10 Lamperti transformation

Def^n A $[0,\infty)$-valued strong Markov process $Y = \{Y_t : t \geq 0\}$ with probs $\{\mathbb{P}_x : x \geq 0\}$ is called a cts-state branching process if its paths are a.s. right cts with left limits and observes the branching property as given in last lecture except $y_1, y_2 \in [0,\infty)$

Another way of phrasing the br. property:

$\forall \theta \geq 0$ $\quad E_{x+y}\left[e^{-\theta Y_t}\right]$

$x, y \geq 0$

$$= E_x\left(e^{-\theta Y_t}\right) E_y\left(e^{-\theta Y_t}\right) (\dagger)$$

In particular $E_x\left(e^{-\theta Y_t}\right) = E_{x/n}\left(e^{-\theta Y_t}\right)^n$ $\circledast$

(and so Y_t is an i.d. r.v!)

Introduce $g(t, \theta, x) = -\log E_x\left(e^{-\theta Y_t}\right)$

$\circledast \implies$ for any integer $m > 0$ $\qquad m = n \times \frac{m}{n}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = m \times 1$

on one hand $g(t, \theta, m) = n\, g(t, \theta, m/n)$

on other hand $g(t, \theta, m) = m\, g(t, \theta, 1)$

$$\implies g(t, \theta, x) = x \underbrace{g(t, \theta, 1)}_{\substack{\shortparallel \\ u_t(\theta)}} \quad \text{for all } x \in \mathbb{Q}_{>0}$$

By (†) we see that

$$E_{x+y}\left(e^{-\theta Y_t}\right) \leq E_x\left(e^{-\theta Y_t}\right)$$

$$\implies \quad g(t,\theta,x) \leq g(t,\theta,x+y)$$

$$\implies \quad g(t,\theta,x-) \leq g(t,\theta,x+) \leftarrow \text{limits defined by monotonicity}$$

$y_n \uparrow x$

now suppose $x_n \downarrow x$ $\quad x_n \in \mathbb{Q}_{>0}$

$$\begin{tikzcd}
e^{-g(t,\theta,\underset{y_n}{x_n})} \arrow[d, "x_n\downarrow x", "y_n\uparrow x"'] & = E_{\underset{y_n}{x_n}}\left(e^{-\theta Y_t}\right) = e^{-\underset{y_n}{x_n} u_t(\theta)} \arrow[r, "x_n\downarrow x", "y_n\uparrow x"'] & e^{-x u_t(\theta)} \\
e^{-g(t,\theta,\underset{x-}{x+})} \arrow[rr, dash, bend right] & &
\end{tikzcd}$$

