Multilevel Weiner-Hopf Monte-Carlo simulation for Lévy processes

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What is a (one-dimensional) Lévy process?

- **Formally:** A stochastic process \( \{X_t : t \geq 0\} \) which satisfies
  - \( X_0 = 0 \),
  - \( X \) has paths that are right-continuous with left limits (almost surely),
  - For any \( 0 \leq s \leq t \), \( X_t - X_s \) is equal in distribution to \( X_{t-s} \),
  - For any \( 0 \leq s \leq t \), \( X_t - X_s \) is independent of \( \{X_u : u \leq s\} \).
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- **Informally:** Some familiar Lévy processes include
  - Linear Brownian motion: \( \sigma B_t + \mu t, t \geq 0 \), where \( \sigma^2 \geq 0 \) and \( \mu \in \mathbb{R} \),
  - Compound Poisson processes: \( \sum_{i=1}^{N_t} \xi_i \), where \( \{N_t : t \geq 0\} \) is a Poisson arrival process and \( \{\xi_i : i \in \mathbb{N}\} \) are i.i.d. random variables.
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  - If \( X^{(1)}_t, X^{(2)}_t, \ldots \), are independent Lévy processes then, subject to certain conditions, so is
    \[
    \sum_{i \geq 1} X^{(i)}_t
    \]
Brownian motion
Compound Poisson process
Brownian motion $+$ compound Poisson process
Multilevel Weiner-Hopf Monte-Carlo simulation for Lévy processes

Unbounded variation paths

![Graph showing unbounded variation paths](image-url)
Bounded variation paths
Motivation

- **Lévy process.** A (one dimensional) process with stationary and independent increments which has paths which are right continuous with left limits and therefore includes Brownian motion with drift, compound Poisson processes, stable processes amongst many others).

- A popular (and often criticised) model in mathematical finance for the evolution of a risky asset is

\[
S_t := e^{X_t}, \quad t \geq 0
\]

where \( \{X_t : t \geq 0\} \) is a Lévy process. Also used in insurance risk models!

- **Barrier options:** The value of up-and-out barrier option with expiry date \( T \) and barrier \( b \) is typically priced as

\[
\mathbb{E}_s(f(X_1)1_{\{X_1 \leq b\}})
\]

where \( \overline{X}_1 = \sup_{u \leq 1} X_u \), \( f \) is some nice function.

- One is fundamentally interested in the joint distribution

\[
P(X_1 \in dx, \overline{X}_1 \in dy)
\]

for any Lévy process \((X, \mathbb{P})\).
Consider a Poisson process with arrival rate $n$. Denote by $\tau_1, \tau_2, \cdots$ the arrival times. Note that $\tau_n$ is the sum of $n$ i.i.d exponential random variables, each with mean $1/n$. We could therefore write $\tau_n = n \sum_{i=1}^{\tau_n} \frac{1}{n} e(i)$, where $e(i)$ are i.i.d. exponential random variables with unit mean. Hence by the SLLN $\tau_n \to 1$ almost surely.

Hence for a suitably large $n$, we have in distribution $(X_{\tau_n}, X_{\tau_n}) \to (X_1, X_1)$ as $n \to \infty$. Indeed since $1$ is not a jump time with probability 1, we have that $(X_{\tau_n}, X_{\tau_n}) \to (X_1, X_1)$ almost surely as $n \to \infty$.

Original WHMC method: Kuznetsov et al. (2011)
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$$\tau_n \rightarrow 1 \quad \text{almost surely.}$$
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Hence for a suitably large $n$, we have in distribution

$$(X_{\tau_n}, X_{\tau_n}) \sim (X_1, X_1).$$

Indeed since 1 is not a jump time with probability 1, we have that

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- A reformulation of the Wiener-Hopf factorization states that

\[ X_{e_q} \overset{d}{=} S_q + I_q \]

where \( S_q \) is independent of \( I_q \) and they are respectively equal in distribution to \( \overline{X}_{e_q} \) and \( \underline{X}_{e_q} \). Here \( \overline{X}_t = \inf_{s \leq t} X_s \).
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- Taking advantage of the above, the fact that \( X \) has stationary and independent increments and the fact that, as a time, \( \tau_n \) can be seen as the sum of independent exponential time periods we have the following:
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- For all $n \in \{1, 2, \cdots \}$ and $n > 0$,

