9 Continuous state Branching Processes

Beyanamé-Galton-Watson process

Markov chain $Z := \{ Z_n : n \geq 0 \}$ (Z_n = population at gen! n)

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_i^{(n)}$$

where for each $n = 1, 2, \ldots$ $\{ \xi_i^{(n)} : i \geq 1 \}$ are i.i.d (also independent across the n parameter) Take $\sum_{i=1}^{0} = 0$

and $\xi_i^{(n)}$'s are $\{0, 1, 2, 3, \ldots\}$-valued.

Note when $Z_n = 0$ then $Z_{n+k} = 0$ $\forall k = 1, 2, 3, \ldots$

The process Z exhibits the branching property: If $Z_0 = x + y$ $\quad (x, y \in \{0, 1, 2, 3, \dots\})$

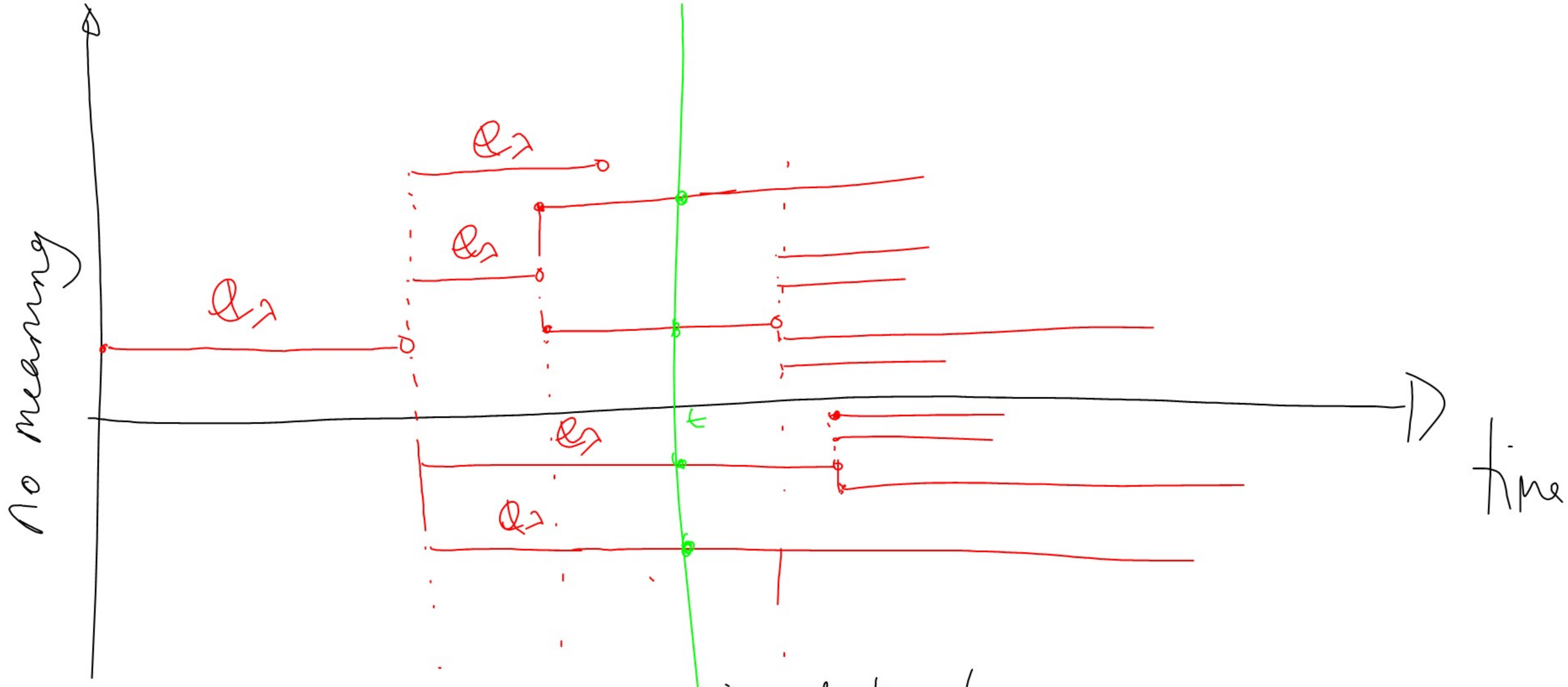
then $Z_n \overset{d}{=} Z_n^{(1)} + Z_n^{(2)}$

