Multil-level Weiner-Hopf Monte-Carlo simulation for Lévy processes

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Motivation

- Lévy process. A (one dimensional) process with stationary and independent increments which has paths which are right continuous with left limits and therefore includes Brownian motion with drift, compound Poisson processes, stable processes amongst many others).
- A popular (and often criticised) model in mathematical finance for the evolution of a risky asset is

$$S_t := e^{X_t},\ t \geq 0$$

  where $\{X_t : t \geq 0\}$ is a Lévy process.
- Barrier options: The value of up-and-out barrier option with expiry date T and barrier b is typically priced as

$$\mathbb{E}_s(f(X_1)\mathbf{1}_{\{\overline{X}_1 \leq b\}})$$

  where $\overline{X}_1 = \sup_{u \leq 1} X_u$, f is some nice function.
- One is fundamentally interested in the joint distribution

$$\mathbb{P}(X_1 \in \mathrm{d}x,\ \overline{X}_1 \in \mathrm{d}y)$$

  for any Lévy process $(X, \mathbb{P})$.

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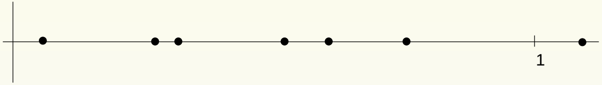

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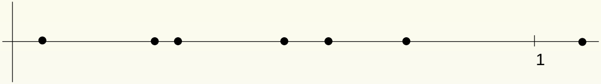

- Note that τ_n is the sum of n i.i.d exponential random variables, each with mean $1/n$. We could therefore write

$$\tau_n = \sum_{i=1}^{n} \frac{1}{n} \mathbf{e}^{(i)},$$

where $\mathbf{e}^{(i)}$ are i.i.d. exponential random variables with unit mean. Hence by the SLLN

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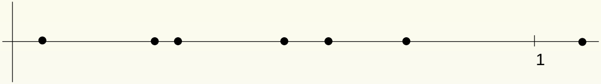

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$$\tau_n \to 1 \qquad \text{almost surely.}$$

- Hence for a suitably large n, we have in distribution

$$(X_{\tau_n}, \overline{X}_{\tau_n}) \simeq (X_1, \overline{X}_1).$$

Indeed since 1 is not a jump time with probability 1, we have that $(X_{\tau_n}, \overline{X}_{\tau_n}) \to (X_1, \overline{X}_1)$ almost surely as $n \to \infty$.

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- A reformulation of the Wiener-Hopf factorization states that

$$X_{\mathbf{e}_q} \stackrel{d}{=} S_q + I_q$$

where S_q is independent of I_q and they are respectively equal in distribution to $\overline{X}_{\mathbf{e}_q}$ and $\underline{X}_{\mathbf{e}_q}$. Here $\underline{X}_t = \inf_{s \leq t} X_s$.

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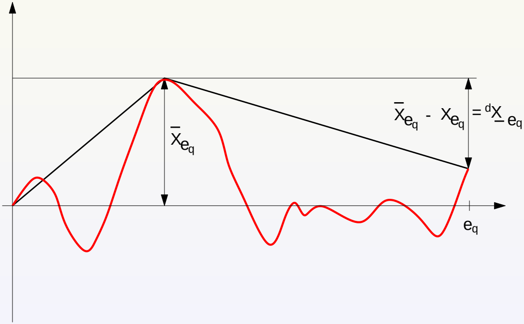

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  where S_q is independent of I_q and they are respectively equal in distribution to $\overline{X}_{\mathbf{e}_q}$ and $\underline{X}_{\mathbf{e}_q}$.
- Taking advantage of the above, the fact that X has stationary and independent increments and the fact that, as a time, τ_n can be seen as the sum of independent exponential time periods we have the following:

