~~Proof~~ $\mathcal{E}_t(\lambda) = e^{\lambda X_t - \psi(\lambda) t}$

$$d\left[e^{-\lambda(\bar{X}_t \vee x) + \psi(\lambda) t}\, \mathcal{E}_t(\lambda)\right]$$

$$= e^{-\lambda((\bar{X}_t \vee x) + \psi(\lambda) t)}\, d\,\mathcal{E}_t(\lambda)$$

$$+ \mathcal{E}_t(\lambda)\, e^{-\lambda((\bar{X}_t \vee x) + \psi(\lambda) t)}\, d\left[-\lambda(\bar{X}_t \vee x) + \psi(\lambda) t\right]$$

+ cross ~~product~~ term

$$dM_t^x = \psi(\lambda)\, e^{-\lambda(\bar{X}_t \vee x) + \psi(\lambda) t}\, \mathcal{E}_t(\lambda)\, dt$$

$$- d\left[e^{-\lambda(\bar{X}_t \vee x) + \psi(\lambda) t}\, \mathcal{E}_t(\lambda)\right] - \lambda d(\bar{X}_t \vee x)$$

$$= \cancel{\boxed{\psi(\lambda)\, e^{-\lambda(\bar{X}_t \vee x) + \psi(\lambda) t}\, \mathcal{E}_t(\lambda)\, dt}}$$

$$- e^{-\lambda(\bar{X}_t \vee x) + \psi(\lambda) t}\, d\,\mathcal{E}_t(\lambda)$$

$$+ \underbrace{e^{-\lambda(\bar{X}_t \vee x) + \psi(\lambda) t}\, \mathcal{E}_t(\lambda)}_{\to = 1 \text{ on } \{t :\, \bar{X}_t = X_t\}}\left[\lambda \cancel{\boxed{d(\bar{X}_t \vee x)}}\ \cancel{\boxed{-\psi(\lambda) dt}}\right]$$

$$- \cancel{\boxed{\lambda d(\bar{X}_t \vee x)}}$$

$\bar{X}_t \vee x = x$ when $t < \tau_x^+$ in that case "$d(\bar{X}_t \vee x) = 0$"

when $t \geq \tau_x^+$ then "$d(\bar{X}_t \vee x) = d\bar{X}_t$"

and the measure $d\bar{X}_t$ is supported by $\{t : \bar{X}_t = X_t\}$

In conclusion:

$$dM_t^x = -e^{-\lambda(\bar{X}_t \vee x) + \psi(\lambda)t} \, d\mathcal{E}_t(\lambda)$$

$$\Rightarrow \underbrace{M_t^x}_{} = 0 + \underbrace{\int_0^t -e^{\cdots\cdots} \, d\mathcal{E}_t(\lambda)}_{\text{local mg}}$$

To prove that $M^x_\cdot$ is a mg (not strictly a local mg) it suffices to show that for each $t>0$

$$\mathbb{E}\left(\sup_{s\le t} |M_s|^x\right) < \infty \quad (*)$$

Note

$$\mathbb{E}\left(M^x_{t\wedge T_n} \middle| \mathcal{F}_s\right) = M^x_{s\wedge T_n}$$ for some localising sequence of s.t. $T_n \xrightarrow{n\uparrow\infty} \infty$

with $(*)$ DCT $(n\uparrow\infty) \Rightarrow \mathbb{E}(M_t \mid \mathcal{F}_s) = M_s$

To prove ⊛

Recall $\overline{X}_{e_q} \sim \exp(\Phi(q))$

$$\mathbb{E}(\overline{X}_{e_q}) = \int_0^\infty q e^{-qt}\, \mathbb{E}(\overline{X}_t)\, dt = \frac{1}{\Phi(q)}$$

$$\Longrightarrow \quad \mathbb{E}(\overline{X}_t) < \infty \quad \text{a.e. on } (0,\infty)$$

but $\overline{X}_t$ is a cts monotone process

$$\Longrightarrow \quad \mathbb{E}(\overline{X}_t) < \infty \quad \forall\, t \geq 0 \quad (\ast\ast)$$

$$\mathbb{E}\Big(\sup_{s \leq t} |M_s^\lambda|\Big) \leq f(\lambda)t + 1 + 1 + \lambda\mathbb{E}(\overline{X}_t \vee a)$$

$< \infty$ using $(\ast\ast)$ ∎

Most of time we are interested in $x=0$, then we write $M_t^0 =: M_t$

Theorem Let $\underline{X}_t := \inf_{s \le t} X_s$ and e_q is an indep. & exp-distributed r.v. rate q. For all $\alpha > 0$

$$\mathbb{E}\left(e^{-\alpha(-\underline{X}_{e_q})}\right) = \frac{q(\alpha - \Phi(q))}{\Phi(q)(\psi(\alpha) - q)}$$

Note: When $\alpha = \Phi(q)$ we understand RHS in the limiting sense

Pf Let $Z_t = \overline{X}_t - X_t$

Using Fubini's Theorem

$$\mathbb{E}\int_0^{e_q} e^{-\alpha Z_s}\, ds = \mathbb{E}\int_0^\infty \mathbb{1}_{(s<e_q)} e^{-\alpha Z_s}\, ds$$

