Lévy Processes and Continuous state branching processes

Start by recalling the defn of two familiar processes

Brownian motion

A real valued process

BM : A real valued process $\underline{B} = \{B_t : t \geq 0\}$.

(i) Paths are a.s. cts

(ii) $B_0 = 0$ a.s.

(iii) $0 \leq s \leq t < \infty$ $\quad B_t - B_s \perp\!\!\!\perp \{B_u : u \leq s\}$

(iv) —— " —— $\quad B_t - B_s \overset{d}{=} B_{t-s}$

(v) $\forall t \geq 0$ $\quad B_t \sim N(0, t)$

PP : $\mathbb{R}_+$-valued process $N = \{N_t : t \geq 0\}$ parameterised by $\lambda > 0$

(i) N has right cts paths with left limits a.s.

(ii) $N_0 = 0$ a.s.

(iii) $0 \leq s \leq t < \infty$ $\quad N_t - N_s \perp\!\!\!\perp \{N_u : u \leq s\}$

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(v) $N_t \sim Po(\lambda t)$

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Def^n (Lévy process) A $\mathbb{R}$-valued process $X = \{X_t : t \geq 0\}$ is said to be a Lévy process if

(I) Paths of X are right cts with left limits a.s.

(II) $X_0 = 0$ a.s.

(III) Indep incr. $X_t - X_s \perp\!\!\!\perp \{X_u : u \leq s\}$

(IV) St. incr. $X_t - X_s \overset{d}{=} X_{t-s}$

Set of LP is not empty because of BM & PP.

But how big is the family of LP?

Infinitely divisible r.v.

<u>Defn</u> an $\mathbb{R}$-valued r.v. Θ is infinitely divisible if for each $n = 1, 2, 3, \dots$

there exist r.v. $\Theta_{n,1}, \dots\dots, \Theta_{n,n}$ which are independent s.t.

$$\Theta \stackrel{d}{=} \Theta_{n,1} + \dots\dots + \Theta_{n,n}$$

$$Po(\lambda) \stackrel{d}{=} \sum_{i=1}^{n} \underbrace{Po^{(i)}(\lambda/n)}_{\text{i.i.d}}$$

Hint at relationship between LPs & i.d. distr^{n}s:

$$X_t = \underbrace{\left(X_{t/n} - X_0\right)}_{\overset{d}{=} X_{t/n}} + \underbrace{\left(X_{2t/n} - X_{t/n}\right)}_{\overset{d}{=} X_{t/n}} + \cdots + \underbrace{\left(X_t - X_{\frac{(n-1)t}{n}}\right)}_{\overset{d}{=} X_{t/n}}$$

$\perp\!\!\!\perp$ $\quad$ $\perp\!\!\!\perp$

Hence for all $t \geq 0$, X_t is an i.d. r.v.

We proceed by investigating further the class of i.d. r.v. — how can we characterise all of them?

Theorem (Lévy-Khintchine formula)

A prob. law μ belongs to a i.d r.v. with characteristic exponent $\Psi(\theta)$, i.e.

$$\int_{\mathbb{R}} e^{i\theta x}\,\mu(dx) =: e^{-\Psi(\theta)} \qquad \theta \in \mathbb{R},$$

if and only if $\exists\,(a,\sigma,\Pi)$ where $a,\sigma \in \mathbb{R}$ and Π is a measure concentrated on $\mathbb{R}\setminus\{0\}$ satisfies $\int_{\mathbb{R}}(1\wedge x^2)\,\Pi(dx) < \infty$ s.t.

$$\Psi(\theta) = \underline{\underline{ia\theta + \tfrac{1}{2}\sigma^2\theta^2}} + \int_{\mathbb{R}}\left(1 - e^{i\theta x} + i\theta x\,\mathbb{1}_{(|x|<1)}\right)\Pi(dx) \quad \forall\,\theta\in\mathbb{R}$$

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