then in the previous theorem

- ▶ \mathbb{N}^0 is a Poisson random measure on $(0, \infty)^3$ with intensity measure $ds \otimes e^{-\lambda r} \Pi(dr) \otimes d\nu$, $\tilde{\mathbb{N}}^0$ is the associated compensated version of \mathbb{N}^0 ,
- ▶ $\mathbb{N}^1(ds, dr, dj)$ is a Poisson point process on $(0, \infty)^2 \times \mathbb{N}$ with intensity $ds \otimes re^{-\lambda r} \Pi(dr) \otimes \sharp(dj)$,
- ▶ $\mathbb{N}^2(ds, dr, dk, dj)$ is a Poisson point process on $(0, \infty)^2 \times \mathbb{N}_0 \times \mathbb{N}$ with intensity $\psi'(\lambda) ds \otimes \eta_k(dr) \otimes p_k \sharp(dk) \otimes \sharp(dj)$, and
- ▶ $W(ds, du)$ is the white noise process on $(0, \infty)^2$ based on the Lebesgue measure $ds \otimes du$.

SUBCRITICAL CSBP.



RAY-KNIGHT REPRESENTATION.

Assume that **Grey's condition** is satisfied, ie.

$$\int^{\infty} \frac{1}{\psi(\theta)} du < \infty.$$

Let

- ▶ $(\xi_t, t \geq 0)$ be a spectrally positive Lévy process with Laplace exponent ψ ,
- ▶ $(\hat{\xi}_r^{(t)}, 0 \leq r \leq t)$, where $\hat{\xi}_r^{(t)} := \xi_t - \xi_{(t-r)-}$, the time reversed process at time t ,
- ▶ $\hat{S}_r^{(t)} := \sup_{s \leq r} \hat{\xi}_s^{(t)}$.

The process $(H_t, t \geq 0)$ is called the **height process** if H_t is the local time at level 0, at time t of $\hat{S}^{(t)} - \hat{\xi}^{(t)}$.

Denote by L_t^a the local time up to time t of H at level $a \geq 0$, and let $T_x := \inf\{t \geq 0 : \xi_t = -x\}$.

Then the generalised **Ray-Knight theorem** for the ψ -CSBP process states that $(L_{T_x}^a, a \geq 0)$ has a càdàg modification for which

$$(L_{T_x}^t, t \geq 0) \stackrel{d}{=} (X, \mathbb{P}_x),$$

that is, the two processes are equal in law.

GENEALOGY OF SUBCRITICAL CSBP.

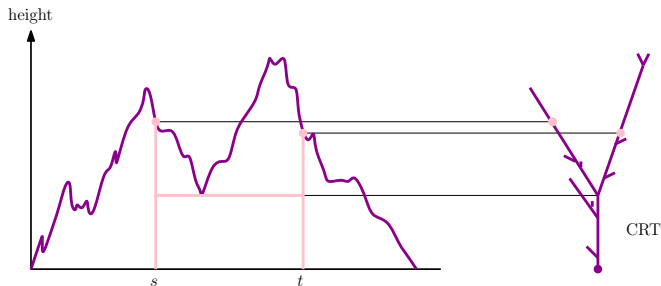
Excursions of H away from 0 form a PPP, denote by \mathfrak{n} its intensity, and let ϵ be a canonical excursion under \mathfrak{n} .

Let $\zeta = \inf\{s > 0, \epsilon_s = 0\}$, and define

$$d_\epsilon(s, t) = \epsilon_s + \epsilon_t - \inf_{s \wedge t \leq r \leq s \vee t} \epsilon_r, \quad (s, t) \in [0, \zeta]^2.$$

Then we can define the equivalence relation \sim_ϵ , such that $(s \sim_\epsilon t)$ is and only if $d_\epsilon(s, t) = 0$, and $\mathcal{T}_\epsilon = [0, \zeta] \setminus \sim_\epsilon$.

The compact metric space $(\mathcal{T}_\epsilon, d_\epsilon)$ is called a **Lévy random tree**.



T-SKELETON.

Fix $T > 0$.

