conservative $\not\Longleftrightarrow$ $\displaystyle\int_{0^+} \frac{1}{|\psi(\xi)|}\, d\xi = \infty$ (*)

$\Updownarrow$

$\forall t \quad u_t(\theta) \xrightarrow[\theta \downarrow 0]{} 0$

Assume henceforth that (*) in force.

Extinction probabilities

Recall that $\zeta = \inf\{t > 0 : Y_t = 0\}$ and if $\{Y_t = 0\}$ then $\{Y_{t+s} = 0\}$ $\forall s \geq 0$

want to study $P_x(\zeta < \infty)$

$$E_x\left(e^{-\theta Y_t}\right) = e^{-x u_t(\theta)}$$

Differentiating in θ we see

$$E_x\left(e^{-\theta Y_t} Y_t\right) = e^{-x u_t(\theta)} x \frac{\partial u_t}{\partial \theta}(\theta) \quad (**)$$

take limits as $\theta \downarrow 0$

MCT: $$E_x(Y_t) = x \underbrace{\frac{\partial u_t}{\partial \theta}(0^+)}_{:= \lim_{\theta \downarrow 0} \frac{\partial u_t}{\partial \theta}(\theta)} \quad (***)$$

both sides are understood to be ∞ simultaneously.

$$\frac{\partial u_t(\theta)}{\partial t} + \psi(u_t(\theta)) = 0 \quad \text{diff in } \theta$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u_t}{\partial \theta}(\theta)\right) + \psi'(u_t(\theta))\left(\frac{\partial u_t}{\partial \theta}(\theta)\right) = 0$$

Solving using standard techniques from 1st order ODEs.

$$\frac{\partial u_t}{\partial \theta}(\theta) = c e^{-\int_0^t \psi'(u_s(\theta))\, ds} \quad (*)$$

c is an unknown constant.

Taking $t \downarrow 0$ in $(**)$ using $u_0(\theta) = \theta$ follows that $\frac{\partial u_t}{\partial \theta}(\theta) \xrightarrow{t \downarrow 0} 1$

By inspection in $(*)$ $\Longrightarrow c = 1$

from $\cancel{\ast\ast\ast}$

$$E_x(Y_t) = x \frac{\partial u_t}{\partial \theta}(0^+)$$

and $\ast$ taking $\theta \downarrow 0$

$$\frac{\partial u_t}{\partial \theta}(0^+) = e^{-\lim_{\theta \downarrow 0} \int_0^t \psi'(u_s(\theta))\,ds}$$

if $|\psi'(0^+)| < \infty$ then DCT and that $u_s(\theta) \xrightarrow{\theta \downarrow 0} 0$

to deduce that $\frac{\partial u_t}{\partial \theta}(0^+) = e^{-\psi'(0^+)t}$

and hence $\boxed{E_x(Y_t) = x e^{-\psi'(0^+)t}}$ $(\dagger)$

if $|\psi'(0+)| = \infty$ by monotonicity and $u_s(\theta) \xrightarrow{\theta \downarrow 0} 0$

$$\frac{\partial u_t}{\partial \theta}(0^+) = e^{-\psi'(0^+)t} = \infty$$

and $(\dagger)$ still holds but RHS (and hence LHS) $= \infty$

Induces the following def^n:

A CSBP with branching mech ψ is called

(1) subcritical if $\psi'(0+) > 0$

(2) critical if $\psi'(0+) = 0$

(3) supercritical if $\psi'(0+) < 0$

Theorem Suppose that Y is a CSBP with br. mech. ψ. Let $p(x) = P_x(\zeta < \infty)$.

[subordinator case]

(i) If $\psi(\infty) < \infty$ then $\forall x > 0$ $p(x) = 0$ obvious!

