

# Lévy Processes and Continuous state branching

processes

Start by recalling the def<sup>n</sup> of two families

processes

Brownian motion

A real valued process

B<sub>M</sub>: A real valued process  $B = \{B_t : t \geq 0\}$ .

(i) Paths are a.s. cts

(ii)  $B_0 = 0$  a.s.

(iii)  $0 \leq s \leq t < \infty$

(iv)

(v)

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$\forall t \geq 0 \quad B_t \sim N(0, t)$

$B_t - B_s \perp\!\!\!\perp \{B_u : u \leq s\}$

$B_t - B_s \stackrel{d}{=} B_{t-s}$

PP:  $\mathbb{R}_+$ -valued process  $N = \{N_t : t \geq 0\}$  parameterised by  $\lambda > 0$

(i)  $N$  has right cts paths with left limits a.s.

(ii)  $N_0 = 0$  a.s.

(iii)  $0 \leq s \leq t < \infty$

(iv)

(v)  $N_t \sim Po(\lambda t)$

$N_t - N_s \perp\!\!\!\perp \{N_u : u \leq s\}$

$N_t - N_s \stackrel{d}{=} N_{t-s}$

BM: A real valued process  $B = \{B_t : t \geq 0\}$ .

(i) Paths are a.s. cts

(ii)  $B_0 = 0$  a.s.

(iii)  $0 \leq s \leq t < \infty$

(iv)

(v)

$\forall t \geq 0 \quad B_t \sim N(0, t)$

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Def<sup>n</sup> (Lévy process) A  $\mathbb{R}$ -valued process  $X = \{X_t : t \geq 0\}$  is said to be a Lévy process

if

(i) Paths of  $X$  are right cts with left limits a.s.

(ii)  $X_0 = 0$  a.s.

(iii) Indep incr.

$$X_t - X_s \perp\!\!\!\perp \{X_u : u \leq s\}$$

(iv) St. incr.

$$X_t - X_s \stackrel{d}{=} X_{t-s}$$

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Set of LP is not empty because of BM & PP.  
But how big is the family of LP?

Infinitely divisible r.v.

Defn an  $\mathbb{R}$ -valued r.v.  $(X)$  is infinitely

divisible if for each  $n = 2, 3, \dots$

there exist r.v.  $(H)_{n,1}, \dots, (H)_{n,n}$   
which are independent s.t.

$$(X) \stackrel{d}{=} (H)_{n,1} + \dots + (H)_{n,n}$$

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$$P_0(\lambda) \stackrel{d}{=} \sum_{l=1}^n \underbrace{P_0^{(l)}\left(\frac{\lambda}{n}\right)}_{\text{i.i.d.}}$$

Hint at relationship between L.P.s & i.d.

dist<sup>n</sup>s:

$$X_t = \underbrace{(X_{t/n} - X_0)}_{\stackrel{d}{=} X_{t/n}} + \underbrace{(X_{2t/n} - X_{t/n})}_{\textcircled{II} \stackrel{d}{=} X_{t/n}} + \dots + \underbrace{(X_t - X_{(n-1)t/n})}_{\textcircled{III} \stackrel{d}{=} X_{t/n}}$$

Hence for all  $t \geq 0$ ,  $X_t$  is an i.d. r.v.

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we proceed by investigating further the class of  
 i. d. r.v. — how can we characterize all of  
 them?

# Theorem (Lévy-Kinchine formula)

A prob. law.  $\mu$  belongs to a i.d. r.v. with characteristic exponent  $\Psi(\theta)$ , i.e.

$$\int_{\mathbb{R}} e^{i\theta x} \mu(dx) =: e^{-\Psi(\theta)} \quad \theta \in \mathbb{R},$$

if and only if  $\exists (a, \sigma, \Pi)$  where  $a, \sigma \in \mathbb{R}$  and  $\Pi$  is a measure concentrated on  $\mathbb{R} \setminus \{0\}$

satisfies  $\int_{\mathbb{R}} (\ln x^2) \Pi(dx) < \infty$  s.t.

$$\underline{\underline{\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbb{1}_{(|x| < 1)}) \Pi(dx)}} \quad \forall \theta \in \mathbb{R}$$

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