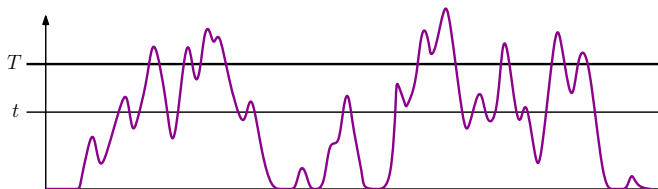
Define $(Z_t^T, 0 \leq t < T)$ as **the process that counts the number of excursions above level t that hit level T .**

Then Z^T is a **time-dependent continuous-time Galton-Watson process** which at time t branching at rate

$$q^{T-t} = \frac{u_{T-t}(\infty)\psi'(u_{T-t}(\infty)) - \psi(u_{T-t}(\infty))}{u_{T-t}(\infty)}, \quad t \in [0, T),$$

and its offspring distribution $(p_k^{T-t}, k \geq 0)$ is given by $p_0^{T-t} = p_1^{T-t} = 0$,

$$p_k^{T-t} = \frac{1}{u_{T-t}(\infty)q^{T-t}} \times \left\{ \beta u_{T-t}^2(\infty) \mathbf{1}_{\{k=2\}} + \int_0^\infty \frac{(u_{T-t}(\infty)x)^k}{k!} e^{-u_{T-t}(\infty)x} \Pi(dx) \right\}.$$



T-SKELETON.

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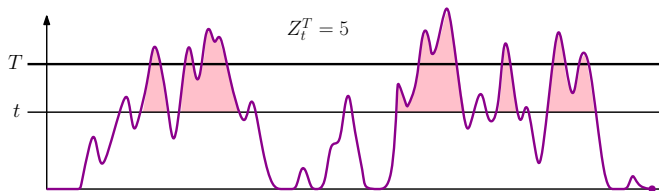
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T-SKELETON.

Fix $T > 0$.

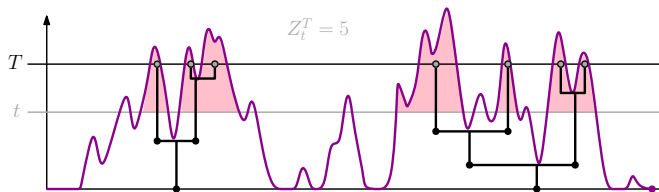
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$$q^{T-t} = \frac{u_{T-t}(\infty)\psi'(u_{T-t}(\infty)) - \psi(u_{T-t}(\infty))}{u_{T-t}(\infty)}, \quad t \in [0, T),$$

and its offspring distribution $(p_k^{T-t}, k \geq 0)$ is given by $p_0^{T-t} = p_1^{T-t} = 0$,

$$p_k^{T-t} = \frac{1}{u_{T-t}(\infty)q^{T-t}} \times \left\{ \beta u_{T-t}^2(\infty) \mathbf{1}_{\{k=2\}} + \int_0^\infty \frac{(u_{T-t}(\infty)x)^k}{k!} e^{-u_{T-t}(\infty)x} \Pi(dx) \right\}.$$



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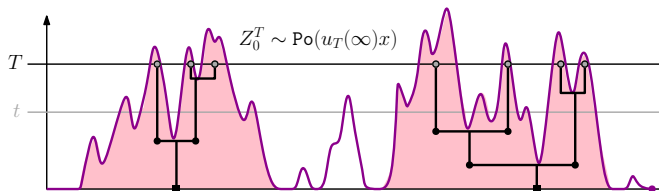
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and its offspring distribution $(p_k^{T-t}, k \geq 0)$ is given by $p_0^{T-t} = p_1^{T-t} = 0$,

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IMMIGRATION.

As

$$\mathbb{P}[X_T = 0 | \mathcal{F}_t] = e^{-X_t u_{T-t}(\infty)},$$

the law of X conditioned to die out by time T can be obtained by the following change of measure

$$\left. \frac{d\mathbb{P}_x^T}{d\mathbb{P}_x} \right|_{\mathcal{F}_t} = \frac{e^{-X_t u_{T-t}(\infty)}}{e^{-x u_T(\infty)}}, \quad t \geq 0, x > 0.$$

We get that (X, \mathbb{P}_x^T) is a **time-dependent CSBP** with Laplace transform

$$\mathbb{E}_x^T[e^{-\theta X_t}] = e^{-x V_t^T(\theta)}, \quad 0 \leq t < T, x, \theta \geq 0,$$

where

$$V_t^T(\theta) = u_t(\theta + u_{T-t}(\infty)) - u_T(\infty).$$

Note that

$$\lim_{t \rightarrow T} u_{T-t}(\infty) = \infty, \quad \text{and} \quad \lim_{T \rightarrow \infty} u_{T-t}(\infty) = 0.$$