$$\implies g(t,\theta,x) = x u_t(\theta) \quad \forall x > 0$$

We now can write $E_x(e^{-\theta Y_t}) = e^{-x u_t(\theta)}$

$\forall x \geqslant 0$
$t \geqslant 0$
$\theta \geqslant 0$

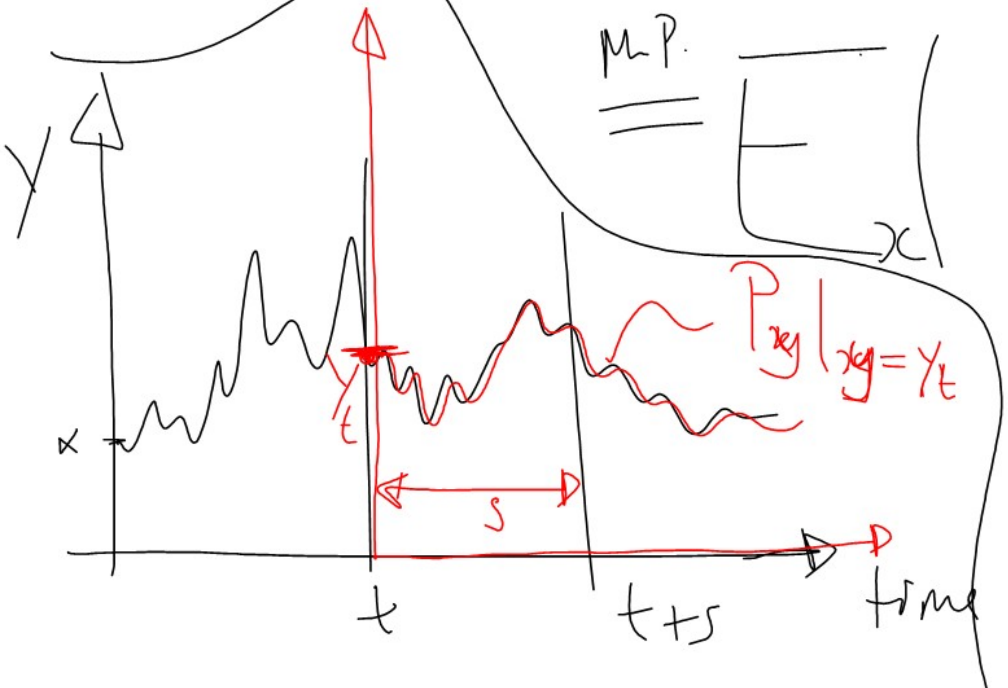
Use the Markov property

$$e^{-x u_{t+s}(\theta)} = E_x\left(e^{-\theta Y_{t+s}}\right) = E_x\left(E\left(e^{-\theta Y_{t+s}} \middle| Y_t\right)\right)$$

$$\overset{M.P.}{=} E_x\left(E_{y}\left(e^{-\theta Y_s}\right)\Big|_{y=Y_t} \right)$$

$$= E_x\left(e^{-Y_t u_s(\theta)}\right)$$

$$e^{-u_t(u_s(\theta))x}$$



Y is a markov Process

$$E_x\left(g(Y_{t+s}) \middle| \{Y_u : u \le t\}\right)$$

$$= E_x\left(g(Y_{t+s}) \middle| Y_t\right)$$

$$= E_y\left(g(Y_s)\right)\Big|_{y=Y_t} \quad \forall g$$

cts and bounded

$$\underbrace{u_{t+s}(\theta) = u_t(u_s(\theta))}_{\text{semi group property of Laplace exponent.}} \quad \forall\, t,s \geq 0, \ \theta \geq 0$$

Strongly suggests we should try and look at

$$"\frac{\partial u_t(\theta)}{\partial t}" = "\lim_{\Delta \downarrow 0} \frac{u_{t+\Delta}(\theta) - u_t(\theta)}{\Delta}$$

Theorem For $t, \theta \geq 0$, suppose $u_t(\theta)$ is the Laplace exponent of a CSBP. Then it is diff'able in t and satisfies

$$\frac{\partial u_t(\theta)}{\partial t} + \psi(u_t(\theta)) = 0 , \quad u_0(\theta) = \theta$$