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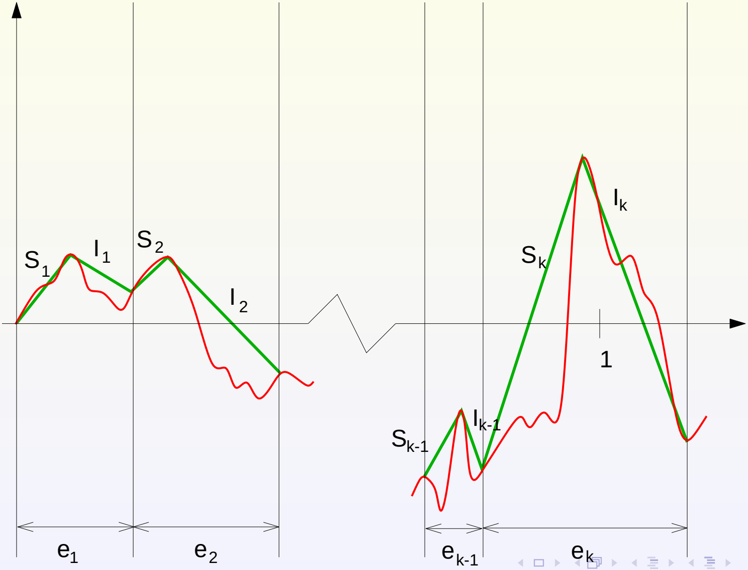

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- For all $n \in \{1, 2, \cdots\}$ and $n > 0$,

$$(X_{\tau_n}, \overline{X}_{\tau_n}) \stackrel{d}{=} (V_n, J_n)$$

where

$$V_n = \sum_{j=1}^{n} \{S_n^{(j)} + I_n^{(j)}\}$$

$$J_n := \bigvee_{i=0}^{n-1} \left(V_i + S_n^{(i+1)} \right).$$

Here, $V_0 = S_n^{(0)} = I_n^{(0)} = 0$, $\{S_n^{(j)} : j \geq 1\}$ are an i.i.d. sequence of random variables with common distribution equal to that of $\overline{X}_{\mathbf{e}_n}$ and $\{I_n^{(j)} : j \geq 1\}$ are another i.i.d. sequence of random variable with common distribution equal to that of $\underline{X}_{\mathbf{e}_n}$.

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- $(V_n, J_n) \stackrel{n\uparrow\infty}{\longrightarrow} (X_1, \overline{X}_1)$ in distribution.

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- Sample repeatedly and independently from the distribution $\overline{X}_{\mathbf{e}_n}$ and $\underline{X}_{\mathbf{e}_n}$ and then construct m independent versions of the variables V_n and J_n, say

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- Sampling from $\overline{X}_{\mathbf{e}_n}$ and $\underline{X}_{\mathbf{e}_n}$ is generally impossible for a given Lévy process, but not for a 10 parameter family of processes known as Kuznetsov's β-class (ask me afterwards if interested in the details!).

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- **Notation.**
  - Write $a \lesssim b$ for two positive quantities a and b, if a/b is uniformly bounded (independent of n, M, or any other parameters).
  - Write $F^{n,(i)} := F(V_n^{(i)}, J_n^{(i)})$ for the i-th sample of $F^n := F(V_n, J_n)$ (using the Wiener-Hopf random walk).
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- Then we have the following convergence/complexity theorem ...

Numerical Analysis WHMC

Theorem (Single-level WHMC)

Assume that $\exists\alpha > 0$ *s.t.*

(i) $\mathbb{E}[|F^n - F(X_t, \overline{X}_t)|] \lesssim n^{-\alpha}$ *and*

(ii) $\mathbb{E}[\mathcal{C}(F^n)] \lesssim n$ (where $\mathcal{C}(F^n)$ is the cost to compute a single sample from F^n)

Then, $\forall\nu \in \mathbb{N} \;\; \exists n, M \in \mathbb{N}$ *s.t.*

$$\mathbb{E}\left[\mathcal{C}(\widehat{F}^{n,M}_{MC})\right] \lesssim \nu \quad \text{and } L^2 \text{ error} \quad e(\widehat{F}^{n,m}_{MC}) \lesssim \nu^{-\frac{1}{2+1/\alpha}}.$$

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Using the forthcoming analysis we shall shortly present, it will turn out that:

- when X has paths of unbounded variation, $\alpha = \frac{1}{4} \Rightarrow \mathcal{O}(\nu^{-\frac{1}{6}})$ convergence!
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The best one can hope for with such Monte-Carlo schemes is an $\mathcal{O}(\nu^{-\frac{1}{2}})$ converegence.