Theorem

Suppose that ψ corresponds to a (sub)critical branching mechanism (i.e. $\alpha \leq 0$) which satisfies Grey's condition. Fix a time horizon $T > 0$ and consider the coupled system of SDEs

$$\begin{aligned}
 \begin{pmatrix} \Lambda_t^T \\ Z_t^T \end{pmatrix} &= \begin{pmatrix} \Lambda_0^T \\ Z_0^T \end{pmatrix} - \int_0^t \psi'(u_{T-s}(\infty)) \begin{pmatrix} \Lambda_{s-}^T \\ 0 \end{pmatrix} ds + \sqrt{2\beta} \int_0^t \int_0^{\Lambda_{s-}^T} \begin{pmatrix} 1 \\ 0 \end{pmatrix} W(ds, du) \\
 &+ \int_0^t \int_0^\infty \int_0^{\Lambda_{s-}^T} \begin{pmatrix} r \\ 0 \end{pmatrix} \tilde{N}_T^0(ds, dr, d\nu) \\
 &+ \int_0^t \int_0^\infty \int_1^{Z_{s-}^T} \begin{pmatrix} r \\ 0 \end{pmatrix} N_T^1(ds, dr, dj) \\
 &+ \int_0^t \int_0^\infty \int_0^\infty \int_1^{Z_{s-}^T} \begin{pmatrix} r \\ k-1 \end{pmatrix} N_T^2(ds, dr, dk, dj) \\
 &+ 2\beta \int_0^t \begin{pmatrix} Z_{s-}^T \\ 0 \end{pmatrix} ds, \quad 0 \leq t < T.
 \end{aligned} \tag{3}$$

with $\Lambda_0^T \geq 0$ given and fixed. Under the assumption that Z_0^T is an independent random variable which is Poisson distributed with intensity $u_T(\infty)\Lambda_0^T$ the system (3) has a unique strong solution such that:

- (i) For $T > t \geq 0$, $Z_t^T | \mathcal{F}_t^{\Lambda^T}$ is Poisson distributed with intensity $u_{T-t}(\infty)\Lambda_t^T$, where $\mathcal{F}_t^{\Lambda^T} := \sigma(\Lambda_s^T : s \leq t)$;
- (ii) Conditional on $(\mathcal{F}_t^{\Lambda^T}, 0 \leq t < T)$, the process $(\Lambda_t^T, 0 \leq t < T)$ is a weak solution to (1).

(DRIVING FORCES OF RANDOMNESS II.)

In the previous theorem

- ▶ N_T^0 is a Poisson random measure on $[0, \infty)^3$ with intensity $ds \otimes e^{-u_{T-s}(\infty)r} \Pi(dr) \otimes d\nu$.

- ▶ N_T^1 is a Poisson process on $[0, \infty)^2 \times \mathbb{N}_0$ with intensity $ds \otimes re^{-u_{T-s}(\infty)r} \Pi(dr) \otimes \sharp(dj)$,

- ▶ $N_T^2(ds, dr, dk, dj)$ is a Poisson process on $[0, \infty)^2 \times \mathbb{N}_0 \times \mathbb{N}$ with intensity

$$\left\{ \frac{u_{T-s}(\infty)\psi'(u_{T-s}(\infty)) - \psi(u_{T-s}(\infty))}{u_{T-s}(\infty)} \right\} ds \otimes \eta_k^{T-s}(dr) \otimes p_k^{T-s} \sharp(dk) \otimes \sharp(dj),$$

where, for $k \geq 2$,

$$\eta_k^{T-s}(dr) = \frac{\beta u_{T-s}^2(\infty) \mathbf{1}_{\{k=2\}} \delta_0(dr) + (u_{T-s}(\infty)r)^k e^{-u_{T-s}(\infty)r} \Pi(dr)/k!}{p_k^{T-s} (u_{T-s}(\infty)\psi'(u_{T-s}(\infty)) - \psi(u_{T-s}(\infty)))}, \quad r \geq 0,$$

- ▶ $W(ds, du)$ is the white noise process on $(0, \infty)^2$ based on the Lebesgue measure $ds \otimes du$.

CONDITIONING ON SURVIVAL.

The law of $(\Lambda_t^T, 0 \leq t < T)$ conditional on $(\mathcal{F}_t^{\Lambda^T} \cap \{Z_0^T \geq 1\}, 0 \leq t < T)$ is that of the law of the ψ -CSBP, X , conditioned to survive until time T .

- ▶ This law is can be obtained by the following change of measure for $t \geq 0, x > 0$

$$\frac{d\tilde{\mathbb{P}}_x^T}{d\mathbb{P}_x} \Big|_{\mathcal{F}_t} = \frac{1 - e^{-X_t u_{T-t}(\infty)}}{1 - e^{-x u_T(\infty)}}.$$

- ▶ We have for $k \geq 1$

$$\mathbf{P}_x^T[Z_0 = k | Z_0 \geq 1] = \frac{(u_T(\infty)x)^k}{k!} \frac{e^{-u_T(\infty)x}}{1 - e^{-u_T(\infty)x}}.$$

- ▶ If n_T denotes the conditional probability $n(\cdot | \sup_{s \geq 0} \epsilon_s \geq T)$, then the first branch time γ_T of the individual corresponding to the excursion ϵ is given by

$$n_T(\gamma_T > t) = \frac{\psi(u_T(\infty))}{u_T(\infty)} \frac{u_{T-t}(\infty)}{\psi(u_{T-t}(\infty))},$$

for $t \in [0, T)$.

