CSBP with immigration

We define a Markov process $Y = \{Y_t : t \geq 0\}$ with probabilities

\[ P_x : x \geq 0 \] is a CSBP with branching rate $\gamma$ and immigration rate $\phi$ (i.e., CSBP with immigration).

If it has right continuous with left limits and $\gamma, \phi \geq 0$,

\[
E_x \left( e^{-\theta Y_t} \right) = \exp \left\{ -\gamma \bar{u}_t (\theta) - \int_0^t \phi \bar{u}_{t-s} (\theta) \, ds \right\}
\]

where $\gamma$ is of the same class as before, $\bar{u}_t (\theta)$ solves

\[
\frac{d}{dt} \bar{u}_t (\theta) + \gamma (\bar{u}_t (\theta)) = 0 \quad \text{and} \quad \phi \text{ takes the form}
\]

\[
\phi (\theta) = \delta \theta + \int_{(0, \infty)} (1 - e^{-\theta x}) \Lambda (dx)
\]

for some $\delta \geq 0$ and measure $\Lambda$ concentrated on $(0, \infty)$ such that

\[
\int_{(0, \infty)} (1/x) \Lambda (dx) < \infty.
\]
Are there any CSBP$^+(4, \phi)$? We justify (prove!) the existence of CSBP$^+(4, \phi)$ for the case that $f = 0$.

The basic idea: take a subordinator $S$ whose Laplace exponent is $\phi$ (i.e. $E[e^{-\phi(S)}] = e^{-\phi(t)k}$).

Let

$$Y^* := Y_t + \sum_{u \leq k} Y_{t-u} \mathbb{1}(\Delta S_u > 0)$$

where $\{Y_t\}_{t \geq 0}$ is a CSBP$^+(4)$ of everything else started from $S_0$, $\{Y_{\Delta S_u}\}_{u \geq 0}$ is another CSBP$^+(4)$ of everything else started from $\min(S_u - S_0)$, $\Delta S_u = S_u - S_{u-}$.

For each $u : \Delta S_u > 0$, process $Y_{\Delta S_u}$ is independent of everything else.

Claim: above $Y^*$ is a CSBP$^+(4, \phi)$ according to Definition on previous slide.
To prove the claim, we rephrase it to CPP and the following

**Lemma:** Suppose that \( N \) is a CPP rate \( \lambda \) and \( \{ T_i : i \geq 1 \} \) are iid (positive) r.v. with common density \( f \).

Then

\[
E(e^{\sum_{i=1}^{N_\lambda} \alpha(t-T_i)}) = \exp\left\{ \int_0^t \left( 1 - e^{-\lambda x} \right) F(dx) dt \right\}
\]

where \( \{ T_i : i \geq 1 \} \) are renewal times and \( \alpha : [0, \infty) \to [0, \infty) \)

is a c.d.f.

**Proof:**

LHS = \[ E \left( \prod_{i=1}^{N_\lambda} E(e^{-\lambda \theta}) \right) \]

\[ = \sum_{n=0}^{\infty} E \left( \prod_{i=1}^{n} E(e^{-\lambda \theta}) \right) \]

\[ = \sum_{n=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^n}{n!} \left( \int_0^t \left( 1 - e^{-\lambda x} \right) F(dx) dt \right)^n \]

[Recall that \( \{ T_1, \ldots, T_n \} \) given \( N_\lambda = n \) is that of ordered iid uniform distributed \( T_i \) s on \( [0, t] \)]

\[ = \sum_{n=0}^{\infty} e^{-\lambda t} \left( \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \left( \int_0^t \left( 1 - e^{-\lambda x} \right) F(dx) dt \right)^n \right) \]

\[ = \sum_{n=0}^{\infty} e^{-\lambda t} \left[ \frac{1}{n!} \int_0^t \left( 1 - e^{-\lambda x} \right) F(dx) dt \right]^n \]

\[ = \exp \left\{ -\int_0^t \left( 1 - e^{-\lambda x} \right) F(dx) dt \right\} \]
If \( \text{of claim: } (Y, \mathcal{F}, P) \equiv (Y_t, P_x, (\Delta S_t)_{t \geq 0}) \)
\[ Y^*_t = Y_t + \sum_{u \in \mathcal{T}} Y_{t-u} \]

First note that since \( \text{sum is empty at } t=0 \)
then \( Y^*_0 = Y_0 = x. \) Call \( P \) measure on \( \mathbb{R}_+ \times \mathbb{R} \) with \( \mathbb{Q} \) a common
\[ \mathbb{E}(e^{-\Theta Y^*_t}) = \mathbb{E}_x(e^{-\Theta Y_t}) \mathbb{E}_e(e^{-\Theta \sum_{u \in \mathcal{T}} Y_{t-u}}) \]

\[ = e^{-x \phi_x(0)} \mathbb{E}_e(e^{-\Theta \sum_{u \in \mathcal{T}} Y_{t-u}}) \]

Hence need to show
\[ \mathbb{E}_e(e^{-\Theta \sum_{u \in \mathcal{T}} Y_{t-u}}) = e^{-\int_0^t \phi_x(u) du} \]

The need to show that \( Y^* \) is Markovian!!

1. \[ \mathbb{E}(e^{-\Theta \sum_{u \in \mathcal{T}} Y_{t-u}}) = \mathbb{E}_x(e^{-\Theta \sum_{u \in \mathcal{T}} (Y_{t-u})}) \]
2. \[ = \mathbb{E}_x \prod_{u \in \mathcal{T}} \mathbb{E}(e^{-\Theta Y_{t-u}}) \]
3. \[ = \mathbb{E}_x \prod_{u \in \mathcal{T}} \mathbb{E}(e^{-\Theta Y_{t-u}}) \]
4. \[ = \mathbb{E}(\lim_{\mathcal{T}} \prod_{u \in \mathcal{T}} e^{-\Delta S_u u t-u(0)}) \]
5. \[ = \mathbb{E}(\lim_{\mathcal{T}} e^{-\sum_{u \in \mathcal{T}} \Delta S_u 1(\Delta S_u > 0) u t-u(0)}) \]
6. \[ = \lim_{\mathcal{T}} \mathbb{E}(e^{-\sum_{u \in \mathcal{T}} \Delta S_u 1(\Delta S_u > 0) u t-u(0)}) \]

\[ = \lim_{\mathcal{T}} \int_0^t \int (1 - e^{-x \phi_x(u)}) \frac{\Lambda(dx)}{\Lambda(x, u)} ds \]

\[ = \int_0^t \int (1 - e^{-x \phi_x(u)}) \frac{\Lambda(dx)}{\Lambda(x, u)} ds \]

Markovian property of \( Y^* \) is an exercise!