(ii) If $\psi(\infty) = \infty$ then $p(x) > 0$ for some (and then all) $x > 0$

[SPLP case]

$\Longleftrightarrow$

$$\int^{\infty} \frac{1}{\psi(\xi)} d\xi < \infty$$

in which case $p(x) = e^{-\Phi(0)x}$ where

$$\Phi(0) = \sup\{\lambda \geq 0 : \psi(\lambda) = 0\}$$

Pf (ii) $P_x\{Y_t = 0) = P_x(\zeta \leq t)$

$$P_x(Y_t = 0) \nearrow_{t \uparrow \infty} P_x(\zeta < \infty)$$

$$E_x(e^{-\theta Y_t}) = e^{-x u_t(\theta)}$$

$$\Rightarrow P_x(Y_t = 0) = e^{-x u_t(\infty)}$$ where $u_t(\infty) := \lim_{\theta \uparrow \infty} u_t(\theta)$

Recall that $\displaystyle\int_{u_t(\theta)}^{\theta} \frac{1}{\psi(\xi)} d\xi = t \qquad (\Delta)$

If $u_t(\infty) < \infty$ then since RHS of (Δ) is indep. of θ then (taking limits in (Δ) as $\theta \uparrow \infty$)

$$\int^{\infty} \frac{1}{\psi(\xi)} d\xi < \infty$$

Conversely, if the above holds $u_t(\infty) < \infty$ (o/w $0 = t$!)

We now know that if $\int^{\infty} \frac{1}{\psi(x)} < \infty$

then
$$\int_{u_t(\infty)}^{\infty} \frac{1}{\psi(\xi)} d\xi = t$$

$u_t(\infty) = -\frac{1}{x} \log P_x(\zeta \leq t)$

obviously $u_t(\infty) \downarrow$ as $t \uparrow \infty$

as $t \uparrow \infty$, $\boxed{u_t(\infty) \text{ must decrease}}$ to a root of ψ

(because RHS blows up so LHS must blow up — don't forget the shape of ψ!)

In fact the largest root which is $\Phi(0)$.

In conclusion
$$\lim_{t \uparrow \infty} P_x(\zeta \leq t) = e^{-x \lim_{t \uparrow \infty} u_t(\infty)} = e^{-\Phi(0)x}$$
$$\| $$
$$P_x(\zeta < \infty)$$

If $\int^{\infty} \frac{1}{\psi(\xi)} d\xi = \infty$ then $u_t(\infty) = \infty$

$$\Rightarrow P_x(\zeta < \infty) = \lim_{t \uparrow \infty} \boxed{P_x(\zeta \leq t) = 0}$$

