Conservative \[ \Rightarrow \int_0^\infty \frac{1}{U(3)} \, d3 = \infty \]

\[ \forall \theta \quad U_t(\theta) \xrightarrow{\theta \to 0} 0 \]

Assume henceforth that \( \star \) in force.

**Extinction Probabilities**

Recall that \( S = \inf \{ t > 0 : Y_t = 0 \} \) and if \( \{ Y_t = 0 \} \)

Then \( \{ Y_{t+s} = 0 \} \) \( \forall s > 0 \)

Want to study \( P_x(S < \infty) \)
\[ E_x(e^{-\Theta Y_t}) = e^{-x \mu_t(\Theta)} \]

Differentiating in $\Theta$ we see

\[ E_x(e^{-\Theta Y_t} Y_t) = e^{-x \mu_t(\Theta)} \mu_t(\Theta) \]

Take limit as $\Theta \to 0$

**MCT:** \[ E_x(Y_t) = \left( \frac{\mu_t(0^+)}{\mu_t(0)} \right) = \lim_{\Theta \to 0^+} \frac{\mu_t(\Theta)}{\Theta} \]

Both sides are understood to be $\infty$ simultaneously.

\[ \frac{\mu_t(\Theta)}{\Theta} + \psi(\mu_t(\Theta)) = 0 \quad \text{diff in } \Theta \]

\[ \frac{d}{dt} \left( \frac{\mu_t(\Theta)}{\Theta} \right) + \psi'(\mu_t(\Theta)) \left( \frac{\mu_t(\Theta)}{\Theta} \right) = 0 \]

Solving using standard techniques from 1st order ODEs:

\[ \frac{\mu_t(\Theta)}{\Theta} = ce^{-\int \theta \psi'(u_s(\Theta)) \, ds} \]

$c$ is an unknown constant.

Taking $\Theta = 0$ in $\Box$, we have $\mu_0(\Theta) = 0$ tells us

\[ \frac{\mu_t(\Theta)}{\Theta} \xrightarrow{\Theta \to 0} 1 \]

By inspection in $\Box$ implies $c = 1$.
\[
E_x(Y_t) = x \frac{\partial u_t(0^+)}{\partial x}
\]
and taking \( \theta \downarrow 0 \)

\[
\frac{\partial u_t(0^+)}{\partial x} = \int_0^{\theta} f'(u_s(0)) \, ds
\]

\[
\frac{\partial u_t(0^+)}{\partial x} = e^{-\theta} - \theta f(0)
\]

If \( |f'(0^+)| < \infty \) then DCT and tend \( u_s(0) \theta \to 0 \)

to deduce that

\[
\frac{\partial u_t(0^+)}{\partial x} = e^{-\theta} - \theta f(0)
\]

and hence

\[
E_x(Y_t) = x e^{-\theta}
\]

(1)

If \( |f'(0^+)| = \infty \) by monotonicity and tend \( u_s(0) \theta \to 0 \)

\[
\frac{\partial u_t(0^+)}{\partial x} = e^{-\theta} - \theta f(0) = \infty
\]

and (1) will hold but RHBS (and hence LTV)

\[
= \infty
\]
Induces the following def:

A CSBP with branching rate $\gamma$ is called

1. Subcritical if $\gamma'(0^+) > 0$
2. Critical if $\gamma'(0^+) = 0$
3. Supercritical if $\gamma'(0^+) < 0$

Theorem  Suppose that $X$ is a CSBP with br. mech. $\gamma$. Let $P(x) = P_x(\tau < \infty)$. [subordinator case]

1. If $\gamma(\infty) < 0$ then $\forall x > 0 \quad P(x) = 0$
2. $\gamma(\infty) = \infty$ then $P(x) > 0$ for some (and then all) $x > 0$

[SPP case]

\[ \int_{\gamma(0)}^{\infty} \frac{1}{\lambda^3} d\lambda < \infty \]

which case $P(x) = e^{-\Phi(0)x}$ where

$\Phi(0) = \sup \{ \lambda \geq 0 : \gamma(\lambda) = 0 \}$. 