$$\left(X_{\tau_n}, \overline{X}_{\tau_n}\right) \overset{d}{=} \left(V_n, J_n\right)$$

where

$$V_n = \sum_{j=1}^{n} \{S_n^{(j)} + I_n^{(j)}\}$$

$$J_n := \bigvee_{i=0}^{n-1} \left(V_i + S_n^{(i+1)}\right).$$

Here, $V_0 = S_n^{(0)} = I_n^{(0)} = 0$, $\{S_n^{(j)} : j \geq 1\}$ are an i.i.d. sequence of random variables with common distribution equal to that of $\overline{X}_{e_n}$ and $\{I_n^{(j)} : j \geq 1\}$ are another i.i.d. sequence of random variable with common distribution equal to that of $X_{e_n}$. 
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■ \((V_n, J_n) \xrightarrow{n\uparrow\infty} (X_1, \overline{X}_1)\) in distribution.
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- Sample repeatedly and independently from the distribution $\overline{X}_{en}$ and $\underline{X}_{en}$ and then construct $m$ independent versions of the variables $V_n$ and $J_n$, say

$$\{V_n^{(i)} : i = 1, \cdots, m\} \text{ and } \{J_n^{(i)} : i = 1, \cdots, m\}.$$
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- Sample repeatedly and independently from the distribution $\bar{X}_{e_n}$ and $\bar{X}_{e_n}$ and then construct $m$ independent versions of the variables $V_n$ and $J_n$, say

  \[ \{ V_n^{(i)} : i = 1, \cdots, m \} \text{ and } \{ J_n^{(i)} : i = 1, \cdots, m \}. \]

- Then

  \[ \mathbb{E}(F(X_1, \bar{X}_1)) \approx \mathbb{E}(F(X_{\tau_n}, \bar{X}_{\tau_n}) = \mathbb{E}(F(V_n, J_n)) \approx \frac{1}{m} \sum_{i=1}^{m} F(V_n^{(i)}, J_n^{(i)}) =: \hat{F}_{MC}^{n,m}. \]
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\]

- Sampling from $\overline{X}_{e_n}$ and $\underline{X}_{e_n}$ is generally impossible for a given Lévy process, but not for a 10 parameter family of processes known as Kuznetsov’s $\beta$-class (ask me afterwards if interested in the details!).
Numerical Analysis

Henceforth assume that

\[ F : \mathbb{R} \times [0, \infty) \to [0, \infty) \]

is Lipschitz continuous with coefficient 1.

Our underlying Lévy process satisfies

\[ \int |x| \geq 1 x^2 \Pi(dx) < \infty, \]

where \( \Pi \) is its associated jump measure (finite second moments).

Notation.
Write \( a \lesssim b \) for two positive quantities \( a \) and \( b \), if \( a/b \) is uniformly bounded (independent of \( n, M \), or any other parameters).

Write \( F_n(i) := F(V_n(i), J_n(i)) \) for the \( i \)-th sample of \( F_n := F(V_n, J_n) \) (using the Wiener-Hopf random walk).

Define the mean square error as

\[ e(\hat{F}_n, m_{MC})^2 = E[(\hat{F}_n, m_{MC} - E[F(X_1, X_1)])^2] = m - 1 V(F_n) + (E[F_n] - E[F(X_1, X_1)])^2 \]

Then we have the following convergence/complexity theorem...
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e(\hat{F}_{MC}^{n,m})^2 := \mathbb{E}[(\hat{F}_{MC}^{n,m} - \mathbb{E}[F(X_1, X_1)])^2] = m^{-1} \mathbb{V}(F^n) + (\mathbb{E}[F^n - F(X_1, X_1)])^2
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Then we have the following convergence/complexity theorem . . .
Numerical Analysis WHMC

**Theorem (Single-level WHMC)**

Assume that \(\exists \alpha > 0\) s.t.