Multi-level Wiener-Hopf Monte-Carlo [Heinrich, 2001], [Giles, 2007], ...

- Computational gains from exploiting the telescopic sum

$$\mathbb{E}[F^{n_L}] = \mathbb{E}[F^{n_0}] + \sum_{\ell=1}^{L} \mathbb{E}[F^{n_\ell} - F^{n_{\ell-1}}],$$

where $n_\ell = 2^\ell n_0$, $\ell = 1, \ldots, L$, for some small $n_0 \in \mathbb{N}$.

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- Suggesting the multilevel estimator

$$\widehat{F}_{ML}^{n_0, L, \{M_\ell\}} := \frac{1}{M_0} \sum_{i=1}^{M_0} F^{n_0,(i)} + \sum_{\ell=1}^{L} \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} (F^{n_\ell,(i)} - F^{n_{\ell-1},(i)}).$$

- Here it is very important that $F^{n_{\ell-1}}$ can be obtained from F^{n_ℓ} by a "deterministic" transformation of the random variables used to obtain F^{n_ℓ}.

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$$e(\widehat{F}_{ML}^{n_0, L, \{M_\ell\}})^2 = \frac{1}{M_0}\mathbb{V}(F^{n_0}) + \sum_{\ell=1}^{L} \frac{1}{M_\ell}\mathbb{V}(F^{n_\ell} - F^{n_{\ell-1}}) + \left(\mathbb{E}[F^n - F(X_1, \overline{X}_1)]\right)^2.$$

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- See also **[Dereich, Heidenreich, 2011], [Dereich, 2011], [Giles, Xia, 2012]**.

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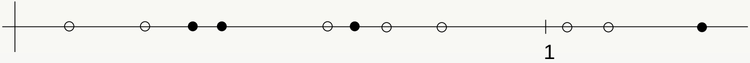
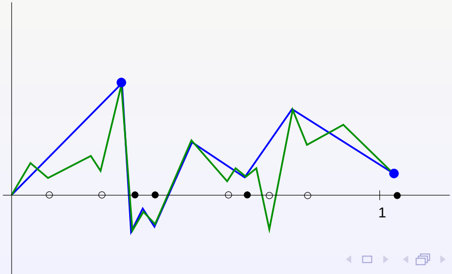
- In the WHMC method how do we introduce "levels"?
- Recall also that it is **crucial** to have a Poisson process for the time randomisations on **all** levels! How do we sample on two consecutive levels?
- Suppose the "level ℓ" grid is based on a Poisson process of rate n_ℓ. Then by tossing a coin and rejecting arrivals with probability $1/2$ we end up with a Poisson process of rate $n_{\ell-1}$: our new coarser "level $\ell-1$" Poisson grid. (**Not** a new idea! Also used by **[Glasserman, Merener, 2003], [Giles, Xia, 2012], ...**)





Numerical Analysis (multilevel case)

Theorem (Multilevel WHMC)

Assume $\exists \alpha, \beta > 0$ *with* $\alpha \geq \frac{1}{2}\max\{\beta, 1\}$ *such that*

(i) $|\mathbb{E}[F^{n_\ell} - F(X_t, \overline{X}_t)]| \lesssim n_\ell^{-\alpha}$

(ii) $\mathbb{V}[F^{n_\ell} - F^{n_{\ell-1}}] \lesssim n_\ell^{-\beta}$

(iii) $\mathbb{E}[\mathcal{C}_{n_\ell}] \lesssim n_\ell$.

