$$\frac{\partial u_t}{\partial t}(\theta) + \psi(u_t(\theta)) = 0 \qquad \circledast$$

where $\mathbb{E}_x\left(e^{-\theta Y_t}\right) = e^{-x u_t(\theta)}$

Note that $\circledast$ can be re-written or "solved"

$$-\int_\theta^{u_t(\theta)} \frac{1}{\psi(\xi)}\, d\xi = t$$

Theorem Given a general branching mechanism ψ [where ψ was the general expression appearing in the previous theorem]

(1) Suppose $X = \{X_t, t \geq 0\}$ is a LP with no negative jumps [NNJ = {SUB} ∪ {SPLP}] with $X_0 = x$ killed at an independent exp. dist'd time e_q (rate $q \geq 0$), such that $\psi(\lambda) = \log \mathbb{E}_x\left(e^{-\lambda(X_1 - x)}\right)$. Define for $t \geq 0$

$$Y_t := X_{\theta_t \wedge \tau_0^-}$$

where $\tau_0^- = \inf\{t > 0 : X_t < 0\}$

and $\theta_t = \inf\left\{s > 0 : \int_0^s \frac{du}{X_u} > t\right\}$

Then $Y = \{Y_t : t \geq 0\}$ is a CSBP (initiated from $Y_0 = x$)

(ii) Conversely suppose that $Y = \{ Y_t : t \geq 0 \}$ is a CSBP with branching mech. ψ

[i.e. when is look at the PDE of its Lap exp I see ψ in the PDE]. Define

$$X_t := Y_{\varphi_t}$$

where $\varphi_t = \inf \left\{ s > 0 : \int_0^s Y_u \, du > t \right\}$

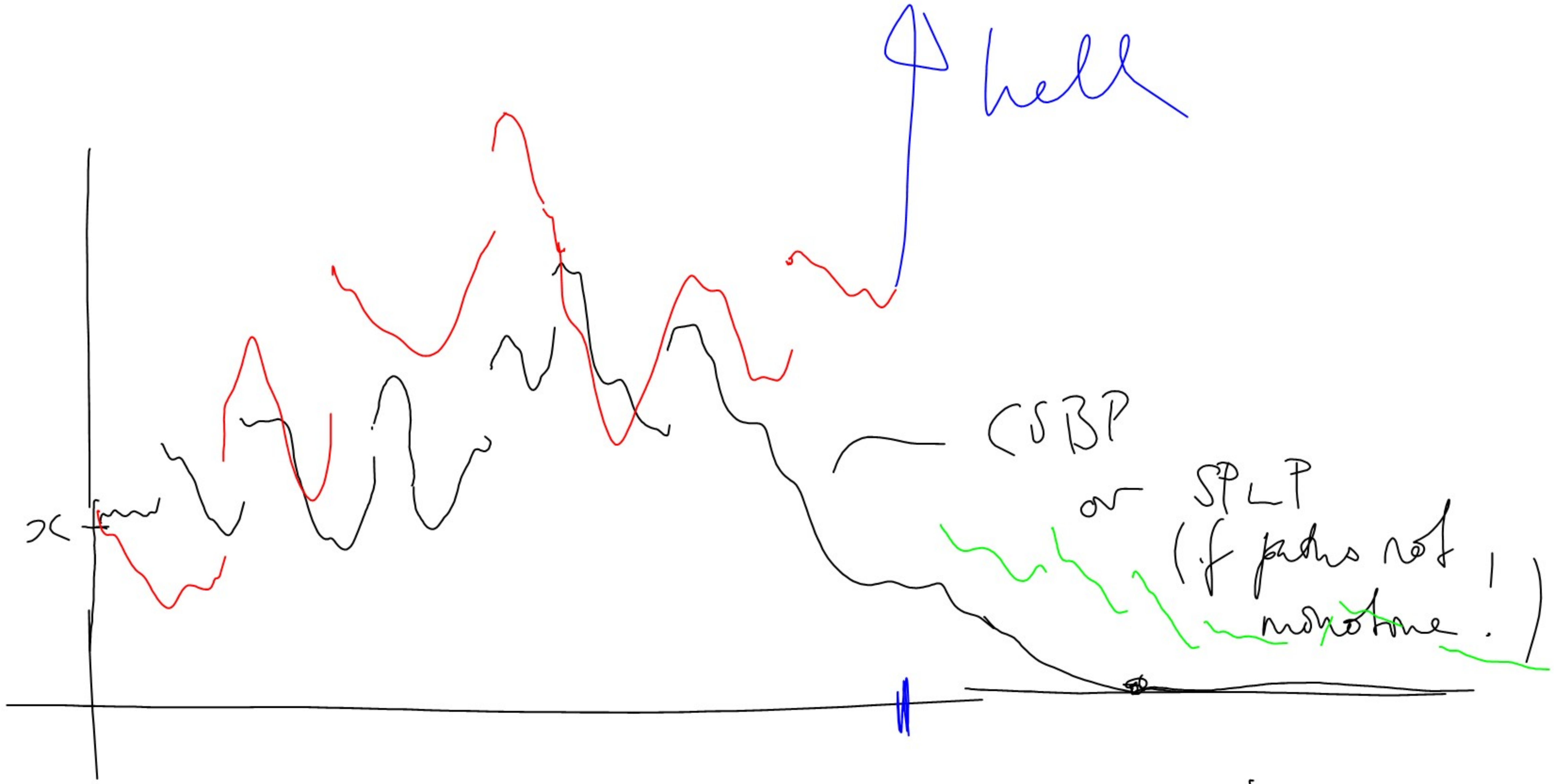
Then $X = \{ X_t : t \geq 0 \}$ is a Levy process with nonnegative jumps killed at the minimum of

