Remainder of course: interested in understanding construction and path properties of CSBP under conditioning on survival.

$$\zeta > T+s$$

$$\lim_{s\to\infty} P_x\left(A \mid Y_{T+s} > 0\right)$$

$A \in \sigma(Y_t : t \le T)$

Step back: Spectrally positive Lévy Processes conditioned to stay positive.

Recall X is SPLP then $-X$ is SNLP

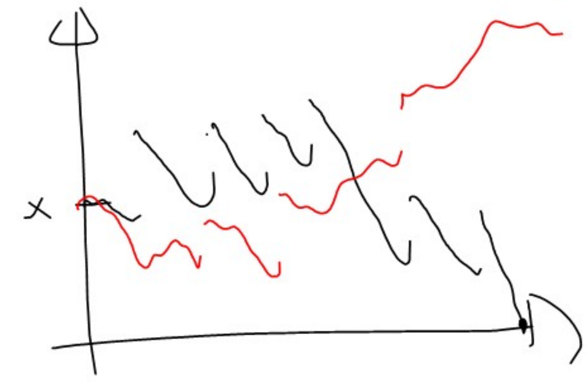
Laplace exponent of X is defined by $\psi(\lambda) = \log \mathbb{E}(e^{-\lambda X_1})$

Recall then that X drifts to $\pm\infty$ / oscillates as $\psi'(0^+) \lessgtr$ / $\psi'(0^+) = 0$

Recall: if $\tau_0^- = \inf\{t > 0 : X_t < 0\}$

$$(x>0) \quad \mathbb{E}_x\left(e^{-q\tau_0^-}\right) = e^{-\Phi(q)x}$$

where $\Phi(q) = \sup\{\lambda \geq 0 : \psi(\lambda) = q\}$



This means that if $\psi'(0+) < 0$ $\left(X_t \xrightarrow[t\to\infty]{} \infty\right)$

$$\mathbb{P}_x(\tau_0^- = \infty) = 1 - e^{-\Phi(0)x}$$

Could think of a SPLP conditioned to stay +ve in terms of

$$\lim_{t\uparrow\infty} \mathbb{P}_x\left(A \mid \tau_0^- > T + t\right) \quad (*)$$

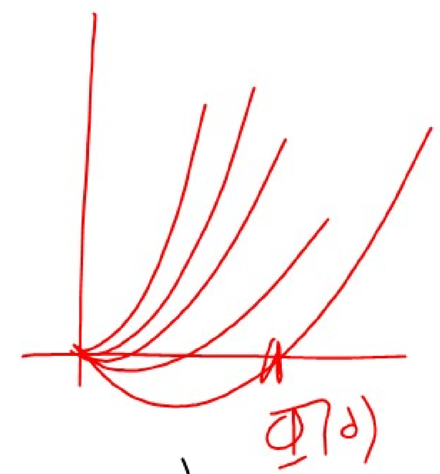
$A \in \mathcal{F}_T$

But in the case $\psi'(0+) < 0$, don't need this limiting sense: can just go straight for a def^n of the form

$$\mathbb{P}_x^\uparrow(A) = \mathbb{P}_x\left(A \mid \tau_0^- = \infty\right) \quad (**)$$

In the case that $\psi'(0+) \geq 0$ cannot make sense of (**) because $\mathbb{P}_x(\tau_0^- = \infty) = 0$ but (*) does make sense.