where $Z_n^{(1)}$ is a BGW started from $x = Z_0^{(1)}$

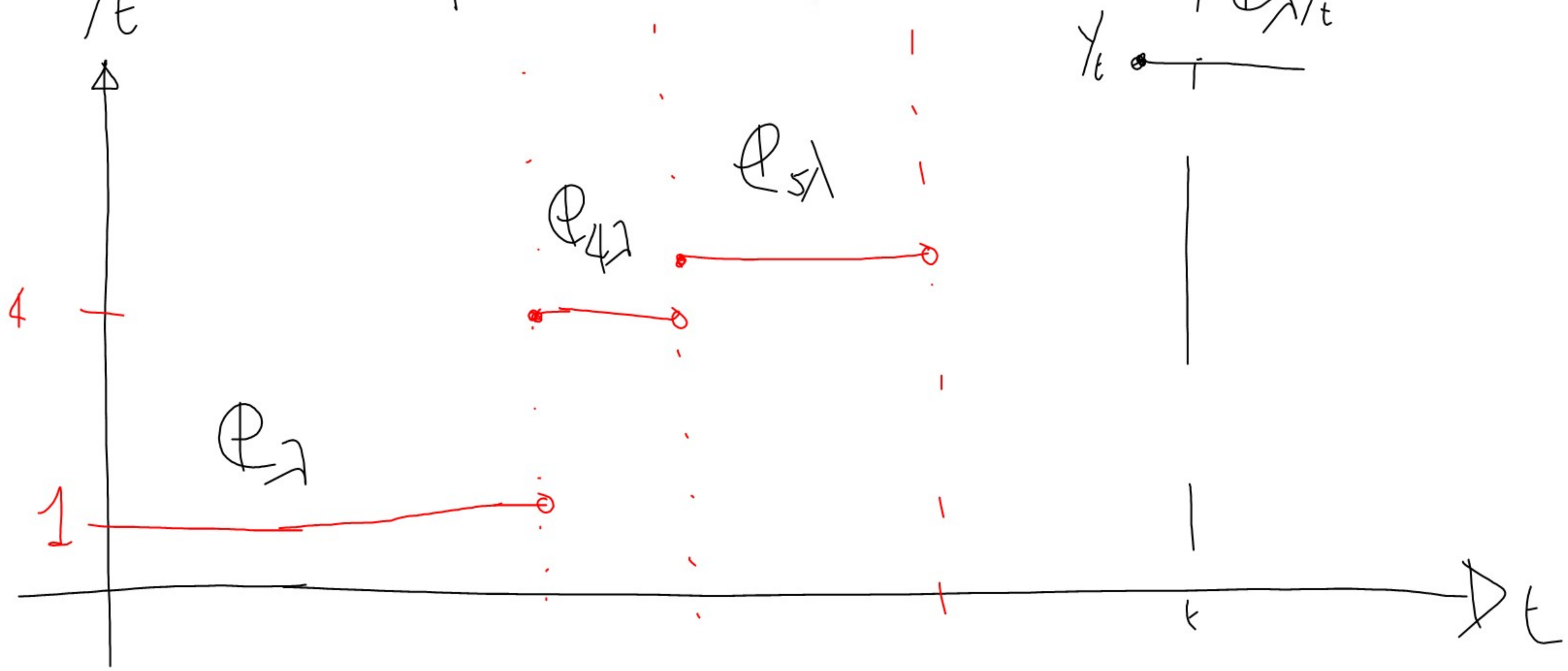
and $Z_n^{(2)}$ is a BGW$\left(\perp\!\!\!\perp Z_\cdot^{(1)}\right)$ started from $y = Z_0^{(2)}$

A ~~mild~~ modification of BGW process is to set it in cts time as follows: each individual reproduces not after 1-unit of time, but after a length of time which is $\exp(\lambda)$-distributed and independent of everything else.