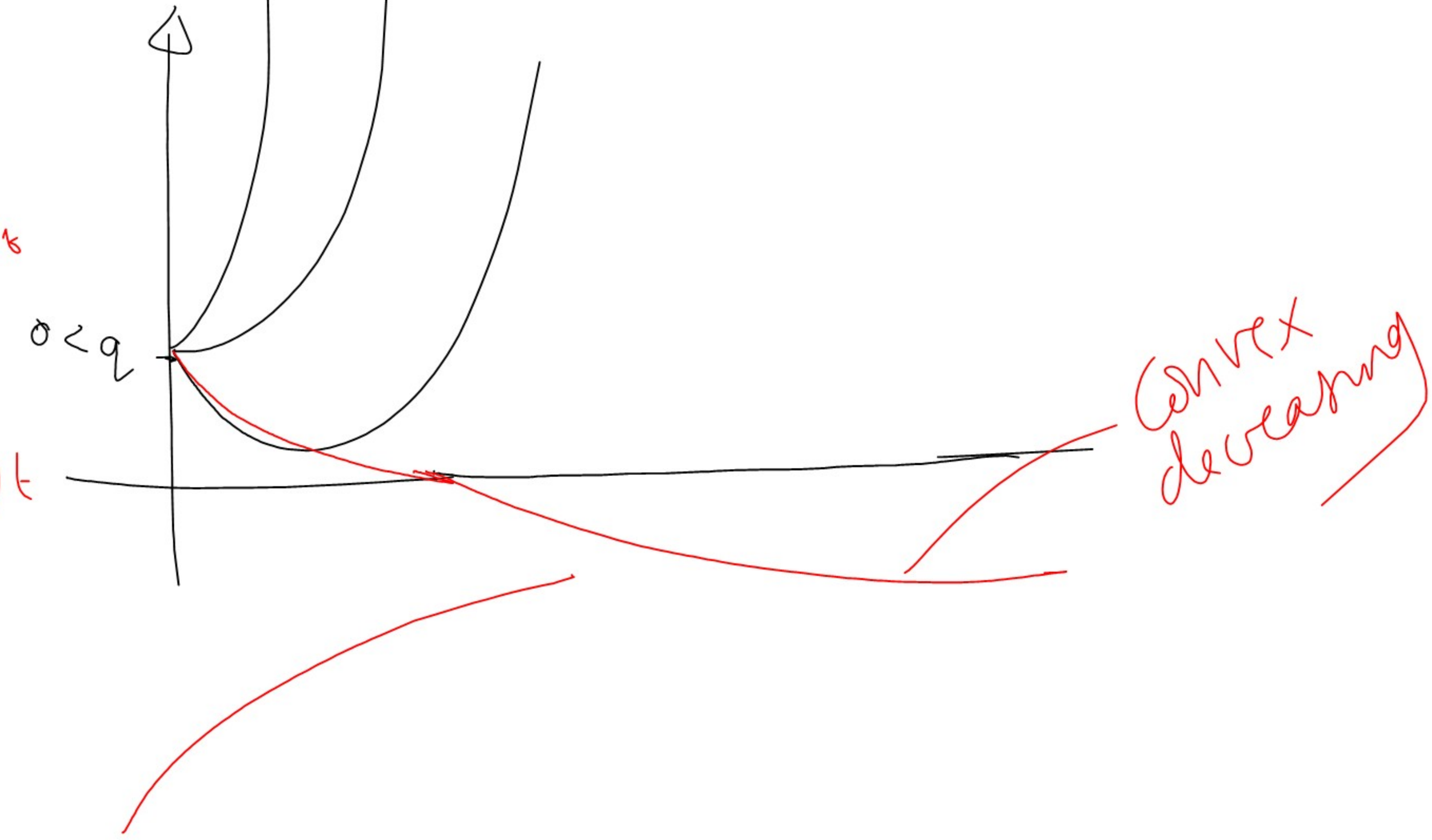
Where for $\lambda \geq 0$

$$\psi(\lambda) = -q - a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x \mathbb{1}_{(x<1)}\right)\Pi(dx)$$

where $q \geq 0$, $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and Π is a measure concentrated in $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge x^2)\Pi(dx) < \infty$

Note that the quantity ψ can be recognised as $\psi(\lambda) = \log \mathbb{E}(e^{-\lambda X_1})$ where X is a spectrally positive Lévy process under $\mathbb{P}$ OR a subordinator, killed independently at rate $q \geq 0$.

Indeed ψ is the Laplace exp. of a SNLP or a -ve sub. (killed at rate q) and so we have the picture we should keep in mind



Sub

$\mathbb{E}_{\cancel{x}}(e^{-\lambda Z_t}) = e^{-\Phi(\lambda)t}$

SNLP

$\mathbb{E}_x(e^{+\lambda X_t}) = e^{\psi(\lambda)t}$