9 Continuous state Branching Processes

Begyama Galton-Watson Process

Markov chain $\mathcal{Z} := \{ Z_n : n \geq 0 \}$ (population at gen. $n$)

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_i$$

where for each $n=1,2,\ldots$, $\{ \xi_i : i \geq 1 \}$ are i.i.d.
(also independent across the $n$ parameter) Take $Z_{-1} = 0$
and $\xi_i$'s are $\{0,1,2,3,\ldots\}$-valued.

Note that if $Z_n = 0$ then $Z_{n+k} = 0 \ \forall k = 1,2,3,\ldots$
The process $Z$ exhibits the branching property: If $Z_0 = x + y$ \((x, y \in \{0, 1, 2, 3, -1\})\), then

$$Z_n = d \cdot Z_n^{(1)} + Z_n^{(2)}$$

where $Z_n^{(1)}$ is a BGW started from $x = Z_0$

and $Z_n^{(2)}$ is a BGW ($Z_n^{(1)}$) started from $y = Z_0$

A mild modification of BGW process is to set it in

its time as follows: Each individual reproduces not after

1-unit of time, but after a length of time which is exp(1) -

distributed and independent of everything else.
Paths of the DB time BEW process,

No meaning

number of individuals alive at time $t$

$Y_t$
Thanks to lack of memory property

\[ Y_t = \sum_{i=1}^{Y_s} Y_{t+s} \]

(\(s \leq s \leq t\))

Where \(\{Y_{t+s} : i=1, \ldots, Y_s\}\)

(given \(\{Y_n : n \leq s\}\)) are independent with the same

distribution as \(Y_0\) initiated from \(Y_0 = 1\).

Again note that 0 is an absorbing state:

if \(Y_t = 0\) then \(Y_{t+s} = 0\) \(\forall s \geq 0\)

\(Y_0\) is called a 03 time Markov BP
Barycentric property: 

\[ \forall t \geq 0 \text{ and } y_1, y_2 \geq 0 \text{ if } P_y \text{ is the law of } Y \text{ and } y = y_1 + y_2, \text{ then } \]

\[ P_{y_1 + y_2} = P_{y_1} \otimes P_{y_2} \]

meaning that \( Y_t \) under \( P_{y_1 + y_2} \) is equal in distribution to \( Y^{(1)}_t + Y^{(2)}_t \) when \( Y^{(1)} = Y^{(2)} \), and \( Y^{(i)} \) has law \( P_{y_i} \), \( i = 1, 2 \). 

Point of interest: \( Y_t \) is an i.i.d. r.v. for all \( t \) if we define \( Y \) as a \([0, \infty)\)-valued process with the barycentric property:

\[ P_y = P_{y_1} \otimes P_{y_2} \otimes \cdots \otimes P_{y_n} \]
What is the connection of a MBP with Levy processes?

All we need so that \( \min_{i=1, ..., n} \mathbb{E}^{(i)} = \beta \mathbb{E}^{(n)} \)
and \( \beta > 0 \) then \( \beta \mathbb{E}^{(n)} = \mathbb{E}^{(n)} \).

Take MBP \( \{Y_t : t \geq 0\} \) and look at transformation
\[
J_t = \int_0^t Y_u \, du
\]
\( J_t = \inf \{ s \geq 0 : J_s > t \} \).

Now consider new process \( \{X_t = Y_{J_t}, t \geq 0\} \).

Claim: \( X \) is a CPP with intensity \( \lambda \) and
jump distribution
\[
F(dx) = \sum_{i=-1}^{\infty} \pi_i \delta_i(dx)
\]
and \( \pi_i = \mathbb{P}(\xi = i+1) \).
Assume for convenience that \( \pi_0 = 0 \).
Suppose that $Y_0 = y \in \{1, 2, 3, \ldots \}$

First jump of $Y$ is denoted $T_1 \sim \exp(2y)$

Note also that $\bar{T}_{1-} = yT_1$

and $\bar{T}_{1-}$ is the first time that $X$ jumps.

because $P(\bar{T}_{1-} = T_1)$

Note that $\bar{T}_{1-} \sim \exp(2y)$

then does $X$ jump to? The same place that $Y$ jumps to

$Y$ goes from

$Y \rightarrow Y + \lfloor 2 - 1 \rfloor$

i.e. $\Delta Y_{T_1} = 2 - 1$

hence jump drift of $X$ at first jump is indeed $F$.

Now use the SMP of $Y$ to iterate this argument

and deduce that $X$ is a CPP -- stopped when it hits zero!

because of the def: $X_t = 0$ when $\xi_t = \infty$
Exercise

The converse is also possible:

Take a $(\mathcal{F}, \mathbb{P})$ CPT

such as before

\[ I_t = \int_0^t \frac{1}{X_u} \, du \]

\[ \Theta_t = \inf \{ s \geq 0 : I_s > t \} \]

Define \( Z_t = X_{\Theta_t \wedge T_0} \) where

\( T_0 = \inf \{ t > 0 : X_t \leq 0 \} \)

Then \( Z_t \) is a MBP!