\[ \Phi(0) \]
\[ \text{Pf (ii) } \quad P_x (Y_t = 0) = P_x (3 \leq t) \]
\[ P_x ( Y_t = 0 ) \quad \frac{1}{t+1} P_x (3 < \infty) \]

\[ E_x (e^{-\lambda Y_t}) = e^{-\lambda u_t(0)} \]
\[ \Rightarrow \quad P_x (Y_t = 0) = e^{-\lambda u_t(\infty)} \quad \text{where } \quad u_t(\infty) = \ln \frac{u_t(0)}{t+1} \]

Recall that
\[ \int_0^1 \frac{1}{u_t(0)} \, d\bar{t} = t \quad (\Delta) \]

If \( u_t(\infty) < \infty \) then since RHS of (\( \Delta \))

1. independent of \( \theta \) then (taking limit in (\( \Delta \)))
\[ \int_0^\infty \frac{1}{t+1} \, d\bar{t} < \infty \]

Conversely, if the above holds \( u_t(\infty) < \infty \) \( (\theta/1, \theta = t) \)
We now know that if \( \int_0^\infty \frac{1}{u(t)} \, dt < \infty \)

then \( \int_0^\infty \frac{1}{u(t)} \, dt = -\frac{1}{x} \log P(3 \leq t) \)

as \( t \to \infty \), \( u(t) \) must decrease to a root of \( u \) (because RHS blows up so LHS must blow up -- don't forget the shape of \( u ! \))

In fact the largest root should be \( \Phi(0) \)

In conclusion \( \lim_{t \to \infty} P_x(3 \leq t) = e^{-x \ln u(t)} \)

\[ = e^{-\Phi(0)x} \]

\[ = e^{-\Phi(0)x} \]

\[ \int_0^\infty \frac{1}{u(t)} \, dt = \infty \text{ then } u(t) = \infty \]

\[ \Rightarrow P_x(3 < \infty) = \lim_{t \to \infty} P_x(3 \leq t) = 0 \]
The conclusion of last form tells us

\[ \end{array} \]

\[ \begin{array}{ll}
\text{Condition} & \int_0^\infty \frac{f(x)}{x \Gamma(x)} dx = 0 \\
4(\infty) < 0, & \int_0^\infty \frac{4(x)}{x \Gamma(x)} dx = \infty \\
4'(0^+) < 0, & e^{-\int_0^\infty \frac{4(x)}{x \Gamma(x)} dx} < 1 - e^{-\int_0^\infty \frac{4(x)}{x \Gamma(x)} dx}
\end{array} \]
From the case \[ f(\infty) = \infty \] \[ \int_{\text{some interval}} \frac{1}{f(x)} \, dx = \infty \]

AND

\( f'(0^+) > 0 \) \( \Rightarrow \) subcritical

apparently

\[ p(x) = 0 \quad \forall x > 0 \]

\[ E_x(\gamma_t) = x e^{-\gamma/(\gamma^+)} t \]
\[ J_S = \int_0^\infty Y_u \, du \]

Recall that when \( 4(\omega) = \infty \) (i.e., \( \mathcal{P}L \) case)
\[ \Phi(q) := \sup \left\{ \tau > 0 : 4(\tau) = q \right\} \]

**Lemma**

\( Y \) is a \( \mathcal{B}P \), \( \mathcal{B}F \), mean 4, s.t. \( 4(r) = \infty \)

Then (\( x \geq 0 \))
\[ \mathbb{E}_x (e^{-q \int_0^\infty Y_s \, ds}) = e^{-\overline{\Phi}(q)x} \]
From the Lamperti transformation

\[
T_0^- = \inf \{ t > 0 : X_t < 0 \}
\]

Then \( T_0^- = \int_0^\infty Y_u \, du \)

\[
E_{-X} \left( e^{-\int_0^{T_0^-} Y_u \, du} \right) = E_0 \left( e^{-\int_0^{T_0^+} X_u \, du} \right)
\]

\[
T_0^+(-X) = \inf \{ t > 0 : -X > x \}
\]

from previously