Paths of the cts time BGW process.



Y_t number of individuals alive at time t



Thanks to lack of memory property

$$(0 \leq s \leq t) \quad Y_t = \sum_{i=1}^{Y_s} Y_{t-s}^{(i)}$$

where $\{ Y_{\cdot}^{(i)} : i = 1, \dots, Y_s \}$

(given $\{ Y_u : u \leq s \}$) are independent with the same law as $Y_{\cdot}$ initiated from $Y_0 = 1$.

Again note that 0 is an absorbing state:

if $Y_t = 0$ then $Y_{t+s} = 0 \quad \forall s \geq 0$

Y is called a cts time Markov BP

Branching property:

$\boxed{\forall t \geq 0 \text{ and } y_1, y_2 \geq 0}$, if P_y is the law of $Y_\cdot$ with $Y_0 = y$, then

$$P_{y_1+y_2} = P_{y_1} \circledast P_{y_2}$$

meaning that Y_t under $P_{y_1+y_2}$ is equal in dist^n to $Y_t^{(1)} + Y_t^{(2)}$ where $Y_\cdot^{(1)} \perp\!\!\!\perp Y_\cdot^{(2)}$ and $Y_\cdot^{(i)}$ has law P_{y_i} $i = 1, 2$.

Point of interest: Y_t is an i.d. r.v. for all t if we we know that we define Y as a $[0, \infty)$-valued process with the Branching property.

$$P_y = P_{\frac{y}{n}} \circledast P_{\frac{y}{n}} \circledast \cdots \circledast P_{y/n}$$

What is the connection of a MBP with Lévy processes?

All we need is that $\min_{i=1,\dots,n} e_\lambda^{(i)} \overset{d}{=} e_{n\lambda}$

and $\beta > 0$ then $\beta e_\lambda \overset{d}{=} e_{\lambda/\beta}$

Take MBP $\{Y_t : t \geq 0\}$ and look at transformation

$$J_t = \int_0^t Y_u \, du$$

$$\varphi_t := \inf\{s \geq 0 : J_s > t\}$$

$\varphi_{J_{T_1}} = \inf\{s : J_s = J_{T_1}\} = T_1$

J is cts and incr.

Now consider a new process $\boxed{X_t = Y_{\varphi_t}}$ where $X_t = 0$ when $\varphi_t = \infty$

claim: X is a CPP with intensity λ and jump distribution

$$F(dx) = \sum_{i=-1}^{\infty} \pi_i \delta_i(dx)$$

and $\pi_i = \mathbb{P}(\xi = i+1)$

Assume for convenience that $\pi_0 = 0$.

Suppose that $Y_0 = y \in \{1, 2, 3, \dots\}$

First jump of Y is denoted $T_1 \sim \exp(\lambda y)$

Note also that $J_{T_1} = y T_1$

and J_{T_1} is the first time that X jumps.

because $\varphi_{J_{T_1}} = T_1$

Note that $J_{T_1} \sim \exp(\lambda)$ $\left[\begin{array}{l} \text{using } \beta > 0 \\ \beta e_\lambda \overset{d}{=} e_{\lambda/\beta} \end{array} \right]$

Where does X jump to? The same place that Y jumps to

Y goes from $y \longmapsto y + \boxed{\xi - 1}$

i.e. $\Delta Y_{T_1} \overset{d}{=} \xi - 1$

hence jump dist^n of X at first jump is indeed F

Now use the SMP of Y to iterate this argument and deduce that X is a CPP — stopped when it hits zero!

because of the def^n $X_t = 0$ when $\varphi_t = \infty$!

~~Exa~~

Exercise.

The converse is also possible.

Take a (λ, F) CPP

same as before

write $$I_t = \int_0^t \frac{1}{X_u}\, du$$

$$\theta_t = \inf\{s \geq 0 : I_s > t\}$$

Define $Z_t = X_{\theta_t \wedge \tau_0^-}$ where $\tau_0^- = \inf\{t > 0 : X_t < 0\}$

Then Z is a CBP!