$\psi'(0+) < 0$: $A \in \mathcal{F}_t$



$$\mathbb{P}^\downarrow_x(A) := \mathbb{P}_x(A \mid \tau_0^- = \infty)$$

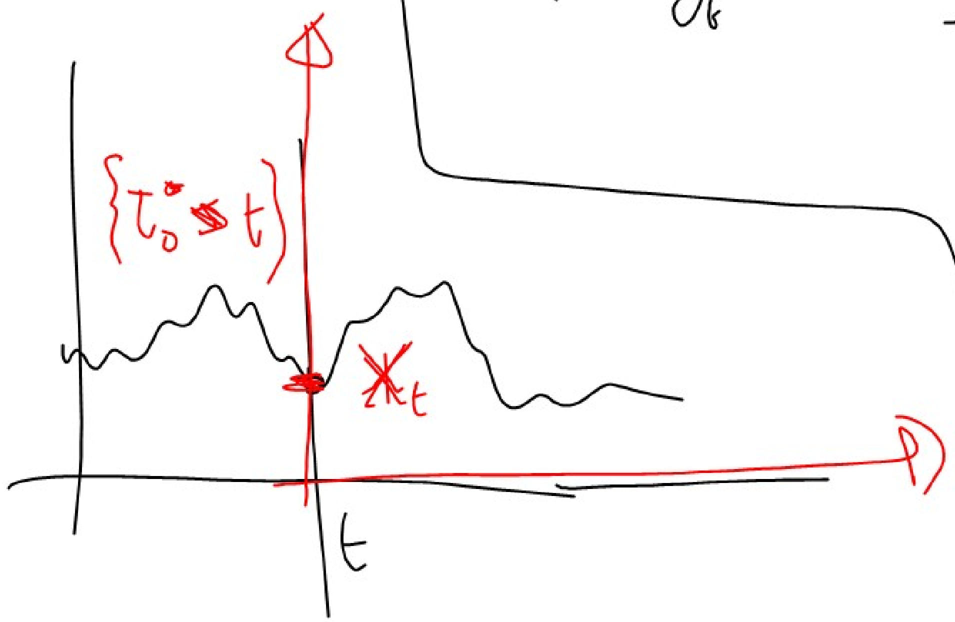
$$\underset{\text{Bayes}}{=} \mathbb{E}_x\left(\mathbb{1}_{(A,\, \overset{\tau_0^- > t,\ \tau_0^- = \infty}{\tau_0^- = \infty})}\right) \Big/ \mathbb{P}_x(\tau_0^- = \infty)$$

$$\underset{\mathbb{E}_x = \mathbb{E}\,\mathbb{E}_{\mathcal{F}_t}}{=} \frac{\mathbb{E}_x\left(\mathbb{1}_{(A,\, \tau_0^- > t)}\, \mathbb{P}_{X_t}(\tau_0^- = \infty)\right)}{\mathbb{P}_x(\tau_0^- = \infty)}$$

$\Phi(0) > 0$

$\Longleftrightarrow \psi'(0^+) < 0$

$\downarrow$



$$= \mathbb{E}_x\left(\mathbb{1}_{(A,\, \tau_0^- > t)} \frac{\left(1 - e^{-\Phi(0) X_t}\right)}{\left(1 - e^{-\Phi(0) x}\right)}\right)$$

This is a Doob h-transform of

$X_t \mathbb{1}_{(t < \tau_0^-)}$

Theorem $(\psi'(0^+) \geq 0)$. We suppose that e_q is an independent and exp. distributed r.v. $\perp\!\!\!\perp$ of SPLPX with rate $q \geq 0$. Assume $\psi'(0^+) \geq 0$. Then $A \in \mathcal{F}_t$

$$\mathbb{P}_x^{\uparrow}(A) := \lim_{q \downarrow 0} \mathbb{P}_x(A, t < e_q \mid \tau_0^- > e_q)$$

exists and satisfies

$$\mathbb{P}_x^{\uparrow}(A) = \mathbb{E}_x\left(\mathbb{1}_{(A, t < \tau_0^-)} \frac{X_t}{x} \right)$$

$$\boxed{e_q \overset{d}{=} \frac{e_1}{q}}$$

Pf. $\mathbb{E}_x(A, t < e_q \mid \tau_0^- > e_q) = \dfrac{\mathbb{P}_x(A, t < e_q, \tau_0^- > e_q)}{\mathbb{P}_x(\tau_0^- > e_q)}$

Theorem ($\psi'(0^+) \geq 0$). We suppose that e_q be an independent and exp. distributed r.v. $\perp$ of SPLPX with rate $q \geq 0$. Assume $\psi'(0^+) \geq 0$. Then $A \in \mathcal{F}_t$