(i) \(\mathbb{E}[|F^n - F(X_t, \overline{X}_t)|] \lesssim n^{-\alpha}\) and

(ii) \(C(F^n) \lesssim n\) (where \(C(F^n)\) is the cost to compute a single sample from \(F^n\))

Then, \(\forall \nu \in \mathbb{N} \ \exists n, M \in \mathbb{N} \ s.t.

\[ C(\hat{F}_{MC}^{n,M}) \lesssim \nu \ \text{and} \ L^2 \text{ error} \ e(\hat{F}_{MC}^{n,m}) \lesssim \nu^{-\frac{1}{2+1/\alpha}}. \]
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- For Kuznetsov’s \( \beta \)-class of Lévy processes also Assumption (ii) holds.
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Using the forthcoming analysis we shall shortly present, it will turn out that:

- when \( X \) has paths of unbounded variation, \( \alpha = \frac{1}{4} \Rightarrow O(\nu^{-\frac{1}{6}}) \) convergence!

- when \( X \) has paths of bounded variation, \( \alpha = \frac{1}{2} \Rightarrow O(\nu^{-\frac{1}{4}}) \) convergence!
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Then, \( \forall \nu \in \mathbb{N} \ \exists n, M \in \mathbb{N} \) s.t.

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C(\hat{F}^{n,M}_{MC}) \lesssim \nu \quad \text{and } L^2 \text{ error } e(\hat{F}^{n,m}_{MC}) \lesssim \nu^{-\frac{1}{2}+\frac{1}{\alpha}}. 
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The best one can hope for with such Monte-Carlo schemes is an \( \mathcal{O}(\nu^{-\frac{1}{2}}) \) convergence.

- Computational gains from exploiting the telescopic sum

\[
\mathbb{E}[F^{n_L}] = \mathbb{E}[F^{n_0}] + \sum_{\ell=1}^{L} \mathbb{E}[F^{n_\ell} - F^{n_{\ell-1}}],
\]

where \( n_\ell = 2^\ell n_0 \), \( \ell = 1, \ldots, L \), for some small \( n_0 \in \mathbb{N} \).
Multi-level Wiener-Hopf Monte-Carlo \cite{Heinrich, 2001}, \cite{Giles, 2007}, \ldots

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- Suggesting the multilevel estimator

\[
\hat{F}_{ML}^{n_0, L, \{M_\ell\}} := \frac{1}{M_0} \sum_{i=1}^{M_0} F^{n_0,(i)} + \sum_{\ell=1}^{L} \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} (F^{n_\ell,(i)} - F^{n_{\ell-1),(i)}}).
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- Here it is very important that \( F^{n_{\ell-1}} \) can be obtained from \( F^{n_\ell} \) by a
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- A little algebra again reveals that the means square error satisfies

\[ e(\hat{F}_{ML}^{n_0,L,\{M_{\ell}\}})^2 = \frac{1}{M_0} \mathbb{V}(F^{n_0}) + \sum_{\ell=1}^{L} \frac{1}{M_{\ell}} \mathbb{V}(F^{n\ell} - F^{n\ell-1}) + (\mathbb{E}[F^n - F(X_1, \overline{X}_1)])^2. \]
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\]

- See also \cite{Dereich, Heidenreich, 2011}, \cite{Dereich, 2011}, \cite{Giles, Xia, 2012}.
Poisson thinning and multilevel WHMC

In the WHMC method how do we introduce "levels"?

Recall also that it is crucial to have a Poisson process for the time randomisations on all levels! How do we sample on two consecutive levels?

Suppose the "level $\ell$" grid is based on a Poisson process of rate $\lambda^\ell$. Then by tossing a coin and rejecting arrivals with probability $1/2$ we end up with a Poisson process of rate $\lambda^{\ell-1}$: our new coarser "level $\ell-1$" Poisson grid.