Then, $\forall \nu \in \mathbb{N}$ $\exists L$ *and* $\{M_\ell\}_{\ell=0}^{L}$ *s.t.* $\mathbb{E}\left[\mathcal{C}\left(\widehat{F}_{\mathrm{ML}}^{n_0, L, \{M_\ell\}}\right)\right] \lesssim \nu$ *and* L^2 *error*

$$e\left(\widehat{F}_{\mathrm{ML}}^{n_0, L, \{M_\ell\}}\right) \lesssim \begin{cases} \nu^{-\frac{1}{2}} & \text{if } \beta > 1\,, \\ \nu^{-\frac{1}{2}} \log^2 \nu & \text{if } \beta = 1\,, \\ \nu^{-\frac{1}{2+(1-\beta)/\alpha}} & \text{if } \beta < 1\,. \end{cases}$$

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- For Kuznetsov's β-class of Lévy processes also Assumption (iii) holds.
- Recall that $\alpha = (1/4) 1/2$ for (un)bounded variation paths & shortly we shall see $\beta = 1/2 \ \Rightarrow\ (\mathcal{O}(\nu^{-\frac{1}{4}}))\ \mathcal{O}(\nu^{-\frac{1}{3}})$.
- Compare with former (single-level) rates $(\mathcal{O}(\nu^{-\frac{1}{6}}))\ \mathcal{O}(\nu^{-\frac{1}{4}})$.

Numerical analysis MLWH

Remains to verify the assumptions (i) $|\mathbb{E}[F^{n_\ell} - F(X_t, \overline{X}_t)]| \lesssim n_\ell^{-\alpha}$ and (ii) $\mathbb{V}[F^{n_\ell} - F^{n_{\ell-1}}] \lesssim n_\ell^{-\beta}$.

- Recalling that F is Lipschitz

$$\begin{aligned} \mathbb{V}(F^{n_\ell} - F^{n_{\ell-1}}) &= \mathbb{V}(F(X_{\tau_{n_\ell}}, \overline{X}_{\tau_{n_\ell}}) - F(X_{\tau_{n_\ell - 1}}, \overline{X}_{\tau_{n_\ell - 1}})) \\ &\leq \mathbb{E}[(X_{\tau_{n_\ell}} - X_{\tau_{n_\ell - 1}})^2] + \mathbb{E}[(\overline{X}_{\tau_{n_\ell}} - \overline{X}_{\tau_{n_\ell - 1}})^2] \end{aligned}$$

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- [Unbounded variation case]:Working with our Lévy process as a Markovian semi-martingale we get for $\mathbf{s}$ and $\mathbf{t}$ random times independent of X,

(1) $\mathbb{E}[(X_{\mathbf{t}} - X_{\mathbf{s}})^2] = \mathbb{V}(X_1)\mathbb{E}[|\mathbf{t} - \mathbf{s}|] + \mathbb{E}[X_1]^2\mathbb{E}[(\mathbf{t} - \mathbf{s})^2]$

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Remains to verify the assumptions (i) $|\mathbb{E}[F^{n_\ell} - F(X_t, \overline{X}_t)]| \lesssim n_\ell^{-\alpha}$ and (ii) $\mathbb{V}[F^{n_\ell} - F^{n_{\ell-1}}] \lesssim n_\ell^{-\beta}$.

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- Via Jensen's inequality we then easily also get $\alpha = \frac{1}{4}$ ...

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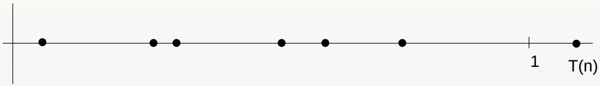
$$(X_{\mathbf{T}(n)}, \overline{X}_{\mathbf{T}(n)}) \stackrel{d}{=} (V_{\kappa_1}, J_{\kappa_1})$$

where

$$V_k := \sum_{i=1}^{k} (S_n^i + I_n^i) \quad \text{and} \quad J_k := \bigvee_{i=1}^{k} (V_{i-1} + S_n^i)$$

and $\{S_n^j : j \geq 1\}$ and $\{I_n^j : j \geq 1\}$ are i.i.d. sequences of random variables with common distribution equal to $\overline{X}_{\mathbf{e}_n}$ and $\underline{X}_{\mathbf{e}_n}$, respectively.



