Note that $\mathbb{P}^{\uparrow}_x(\cdot)$ is only a pr measure when $\psi'(0^+) = 0$ in previous thm:

Recall $\mathbb{P}^{\uparrow}_x(A) = \mathbb{E}_x\left(\mathbb{1}_{(A,\, t<\tau_0^-)} \frac{X_t}{x}\right)$: $A \in \mathcal{F}_t$

need to check if $\mathbb{E}_x\left(\mathbb{1}_{(t<\tau_0^-)} X_t\right) = x$
for $\mathbb{P}^{\uparrow}_x$ to be a pr. measure

We see that

$$\mathbb{E}_x\left(\mathbb{1}_{(t<\tau_0^-)} \frac{X_t}{x}\right) = \lim_{q\downarrow 0} \mathbb{P}_x\left(t < e_q \,\middle|\, \tau_0^- > e_q\right)$$

$$= 1 - \lim_{q\downarrow 0} \mathbb{P}_{\cancel{x}}\left(e_q \le t \,\middle|\, \tau_0^- > e_q\right)$$

$$= 1 - \lim_{q\downarrow 0} \frac{\mathbb{P}_x\left(e_q \le t,\ e_q < \tau_0^-\right)}{\mathbb{P}_x\left(e_q \le \tau_0^-\right)}$$

$$= 1 - \lim_{q\downarrow 0} \frac{\int_0^\infty q e^{-qu}\, \mathbb{1}_{(u\le t)}\, \mathbb{E}_x \mathbb{1}_{(u<\tau_0^-)}\, du}{1 - e^{-\Phi(q)x}}$$

$$= 1 - \lim_{q\downarrow 0} \frac{q}{\Phi(q)x} \int_0^t e^{-qu}\, \mathbb{P}_x(\tau_0^- > u)\, du$$

$$\lim_{q\downarrow 0} \frac{q}{\Phi(q)} = \lim_{q\downarrow 0} \frac{\psi(\Phi(q))}{\Phi(q)} \underset{\substack{\text{since} \\ \psi'(0^+)\geq 0 \\ \Phi(q)\to 0}}{\overset{\theta = \Phi(q)}{=\!=}} \lim_{\theta\downarrow 0} \frac{\psi(\theta)}{\theta} = \psi'(0^+)$$

Hence $\mathbb{E}_x\left(\mathbb{1}_{(t<\tau_0^-)} \frac{X_t}{x}\right) \underset{\forall t\geq 0}{\overset{\forall x>0}{=}} 1 \iff \psi'(0^+) = 0$

o/w < 1

Exercise: $\psi'(0^+) = 0 \implies \mathbb{1}_{(t<\tau_0^-)} X_t$ is a mg

R.2 Conditioning (sub)critical CSBP to avoid extinction

Necessarily restrict ourselves to CSBP with br mech ψ s.t. $\psi(\infty) = \infty$ (i.e. driven by SPLP) (not a subordinator)

Moreover we shall insist that there is extinction w.p. 1 ie $\psi'(0^+) \geq 0$ $\int^\infty \frac{1}{\psi(z)} dz < \infty$

$$\boxed{\text{Define } \rho = \psi'(0^+)}$$

<u>Theorem</u> Suppose that Y is a CSBP with br. mech. ψ satisfying above conditions, then for $A \in \sigma(Y_s : s \leq t)$, $x > 0$

$$P^{\uparrow}_x(A) := \lim_{s \uparrow \infty} P_x(A \mid \zeta > t + s)$$

is well defined as a Pr. measure and satisfies

$$P^{\uparrow}_x(A) = E_x\left(\mathbb{1}_{(A)} e^{\rho t} \frac{Y_t}{x}\right)$$

In particular $P^{\uparrow}_x(\zeta < \infty) = 0$

and $\{e^{\rho t} Y_t : t \geq 0\}$ is a mg.