(Not a new idea! Also used by [Glasserman, Merener, 2003], [Giles, Xia, 2012], ...)
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- In the WHMC method how do we introduce "levels"?
- Recall also that it is **crucial** to have a Poisson process for the time randomisations on **all** levels! How do we sample on two consecutive levels?
- Suppose the "level \( \ell \)" grid is based on a Poisson process of rate \( n_\ell \). Then by **tossing a coin** and rejecting arrivals with probability \( \frac{1}{2} \) we end up with a Poisson process of rate \( n_{\ell-1} \): our new coarser "level \( \ell - 1 \)" Poisson grid.

(Not a new idea! Also used by [Glasserman, Merener, 2003], [Giles, Xia, 2012], ...)

![Diagram showing Poisson thinning and multilevel WHMC](image_url)
Numerical Analysis (multilevel case)

Theorem (Multilevel WHMC)

Assume $\exists \alpha, \beta > 0$ with $\alpha \geq \frac{1}{2} \max\{\beta, 1\}$ such that

(i) $|\mathbb{E}[F_{n\ell} - F(X_t, \overline{X}_t)]| \lesssim n_{\ell}^{-\alpha}$

(ii) $\forall [F_{n\ell} - F_{n\ell-1}] \lesssim n_{\ell}^{-\beta}$

(iii) $\mathbb{E}[C_{n\ell}] \lesssim n_{\ell}$.

Then, $\forall \nu \in \mathbb{N}$ $\exists L$ and $\{M_{\ell}\}_{\ell=0}^{L}$ s.t. $C\left(\hat{F}_{n0,l,\{M_{\ell}\}}\right) \lesssim \nu$ and $L^2$ error

$$e\left(\hat{F}_{n0,l,\{M_{\ell}\}}\right) \lesssim \begin{cases} \nu^{-\frac{1}{2}} & \text{if } \beta > 1, \\ \nu^{-\frac{1}{2}} \log^2 \nu & \text{if } \beta = 1, \\ \nu^{-\frac{1}{2}+\left(1-\beta\right)/\alpha} & \text{if } \beta < 1. \end{cases}$$
**Numerical Analysis (multilevel case)**

**Theorem (Multilevel WHMC)**

Assume $\exists \alpha, \beta > 0$ with $\alpha \geq \frac{1}{2} \max\{\beta, 1\}$ such that

(i) $|E[F_{n,\ell} - F(X_t, X_t)]| \lesssim n^{-\alpha}_\ell$

(ii) $\forall [F_{n,\ell} - F_{n,\ell-1}] \lesssim n^{-\beta}_\ell$

(iii) $E[C_{n,\ell}] \lesssim n_\ell$.

Then, $\forall \nu \in \mathbb{N}$ $\exists L$ and $\{M_\ell\}_{\ell=0}^L$ s.t. $C\left(\hat{F}_{ML}^{n_0, L, \{M_\ell\}}\right) \lesssim \nu$ and $L^2$ error

$$e\left(\hat{F}_{ML}^{n_0, L, \{M_\ell\}}\right) \lesssim \begin{cases} \nu^{-\frac{1}{2}} & \text{if } \beta > 1, \\ \nu^{-\frac{1}{2}} \log^2 \nu & \text{if } \beta = 1, \\ \nu^{-\frac{1}{2}+(1-\beta)/\alpha} & \text{if } \beta < 1. \end{cases}$$

- For Kuznetsov's $\beta$-class of Lévy processes also Assumption (iii) holds.
- Recall that $\alpha = (1/4)1/2$ for (un)bounded variation paths & shortly we shall see $\beta = 1/2 \Rightarrow (O(\nu^{-\frac{1}{4}})) O(\nu^{-\frac{1}{3}})$.
- Compare with former (single-level) rates $O(\nu^{-\frac{1}{6}}) O(\nu^{-\frac{1}{4}})$. 


Remains to verify the assumptions (i) \( |\mathbb{E}[F^{n\ell} - F(X_t, \overline{X}_t)]| \lesssim n^{-\alpha}_{\ell} \) and (ii) \( \nabla[F^{n\ell} - F^{n\ell-1}] \lesssim n^{-\beta}_{\ell} \).