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Pf

$$\mathbb{E}_x\left(A, t < e_q \,\middle|\, \tau_0^- > e_q\right) = \frac{\mathbb{P}_x\left(A, t < e_q, \tau_0^- > e_q\right)}{\mathbb{P}_x\left(\tau_0^- > e_q\right)}$$

$$\overset{\mathbb{E}_x = \mathbb{E}\mathbb{E}_{\mathcal{F}_t}}{=} \frac{\mathbb{E}_x\left[\mathbb{1}_{(A, t < e_q)} \mathbb{1}_{(t < \tau_0^-)} \mathbb{E}\left(\mathbb{1}_{(\tau_0^- > e_q)} \,\middle|\, \mathcal{F}_t\right)\right]}{\mathbb{E}_x\left(1 - e^{-q\tau_0^-}\right)}$$

$$\underset{\text{LoM}}{\overset{\text{MP}}{=}} \frac{\mathbb{E}_x\left(\mathbb{1}_{(A, t < \tau_0^-)} \mathbb{1}_{(t < e_q)} \mathbb{P}_{X_t}\left(\tau_0^- > e_q\right)\right)}{\mathbb{E}_x\left(1 - e^{-q\tau_0^-}\right)}$$

$$= \frac{\mathbb{E}_x\left(\mathbb{1}_{(A, t < \tau_0^-)} \mathbb{1}_{(t < e_q)} \mathbb{E}_{X_t}\left(1 - e^{-q\tau_0^-}\right)\right)}{\mathbb{E}_x\left(1 - e^{-q\tau_0^-}\right)}$$

$$= \mathbb{E}_x\left(\mathbb{1}_{(A, t < \tau_0^-)} e^{-qt} \frac{\left(1 - e^{-\Phi(q) X_t}\right)}{\left(1 - e^{-\Phi(q) x}\right)}\right)$$

Hence

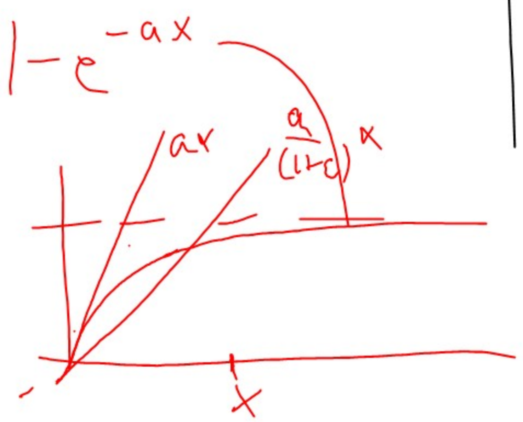
$$\lim_{q\downarrow 0} \mathbb{P}_x\left(A, t<e_q \mid \tau_0^- > e_q\right) = \lim_{q\downarrow 0} \mathbb{E}_x\left(\mathbb{1}_{(A,\, t<\tau_0^-)}\, e^{-qt} \frac{\left(1-e^{-\Phi(q)X_t}\right)}{\left(1-e^{-\Phi(q)x}\right)}\right)$$

In order to apply DCT (and then L'Hôpital's rule) to get the density X_t/x (because $\Phi(q)\to 0$ as $q\to 0$)

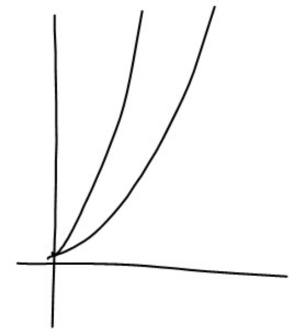
Note that

$$\boxed{\frac{1-e^{-\Phi(q)X_t}}{1-e^{-\Phi(q)x}} \le C\frac{\cancel{\Phi(q)}X_t}{\cancel{\Phi(q)}x}}$$

for q suff small



Note that $\psi'(0^+)\ge 0$



$$\implies \mathbb{E}(|X_t|)<\infty$$

Hence $\square$ gives us a good ~~fn~~ dominating fn for DCT. ∎