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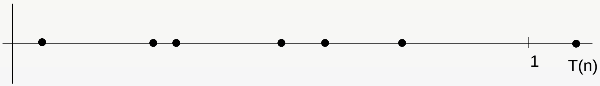

- Note that by the lack of memory property $\mathbf{T}(n) - t$ is still exponentially distributed with parameter n. Moreover

$$\mathbf{T}(n) - t \rightarrow 0 \quad \text{as} \quad n \uparrow \infty \qquad \text{(significantly faster than } \tau_n - t\text{)}$$

- $(V_{\kappa_t}, J_{\kappa_t}) \rightarrow (X_t, \overline{X}_t)$ in distribution as $n \uparrow \infty$ (rates later ...)

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- Now construct M independent versions of V_{κ_t} and J_{κ_t} by sampling repeatedly and indep. from the distributions of $\overline{X}_{\mathbf{e}_n}$ and $\underline{X}_{\mathbf{e}_n}$ and define

$$\widehat{F}^{n,M}_{MC} := \frac{1}{M}\sum_{i=1}^{M} F\left(V^{(i)}_{\kappa_t}, J^{(i)}_{\kappa_t}\right) \approx \mathbb{E}(F(X_{\mathbf{T}(n)}, \overline{X}_{\mathbf{T}(n)})) \quad \text{(sampling error)}$$

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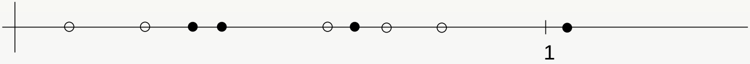

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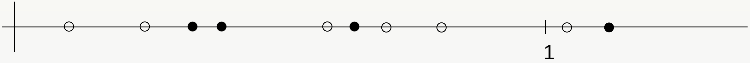

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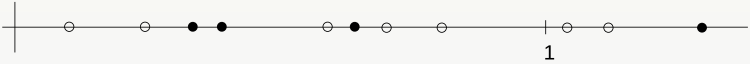

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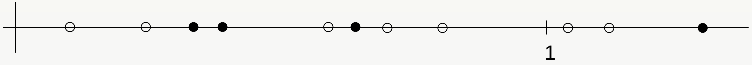

ETC ...

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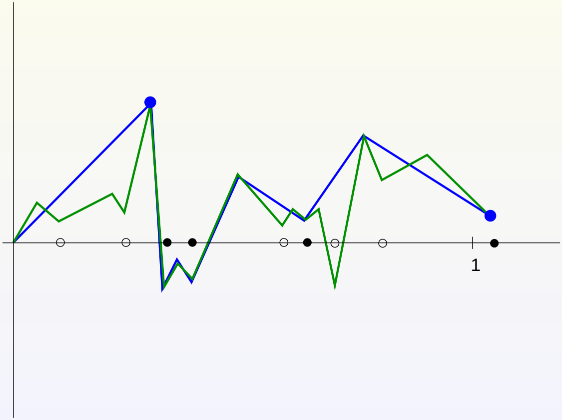
EITHER, with probability 1/2, $\mathbf{T}(n_\ell) = \mathbf{T}(n_{\ell-1})$. Hence, $F^{n_\ell} - F^{n_{\ell-1}} = 0$
(due to the exactness of the Wiener-Hopf factorisation)
and we have to do no work at all (apart from one coin toss) for this sample.