Recalling that \( F \) is Lipschitz

\[
\nabla(F^{n\ell} - F^{n\ell-1}) = \nabla(F(X_{\tau_{n\ell}}, \overline{X}_{\tau_{n\ell}}) - F(X_{\tau_{n\ell}-1}, \overline{X}_{\tau_{n\ell}-1})) \\
\leq \mathbb{E}[(X_{\tau_{n\ell}} - X_{\tau_{n\ell}-1})^2] + \mathbb{E}[(\overline{X}_{\tau_{n\ell}} - \overline{X}_{\tau_{n\ell}-1})^2]
\]

"Better estimates for bounded variation case!"
Numerical analysis MLWH

Remains to verify the assumptions (i) \(|E[F_{n\ell} - F(X_t, \overline{X}_t)]| \lesssim n_\ell^{-\alpha}\) and (ii) \(\nabla[F_{n\ell} - F_{n\ell-1}] \lesssim n_\ell^{-\beta}\).

- Recalling that \(F\) is Lipschitz
  \[\nabla(F_{n\ell} - F_{n\ell-1}) = \nabla(F(X_{\tau_{n\ell}}, \overline{X}_{\tau_{n\ell}}) - F(X_{\tau_{n\ell-1}}, \overline{X}_{\tau_{n\ell-1}})) \leq E[(X_{\tau_{n\ell}} - X_{\tau_{n\ell-1}})^2] + E[(\overline{X}_{\tau_{n\ell}} - \overline{X}_{\tau_{n\ell-1}})^2]\]

- \([\text{Unbounded variation case}]:\) Working with our Lévy process as a Markovian semi-martingale we get for \(s\) and \(t\) random times independent of \(X\),

  (1) \(E[(X_t - X_s)^2] = \nabla(X_1)E[|t - s|] + E[X_1]^2E[(t - s)^2]\)
  
  (2) \(E[(\overline{X}_t - \overline{X}_s)^2] \leq 16\nabla(X_1)E[|t - s|] + 2(\max\{E[X_1], 0\})^2E[(t - s)^2],\)

[Better estimates for bounded variation case!]

[Unbounded variation case]: Working with our Lévy process as a Markovian semi-martingale we get for \(s\) and \(t\) random times independent of \(X\),
Numerical analysis MLWH

Remains to verify the assumptions (i) $|E[F^{n\ell} - F(X_t, \overline{X}_t)]| \lesssim n_{\ell}^{-\alpha}$ and (ii) $\nabla [F^{n\ell} - F^{n\ell-1}] \lesssim n_{\ell}^{-\beta}$.

- Recalling that $F$ is Lipschitz

  $$\nabla (F^{n\ell} - F^{n\ell-1}) = \nabla (F(X_{\tau_{n\ell}}, \overline{X}_{\tau_{n\ell}}) - F(X_{\tau_{n\ell}-1}, \overline{X}_{\tau_{n\ell}-1})) \leq E[(X_{\tau_{n\ell}} - X_{\tau_{n\ell}-1})^2] + E[(\overline{X}_{\tau_{n\ell}} - \overline{X}_{\tau_{n\ell}-1})^2]$$

- [Unbounded variation case]: Working with our Lévy process as a Markovian semi-martingale we get for $s$ and $t$ random times independent of $X$,

  1. $E[(X_t - X_s)^2] = \nabla(X_1)E[|t - s|] + E[X_1]^2E[(t - s)^2]$
  2. $E[(\overline{X}_t - \overline{X}_s)^2] \leq 16\nabla(X_1)E[|t - s|] + 2(\max\{E[X_1], 0\})^2E[(t - s)^2],$

  [Better estimates for bounded variation case!]

- Finally, recalling that $\tau_{n\ell}$ is a $(n_{\ell}, n_{\ell})$-gamma distribution with mean 1,

  $$E[(\tau_{n\ell} - 1)^2] \lesssim n_{\ell}^{-1} \quad \text{and} \quad E[|\tau_{n\ell} - 1|] \lesssim n_{\ell}^{-\frac{1}{2}}.$$
Remains to verify the assumptions (i) $\left| \mathbb{E}\left[F^{n_\ell} - F(X_t, \overline{X}_t)\right]\right| \lesssim n_\ell^{-\alpha}$ and (ii) $\nabla[F^{n_\ell} - F^{n_\ell-1}] \lesssim n_\ell^{-\beta}$.