"naughty"

$$\underline{\overline{\quad}}\ \mathbb{E}\int_0^\infty e^{-qs} e^{-\alpha Z_s}\, ds$$

"Integrating out e_q"

$$= \int_0^\infty e^{-qs}\,\mathbb{E}\left(e^{-\alpha Z_s}\right) ds$$

Duality

$\overline{X}_t - X_t \overset{d}{=} -\underline{X}_t$

$$= \frac{1}{q}\int_0^\infty q e^{-qs}\,\mathbb{E}\left(e^{\alpha \underline{X}_s}\right) ds$$

$$= \frac{1}{q}\mathbb{E}\left(e^{\alpha \underline{X}_{e_q}}\right)$$

Now recall that M_t is a zero mean mg

$$0 = \mathbb{E}(M_{e_q}) = \psi(\alpha)\,\mathbb{E}\left(\int_0^{e_q} e^{-\alpha Z_s}\,ds\right) + 1 - \mathbb{E}\left(e^{-\alpha Z_{e_q}}\right) - \alpha\,\mathbb{E}(\bar{X}_{e_q})$$

$$\underset{\bar{X}_{e_q}\sim \exp(\Phi(q))}{\overset{\text{Duality}}{=}} \frac{\psi(\alpha)}{q}\,\mathbb{E}\left(e^{\alpha \underline{X}_{e_q}}\right) + 1 - \mathbb{E}\left(e^{\alpha \underline{X}_{e_q}}\right) - \frac{\alpha}{\Phi(q)}$$

$$0 = \frac{(\psi(\alpha) - q)}{q}\,\mathbb{E}\left(e^{\alpha \underline{X}_{e_q}}\right) - \frac{(\alpha - \Phi(q))}{\Phi(q)}$$

$\blacksquare$

Remark $\mathbb{E}\left(e^{-\alpha \overline{X}_{e_q}}\right) = \dfrac{\Phi(q)}{\Phi(q)+\alpha}$

because $\overline{X}_{e_q} \sim \exp(\Phi(q))$

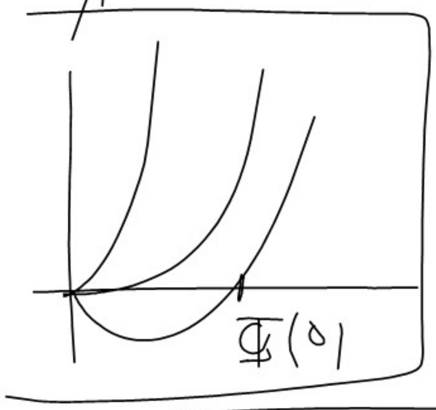
Lemma We have for any SNLP

(i) $\overline{X}_\infty$ (and $-\underline{X}_\infty$) are either ∞ a.s or $<\infty$ a.s.

(ii) $\overline{X}_\infty = \infty \iff \psi'(0^+) \left(= \mathbb{E}(X_1)\right) \geq 0$

(iii) $\underline{X}_\infty = -\infty \iff \psi'(0^+) \leq 0$

Pf/ Important in the proof is to understand



$$\lim_{q \downarrow 0} \frac{q}{\Phi(q)} = \begin{cases} \psi'(0^+) & \psi'(0^+) \geq 0 \quad \Phi(0) = 0 \\ 0 & \psi'(0^+) < 0 \quad \text{i.e. } \Phi(0) > 0 \end{cases}$$

$q = \psi(\Phi(q))$

$$\lim_{q \downarrow 0} \mathbb{E}\left(e^{\alpha \underline{X}_{e_q}}\right) = \lim_{q \downarrow 0} \frac{q}{\Phi(q)} \frac{(\alpha - \Phi(q))}{(\psi(\alpha) - q)}$$

$$\mathbb{E}\left(e^{\alpha \underline{X}_\infty}\right) = \begin{cases} \psi'(0^+) \dfrac{\alpha}{\psi(\alpha)} & \psi'(0^+) \geq 0 \\ 0 & \psi'(0^+) < 0 \end{cases}$$

Hence if $\psi'(0^+) < 0$ then $\mathbb{P}(-\underline{X}_\infty = \infty) = 1$

if $\psi'(0^+) > 0$ then $\mathbb{P}(-\underline{X}_\infty < \infty) = \lim_{\alpha \downarrow 0} \mathbb{E}\left(e^{-\alpha(-\underline{X}_\infty)}\right)$

$$= \psi'(0^+) \lim_{\alpha \downarrow 0} \frac{\alpha}{\psi(\alpha)} = 1$$

Similar anaysis with the identity $\mathbb{E}\left(e^{-\alpha \overline{X}_{e_q}}\right)$ finishes the pf.

i.e consider first $\lim_{q \downarrow 0}$ then $\lim_{\alpha \downarrow 0}$

$\blacksquare$

Final remark; this lemma is weaker than, but supports, the conclusion that

$$\limsup_{t\uparrow\infty} X_t = -\liminf_{t\uparrow\infty} X_t = \infty$$

$\psi'(0^+) = 0$

$$\lim_{t\uparrow\infty} X_t = \infty \quad \text{when } \psi'(0^+) > 0$$

$$\lim_{t\uparrow\infty} X_t = -\infty \quad \text{when } \psi'(0^+) < 0$$

Cannot happen!

$\psi'(0^+) < 0$