(a) first entry of X into $(-\infty, 0)$

(b) an independent exp-dist'd r.v. rate $q \geq 0$

Moreover $\psi(\lambda)$ is the Laplace exponent of $-X$.

$= \log \mathbb{E}_x(e^{-\lambda(X_1 - x)})$ where $x = Y_0$.



Interested in (for now!) four things:

Explosion, Extinction, $\sup_{s\le\infty} Y_s$, $\underbrace{\int_0^{\zeta} Y_u\,du}_{\text{total progeny}}$

(ζ = extinction time)

BGW analogue: $\sum_{k\geq 0} Z_k$

11.1 Explosion (vs Conservative) behavior

A CSBP Y is said to be <u>conservative</u> if $\forall t > 0 \quad P(Y_t < \infty) = 1$.

Theorem A CSBP with br. mechanism ψ is conservative iff

$$\int_{0+} \frac{1}{|\psi(\xi)|} d\xi = \infty$$

i.e. $\exists \varepsilon_0$ s.t $\forall \varepsilon < \varepsilon_0$ $\int_0^{\varepsilon} \frac{1}{|\psi(\xi)|} d\xi = \infty$

Remark: A necessary condition for conservativeness is thus $(q=)\psi(0) = 0$

Because if $\psi(0)(=q) > 0$ then $\frac{1}{|\psi(\xi)|} \underset{\xi \downarrow 0}{\sim} O\left(\frac{1}{q}\right)$ which is integrable.

A sufficient condition is $\psi(0)(=q)=0$ and $E|X_1| < \infty$ ($|\psi'(0)| < \infty$): Because $\frac{1}{\psi(\xi)} \overset{\xi \downarrow 0}{\sim} \frac{1}{|\psi'(0)|\xi}$

Pf/ Recall $E_x(e^{-\theta Y_t}) = e^{-x u_t(\theta)}$

This means that $P_x(Y_t < \infty) \left(= \lim_{\theta \downarrow 0} E_x(e^{-\theta Y_t})\right)$

$$= e^{-x u_t(0+)} \quad \text{where } u_t(0+) := \lim_{\theta \downarrow 0} u_t(\theta)$$

[We are fixing paths so that if $Y_t = 0$ then $Y_{t+s} = 0 \ \forall s \geq 0$ i.e. 0 is the "extinction state"]

Recall also

$$t = \int_{u_t(\theta)}^{\theta} \frac{1}{\psi(\xi)} \, d\xi$$

hence

$$t = \boxed{- \int_{\theta}^{\delta} \frac{1}{\psi(\xi)} \, d\xi \quad 1.} + \boxed{\int_{u_t(\theta)}^{\delta} \frac{1}{\psi(\xi)} \, d\xi \quad \text{for } \delta > 0 \quad 2.}$$

Note LHS is indep of θ so $\boxed{1.}$ "explodes" $\Leftrightarrow$ $\boxed{2.}$ explodes. i.e.

$$u_t(\theta) \xrightarrow{\theta \downarrow 0} 0 \iff \int_{0^+} \frac{1}{|\psi(\xi)|} \, d\xi = \infty$$

In conclusion from the top

$$P_x(Y_t < \infty) = 1 \iff u_t(0+) = 0 \iff \int_{0^+} \frac{1}{|\psi(\xi)|} \, d\xi = \infty$$

∎