- Recalling that $F$ is Lipschitz

$$\nabla(F^{n_\ell} - F^{n_\ell-1}) = \nabla(F(X_{\tau_{n_\ell}}, \overline{X}_{\tau_{n_\ell}}) - F(X_{\tau_{n_\ell}-1}, \overline{X}_{\tau_{n_\ell}-1}))$$

$$\leq \mathbb{E}[(X_{\tau_{n_\ell}} - X_{\tau_{n_\ell}-1})^2] + \mathbb{E}[(\overline{X}_{\tau_{n_\ell}} - \overline{X}_{\tau_{n_\ell}-1})^2]$$

- [Unbounded variation case]: Working with our Lévy process as a Markovian semi-martingale we get for $s$ and $t$ random times independent of $X$,

(1) $\mathbb{E}[(X_t - X_s)^2] = \nabla(X_1)\mathbb{E}[|t - s|] + \mathbb{E}[X_1]^2\mathbb{E}[(t - s)^2]$

(2) $\mathbb{E}[(\overline{X}_t - \overline{X}_s)^2] \leq 16\nabla(X_1)\mathbb{E}[|t - s|] + 2(\max\{\mathbb{E}[X_1], 0\})^2\mathbb{E}[(t - s)^2],$

[Better estimates for bounded variation case!]

- Finally, recalling that $\tau_{n_\ell}$ is a $(n_\ell, n_\ell)$-gamma distribution with mean 1,

$$\mathbb{E}[(\tau_{n_\ell} - 1)^2] \lesssim n_\ell^{-1} \quad \text{and} \quad \mathbb{E}[|\tau_{n_\ell} - 1|] \lesssim n_\ell^{-\frac{1}{2}}.$$

- All together we get $\nabla(F^{n_\ell} - F^{n_\ell-1}) \lesssim n_\ell^{-\frac{1}{2}}$, and so $\beta = \frac{1}{2}$. 
**Numerical analysis MLWH**

Remains to verify the assumptions (i) \( |\mathbb{E}[F^{n\ell} - F(X_t, X_t)]| \lesssim n_{\ell}^{-\alpha} \) and (ii) \( \nabla[F^{n\ell} - F^{n\ell-1}] \lesssim n_{\ell}^{-\beta} \).

- Recalling that \( F \) is Lipschitz
  \[
  \nabla(F^{n\ell} - F^{n\ell-1}) = \nabla(F(X_{\tau_{n\ell}}, X_{\tau_{n\ell}}) - F(X_{\tau_{n\ell-1}}, X_{\tau_{n\ell-1}})) \\
  \leq \mathbb{E}[(X_{\tau_{n\ell}} - X_{\tau_{n\ell-1}})^2] + \mathbb{E}[(X_{\tau_{n\ell}} - X_{\tau_{n\ell-1}})^2]
  \]

- **[Unbounded variation case]:** Working with our Lévy process as a Markovian semi-martingale we get for \( s \) and \( t \) random times independent of \( X \),

  (1) \( \mathbb{E}[(X_t - X_s)^2] = \nabla(X_1)\mathbb{E}[|t - s|] + \mathbb{E}[X_1]^2\mathbb{E}[(t - s)^2] \)

  (2) \( \mathbb{E}[(X_t - \overline{X}_s)^2] \leq 16\nabla(X_1)\mathbb{E}[|t - s|] + 2(\max\{\mathbb{E}[X_1], 0\})^2\mathbb{E}[(t - s)^2], \)

  **[Better estimates for bounded variation case!]**

- Finally, recalling that \( \tau_{n\ell} \) is a \((n_{\ell}, n_{\ell})\)-gamma distribution with mean 1,
  \[
  \mathbb{E}[(\tau_{n\ell} - 1)^2] \lesssim n_{\ell}^{-1} \quad \text{and} \quad \mathbb{E}[|\tau_{n\ell} - 1|] \lesssim n_{\ell}^{-\frac{1}{2}}.
  \]

- All together we get \( \nabla(F^{n\ell} - F^{n\ell-1}) \lesssim n_{\ell}^{-\frac{1}{2}} \), and so \( \beta = \frac{1}{2} \).