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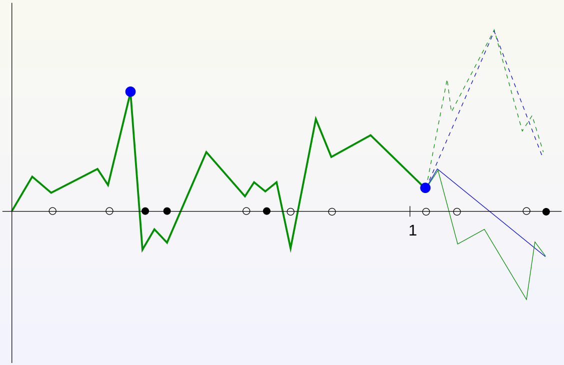
OR, with probability 1/2, $\mathbf{T}(n_{\ell-1}) > \mathbf{T}(n_\ell)$ in which case

$$\begin{aligned}\overline{X}_{\mathbf{T}(n_{\ell-1})} &= \max(\overline{X}_{\mathbf{T}(n_\ell)}, X_{\mathbf{T}(n_\ell)} + S_{n_{\ell-1}/t}) \\ X_{\mathbf{T}(n_{\ell-1})} &= X_{\mathbf{T}(n_{\ell-1})} + S_{n_{\ell-1}/t} + I_{n_{\ell-1}/t}\end{aligned}$$

and so the additional work for the path on level $\ell - 1$ is negligible.



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- Following the numerical analysis through, remembering that $\mathbf{T}(n) - 1$ is exponentially distributed, the key estimates boil down to

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- BAD NEWS! Because this algorithm has "awareness" of each of the exponential "Wiener-Hopf exponential periods" (i.e. we need to see whether the cumulative exponential periods exceed the level 1), it means that to implement the algorithm we need to be able to sample simultaneously from the triple $(\overline{X}_{\mathbf{e}_n}, X_{\mathbf{e}_n} - \overline{X}_{\mathbf{e}_n}, \mathbf{e}_n)$.

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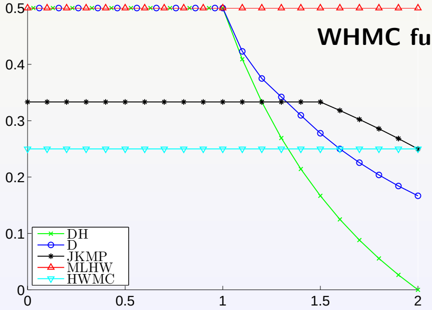

WHMC fully robust w.r.t ρ!

References

- Cliffe, K.A., Giles, M.B., Scheichl, R. and Teckentrup, A.L. (2011) Multilevel Monte Carlo methods and applications to elliptic PDEs with random coefficients. *Comput. Visual. Sci.* **14**(1), 3–15.
- Dereich, S. (2011) Multilevel Monte Carlo Algorithms for Lévy–driven SDEs with Gaussian correction. *Ann. Appl. Probab.* **21**(1), 283–311.
- Dereich, S. and Heidenreich, F. (2011) A multilevel Monte Carlo algorithm for Lévy–driven stochastic differential equations. *Stochastic Process. Appl.* **121**(7), 1565–1587.
- Giles, M.B. (2008) Multilevel Monte Carlo Path Simulation. *Oper. Res.* **56**(3), 607–617.
- Jacod, J., Kurtz, T.G., Méléard, S. and Protter, P. (2005) The approximate Euler method for Lévy driven stochastic differential equations. *Ann. Inst. H. Poincaré Probab. Statist.* **41**, 523–558.
- Kuznetsov (2010) Wiener-Hopf factorization and distribution of extrema for a family of Levy processes *Ann. Appl. Probab.* **20**, 1801–1830.