CONDITIONING ON SURVIVAL.

The law of $(\Lambda_t^T, 0 \leq t < T)$ conditional on $(\mathcal{F}_t^{\Lambda_t^T} \cap \{Z_0^T \geq 1\}, 0 \leq t < T)$ is that of the law of the ψ -CSBP, X , conditioned to survive until time T .

Take $T \rightarrow \infty$.

- ▶ This law is can be obtained by the following change of measure for $t \geq 0, x > 0$

$$\frac{d\tilde{\mathbb{P}}_x^T}{d\mathbb{P}_x} \Big|_{\mathcal{F}_t} = \frac{1 - e^{-X_t u_{T-t}(\infty)}}{1 - e^{-x u_T(\infty)}} \rightarrow e^{-\alpha t} \frac{X_t}{x}.$$

- ▶ We have for $k \geq 1$

$$\mathbf{P}_x^T[Z_0 = k | Z_0 \geq 1] = \frac{(u_T(\infty)x)^k}{k!} \frac{e^{-u_T(\infty)x}}{1 - e^{-u_T(\infty)x}} \rightarrow 0, \text{ unless } k = 1.$$

- ▶ If n_T denotes the conditional probability $n(\cdot | \sup_{s \geq 0} \epsilon_s \geq T)$, then the first branch time γ_T of the individual corresponding to the excursion ϵ is given by

$$n_T(\gamma_T > t) = \frac{\psi(u_T(\infty))}{u_T(\infty)} \frac{u_{T-t}(\infty)}{\psi(u_{T-t}(\infty))} \rightarrow 1,$$

for $t \in [0, T)$.

EMERGENCE OF THE SPINE.

Note that the convergence is in a **weak sense**.

SPINE.

Theorem

Suppose that ψ is a critical or subcritical branching mechanism such that Grey's condition holds. Suppose, moreover, that $((\Lambda_t^T, Z_t^T), 0 \leq t < T)$ is a weak solution to (3) and that Z_0^T is an independent random variable which is Poisson distributed with intensity $u_T(\infty)\Lambda_0^T$. Then, conditional on the event $Z_0^T > 0$, in the sense of weak convergence with respect to the Skorokhod topology on $\mathbb{D}([0, \infty), \mathbb{R}^2)$, for all $t > 0$,

$$((\Lambda_s^T, Z_s^T), 0 \leq s \leq t) \rightarrow ((X_s^\uparrow, 1), 0 \leq s \leq t),$$

where X^\uparrow is a weak solution to

$$\begin{aligned} X_t = x + \alpha \int_0^t X_{s-} ds + \sqrt{2\beta} \int_0^t \int_0^{X_{s-}} W(ds, du) + \int_0^t \int_0^\infty \int_0^{X_{s-}} r \tilde{N}(ds, dr, du) \\ + \int_0^t \int_0^\infty r N^*(ds, dr) + 2\beta t, \quad t \geq 0. \end{aligned}$$

SPINE.

Theorem

Suppose that ψ is a critical or subcritical branching mechanism such that Grey's condition holds. Suppose, moreover, that $((\Lambda_t^T, Z_t^T), 0 \leq t < T)$ is a weak solution to (3) and that Z_0^T is an independent random variable which is Poisson distributed with intensity $u_T(\infty)\Lambda_0^T$. Then, conditional on the event $Z_0^T > 0$, in the sense of weak convergence with respect to the Skorokhod topology on $\mathbb{D}([0, \infty), \mathbb{R}^2)$, for all $t > 0$,

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(DRIVING SOURCES OF RANDOMNESS III.)

In the previous theorem

- ▶ $W(ds, du)$ is a white noise process on $(0, \infty)^2$ based on the Lebesgue measure $ds \otimes du$,
- ▶ $N(ds, dr, d\nu)$ is a Poisson point process on $(0, \infty)^3$ with intensity $ds \otimes \Pi(dr) \otimes d\nu$, and $\tilde{N}(ds, dr, d\nu)$ is the compensated measure of $N(ds, dr, d\nu)$,
- ▶ N^* is a Poisson random measure on $[0, \infty) \times (0, \infty)$ with intensity measure $ds \otimes r\Pi(dr)$.

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