~~Pf~~ Recall $P_x(\zeta > t) = 1 - P_x(\zeta \le t)$ $\overset{Y_t=0}{}$

$$= 1 - e^{-x u_t(\infty)}$$

$E_x(e^{-\theta Y_t}) = e^{-x u_t(\theta)}$

We will need in the proof to know that

$$\boxed{\lim_{s \uparrow \infty} \frac{u_s(\infty)}{u_{t+s}(\infty)} = e^{\rho t}} \quad \text{if } \rho \geq 0$$

Because $(\rho > 0)$

$$\int_{u_t(\infty)}^{\infty} \frac{1}{\psi(\zeta)} d\zeta = t \implies \int_{u_{t+s}(\infty)}^{u_s(\infty)} \frac{\rho}{\psi(\zeta)} d\zeta = \rho t$$

This means that $\log \frac{u_s(\infty)}{u_{t+s}(\infty)} \doteq \int_{u_{t+s}(\infty)}^{u_s(\infty)} \frac{1}{\zeta} d\zeta = \int_{u_{t+s}(\infty)}^{u_s(\infty)} \frac{\cancel{\rho}}{\psi(\zeta)} \boxed{\frac{\psi(\zeta)}{\cancel{\rho}\zeta}} d\zeta$

Since $u_t(\infty) \searrow \Phi(0) = 0$

$$\lim_{s \uparrow \infty} \log \frac{u_s(\infty)}{u_{t+s}(\infty)} = \rho t$$

Similar proof shows $\boxed{}$ is true when $\rho = 0$ $\left(\frac{\psi(\zeta)}{\zeta} \to 0\right)$

$$P_x(A \mid \zeta > t+s) = E_x\left(\frac{1_{(A,\, \zeta > t+s)}}{P_x(\zeta > t+s)}\right)$$

$$= \frac{E_x\left(1_{(A,\, t < \zeta)} P_{Y_t}(\zeta > s)\right)}{P_x(\zeta > t+s)}$$

$$= E_x\left(1_{(A,\, t<\zeta)} \frac{1 - e^{-Y_t u_s(\infty)}}{1 - e^{-x u_{t+s}(\infty)}}\right)$$

In order to apply DCT in limit need to check that

$$\frac{1-e^{-Y_t u_s(\infty)}}{1-e^{-x u_{t+s}(\infty)}} \leq \frac{Y_t u_s(\infty)}{1-e^{-x u_{t+s}(\infty)}} \overset{\square}{\leq} C \frac{Y_t}{x} e^{\rho t}$$

$\sim x u_{t+s}(\infty)$

for s suff large

Note also that $\mathbb{E}_x(Y_t) = e^{-\psi'(0^+)t} x = x e^{-\rho t}$

Now with DCT

$$\lim_{s\uparrow\infty} P_x(A \mid \zeta > t+s) \underset{\mathrm{DCT}}{\overset{\square}{=}} \mathbb{E}_x \mathbb{1}_{(A,\, t<\zeta)} e^{\rho t} \frac{Y_t}{x}$$

That P_x^{ϕ} is a pr.-measure on $\mathcal{F}_t$ follows from the fact that

$$E_x\left(\mathbb{1}_{\substack{(t<\zeta) \\ Y_t \neq 0}} \frac{Y_t e^{\rho t}}{x}\right) = E_x(Y_t) = x e^{-\rho t} \quad (*)$$

note $\mathbb{1}_{(Y_t \neq 0)} Y_t = Y_t$

$P_x^{\uparrow}(\zeta < \infty) = 0$ follows from

$$P_x^{\phi}(\zeta < t) = 0 \quad \forall t$$

$\{e^{\rho t} Y_t : t \geq 0\}$ is a mg by markov property and $(*)$

$$E_x\left(e^{\rho t} Y_t \mid \sigma(Y_u : u \leq s)\right)$$

$$= e^{\rho s} e^{\rho(t-s)} E_x(Y_t \mid Y_s)$$

$$\overset{(*)}{=} e^{\rho s} \cancel{e^{\rho(t-s)}} Y_s \cancel{e^{-\rho(t-s)}}$$

$$= Y_s e^{\rho s}$$

Next: show that (Y, P_x^{ϕ}) is in fact a CSBP with immigration

Moreover when $\psi'(0+)$ (Y, P_x^{ϕ}) is related to $(X, \mathbb{P}_x^{\phi})$ still through the Lamperti transf