$\blacksquare$

The conclusion of last Thm tells us

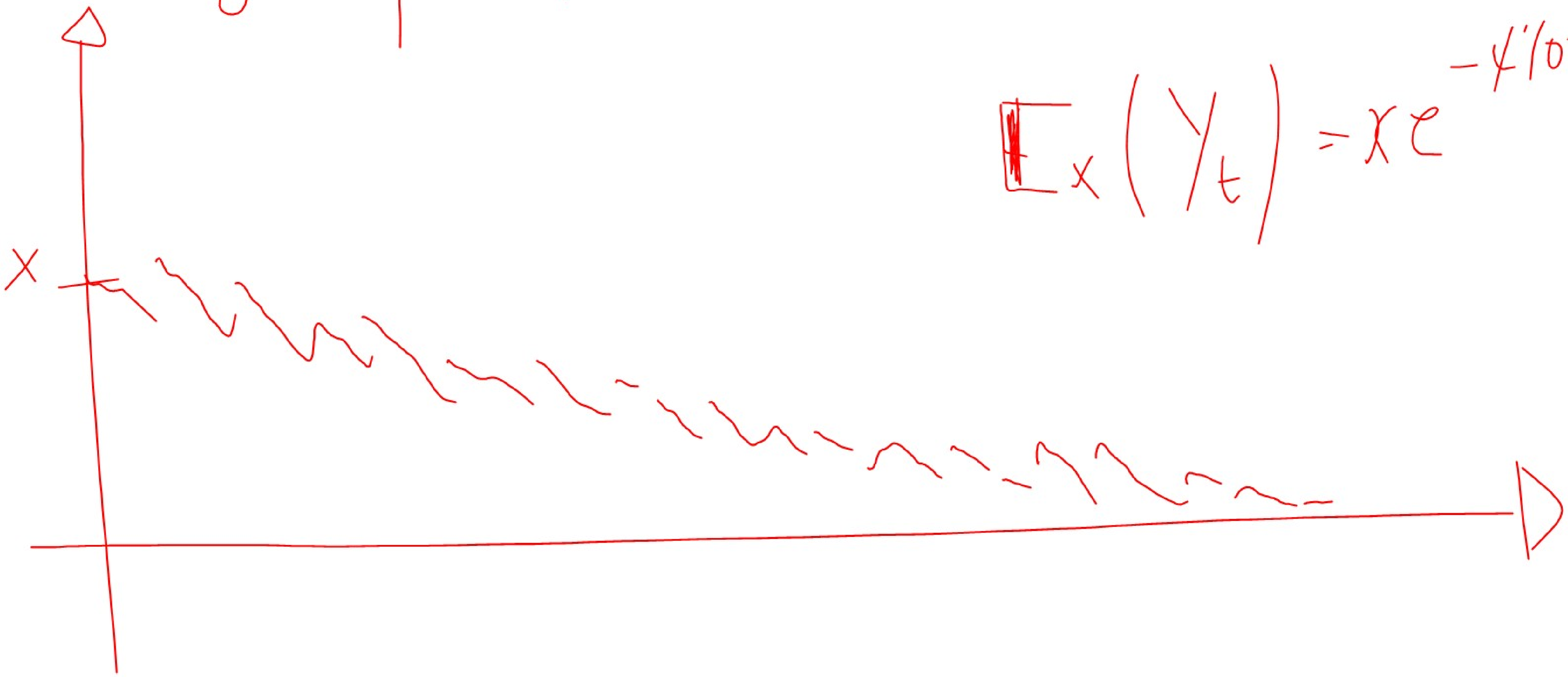
	Condition	$\rho(x)$
Subordinator	$\psi(\infty) < 0$	0
SPLP	$\psi(\infty) = \infty, \quad \int^{\infty} \psi(\xi)^{-1} d\xi = \infty$	0
	$\psi(\infty) = \infty \quad \int^{\infty} \psi(\xi)^{-1} d\xi < \infty$, $\psi'(0^+) < 0$	$e^{-\Phi(0)x} \in (0,1)$
	$\psi(\infty) = \infty \quad \int^{\infty} \psi(\xi)^{-1} d\xi < \infty$, $\psi'(0^+) \geq 0$	$1 = e^{-\Phi(0)x} = e^{-0x}$

From the case $\psi(\infty) = \infty$ $\int^{\infty} \frac{1}{\psi(\xi)} d\xi = \infty$

AND $\psi'(0^+) > 0$ ($\iff$ sub critical)

apparently $p(x) = 0$ $\forall x > 0$

$$\mathbb{E}_x(Y_t) = x e^{-\psi'(0^+)t}$$



Total progeny & supremum

$$J_\zeta = \int_0^\zeta Y_u \, du \qquad \sup_{s \le \zeta} Y_s$$

$$\text{“} \sum_{n \geq 0} Z_n \text{”}$$

Recall that when $\psi(\infty) = \infty$ (i.e. SPLP case)

$$\Phi(q) := \sup\{\lambda > 0 : \psi(\lambda) = q\}$$

Lemma Y is a CSBP, br. mech. ψ s.t. $\psi(\infty) = \infty$

Then $(x > 0)$ $\mathbb{E}_x\left(e^{-q\int_0^\zeta Y_s \, ds}\right) = e^{-\Phi(q)x}$

Pf/ From the Lamperti transformation

$$\tau_0^- := \inf\{t>0 : X_t < 0\}$$

SPLP

then $\tau_0^- = \int_0^{\zeta} Y_u \, du$

$$\mathbb{E}_x\left(e^{-q\int_0^{\zeta} Y_u \, du}\right) = \mathbb{E}_0\left(e^{-q\tau_x^+(-X)}\right) = e^{-\Phi(q)x}$$

from previously

$$\tau_x^+(-X) = \inf\{t>0 : -X > x\}$$