- Via Jensen’s inequality we then easily also get \( \alpha = \frac{1}{4} \ldots \)
Kuznetsov’s $\beta$-class

- The characteristic exponent ($\Psi(\theta) = -\log \mathbb{E}(e^{i\theta X_1}), \theta \in \mathbb{R}$) is given by

$$
\Psi(\theta) = i a z + \frac{1}{2} \sigma^2 z^2 + \frac{c_1}{\beta_1} \left\{ B(\alpha_1, 1 - \lambda_1) - B(\alpha_1 - \frac{i\theta}{\beta_1}, 1 - \lambda_1) \right\} \\
+ \frac{c_2}{\beta_2} \left\{ B(\alpha_2, 1 - \lambda_2) - B(\alpha_2 + \frac{i\theta}{\beta_2}, 1 - \lambda_2) \right\}
$$

where $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ is the Beta function, with parameter range $a \in \mathbb{R}, \sigma, c_i, \alpha_i, \beta_i > 0$ and $\lambda_1, \lambda_2 \in (0, 3) \setminus \{1, 2\}$.

- The corresponding Lévy measure $\Pi$ has density

$$
\pi(x) = c_1 \frac{e^{-\alpha_1 \beta_1 x}}{(1 - e^{-\beta_1 x})^{\lambda_1}} 1_{\{x > 0\}} + c_2 \frac{e^{\alpha_2 \beta_2 x}}{(1 - e^{\beta_2 x})^{\lambda_2}} 1_{\{x < 0\}}.
$$

The $\beta$-class of Lévy processes includes another recently introduced family of Lévy processes known as Lamperti-stable processes.
Meromorphic Lévy processes (contains the $\beta$-class)

(i) The characteristic exponent $\Psi(z)$ is a meromorphic function which has poles at points $\{-i\rho_n, i\hat{\rho}_n\}_{n \geq 1}$, where $\rho_n$ and $\hat{\rho}_n$ are positive real numbers.

(ii) For $q \geq 0$ function $q + \Psi(z)$ has roots at points $\{-i\zeta_n, i\hat{\zeta}_n\}_{n \geq 1}$ where $\zeta_n$ and $\hat{\zeta}_n$ are nonnegative real numbers (strictly positive if $q > 0$). We will write $\zeta_n(q)$, $\hat{\zeta}_n(q)$ if we need to stress the dependence on $q$.

(iii) The roots and poles of $q + \Psi(iz)$ satisfy the following interlacing condition

$$... - \rho_2 < -\zeta_2 < -\rho_1 < -\zeta_1 < 0 < \hat{\zeta}_1 < \hat{\rho}_1 < \hat{\zeta}_2 < \hat{\rho}_2 < ...$$

(iv) The Wiener-Hopf factors are expressed as convergent infinite products,

$$\mathbb{E}\left[e^{-zX}e_q\right] = \prod_{n \geq 1} \frac{1 + \frac{z}{\rho_n}}{1 + \frac{z}{\zeta_n}}$$

$$\mathbb{E}\left[e^{zX}e_q\right] = \prod_{n \geq 1} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 + \frac{z}{\hat{\zeta}_n}}.$$
Distribution of extrema

- For $x \geq 0$

$$
P(X_{e_q} \in dx) = a_0(\rho, \zeta)\delta_0(dx) + \sum_{n=1}^{\infty} a_n(\rho, \zeta)\zeta_n e^{-\zeta_n x} dx
$$

- Here

$$
a_0(\rho, \zeta) = \lim_{n \to +\infty} \prod_{k=1}^{n} \frac{\zeta_k}{\rho_k}, \quad a_n(\rho, \zeta) = \left(1 - \frac{\zeta_n}{\rho_n}\right) \prod_{\substack{k \geq 1 \atop k \neq n}} \frac{1 - \frac{\zeta_n}{\rho_k}}{1 - \frac{\zeta_n}{\zeta_k}}
$$

- A similar expression holds for $P(-X_{e_q} \in dx)$. 
References