Kuznetsov's β-family

- The characteristic exponent $(\Psi(\theta) = -\log \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta X_1}), \theta \in \mathbb{R})$ is given by

$$\Psi(\theta) = \mathrm{i}az + \frac{1}{2}\sigma^2 z^2 + \frac{c_1}{\beta_1}\left\{\mathrm{B}(\alpha_1, 1-\lambda_1) - \mathrm{B}(\alpha_1 - \frac{\mathrm{i}\theta}{\beta_1}, 1-\lambda_1)\right\} + \frac{c_2}{\beta_2}\left\{\mathrm{B}(\alpha_2, 1-\lambda_2) - \mathrm{B}(\alpha_2 + \frac{\mathrm{i}\theta}{\beta_2}, 1-\lambda_2)\right\}$$

  where $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta function, with parameter range $a \in \mathbb{R}, \sigma, c_i, \alpha_i, \beta_i > 0$ and $\lambda_1, \lambda_2 \in (0,3) \setminus \{1,2\}$.
- The corresponding Lévy measure Π has density

$$\pi(x) = c_1 \frac{\mathrm{e}^{-\alpha_1\beta_1 x}}{(1-\mathrm{e}^{-\beta_1 x})^{\lambda_1}} \mathbf{1}_{\{x>0\}} + c_2 \frac{\mathrm{e}^{\alpha_2\beta_2 x}}{(1-\mathrm{e}^{\beta_2 x})^{\lambda_2}} \mathbf{1}_{\{x<0\}}.$$

  The β-class of Lévy processes includes another recently introduced family of Lévy processes known as Lamperti-stable processes.

Meromorphic Lévy processes (equivalent definition)

(i) The characteristic exponent $\Psi(z)$ is a meromorphic function which has poles at points $\{-\mathrm{i}\rho_n, \mathrm{i}\hat{\rho}_n\}_{n\geq 1}$, where ρ_n and $\hat{\rho}_n$ are positive real numbers.

(ii) For $q \geq 0$ function $q + \Psi(z)$ has roots at points $\{-\mathrm{i}\zeta_n, \mathrm{i}\hat{\zeta}_n\}_{n\geq 1}$ where ζ_n and $\hat{\zeta}_n$ are nonnegative real numbers (strictly positive if $q > 0$). We will write $\zeta_n(q)$, $\hat{\zeta}_n(q)$ if we need to stress the dependence on q.

(iii) The roots and poles of $q + \Psi(\mathrm{i}z)$ satisfy the following interlacing condition

$$... - \rho_2 < -\zeta_2 < -\rho_1 < -\zeta_1 < 0 < \hat{\zeta}_1 < \hat{\rho}_1 < \hat{\zeta}_2 < \hat{\rho}_2 < ...$$

(iv) The Wiener-Hopf factors are expressed as convergent infinite products,

$$\mathbb{E}\left[\mathrm{e}^{-z\overline{X}_{\mathbf{e}_q}}\right] = \prod_{n\geq 1} \frac{1 + \frac{z}{\rho_n}}{1 + \frac{z}{\zeta_n}}$$

$$\mathbb{E}\left[\mathrm{e}^{z\underline{X}_{\mathbf{e}_q}}\right] = \prod_{n\geq 1} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 + \frac{z}{\hat{\zeta}_n}}.$$

Distribution of extrema

- For $x \geq 0$

$$\mathbb{P}(\overline{X}_{\mathsf{e}_q} \in \mathrm{d}x) \quad = \quad \mathsf{a}_0(\rho, \zeta)\delta_0(\mathrm{d}x) + \sum_{n=1}^{\infty} \mathsf{a}_n(\rho, \zeta)\zeta_n \mathrm{e}^{-\zeta_n x}\mathrm{d}x$$

- Here

$$\mathsf{a}_0(\rho, \zeta) \quad = \quad \lim_{n \to +\infty} \prod_{k=1}^{n} \frac{\zeta_k}{\rho_k}, \quad \mathsf{a}_n(\rho, \zeta) = \left(1 - \frac{\zeta_n}{\rho_n}\right) \prod_{\substack{k \geq 1 \\ k \neq n}} \frac{1 - \frac{\zeta_n}{\rho_k}}{1 - \frac{\zeta_n}{\zeta_k}}$$

- A similar expression holds for $\mathbb{P}(-\underline{X}_{\mathsf{e}_q} \in \mathrm{d}x)$.