CSBP with immigration

We define a $[0,\infty)$-valued M.P. $Y^* = \{Y^*_t : t \geq 0\}$ with probabilities $\{\mathbb{P}_x : x \geq 0\}$ to be a CSBP with branching mech ψ and immigration mech. ϕ (ie CSBP with immigration) if it has right cts paths with left limits and $\forall x, t \geq 0$ & $\theta \geq 0$

$$\mathbb{E}_x\left(e^{-\theta Y^*_t}\right) = \exp\left\{-x u_t(\theta) - \int_0^t \phi \circ u_{t-s}(\theta)\, ds\right\}$$

Where ψ is of the same class as before, $u_t(\theta)$ solves

$\frac{\partial u_t}{\partial t}(\theta) + \psi(u_t(\theta)) = 0$ and ϕ takes the form

$$\phi(\theta) = \delta\theta + \int_{(0,\infty)} (1 - e^{-\theta x}) \Lambda(dx)$$

for some $\delta \geq 0$ and measure Λ concentrated on $(0,\infty)$ s.t. $\int_{(0,\infty)} (1 \wedge x) \Lambda(dx) < \infty$.

Are there any CSBP(ψ, ϕ)?
We justify (prove!) the existence of CSBP(ψ,ϕ)
for the case that $\delta = 0$.

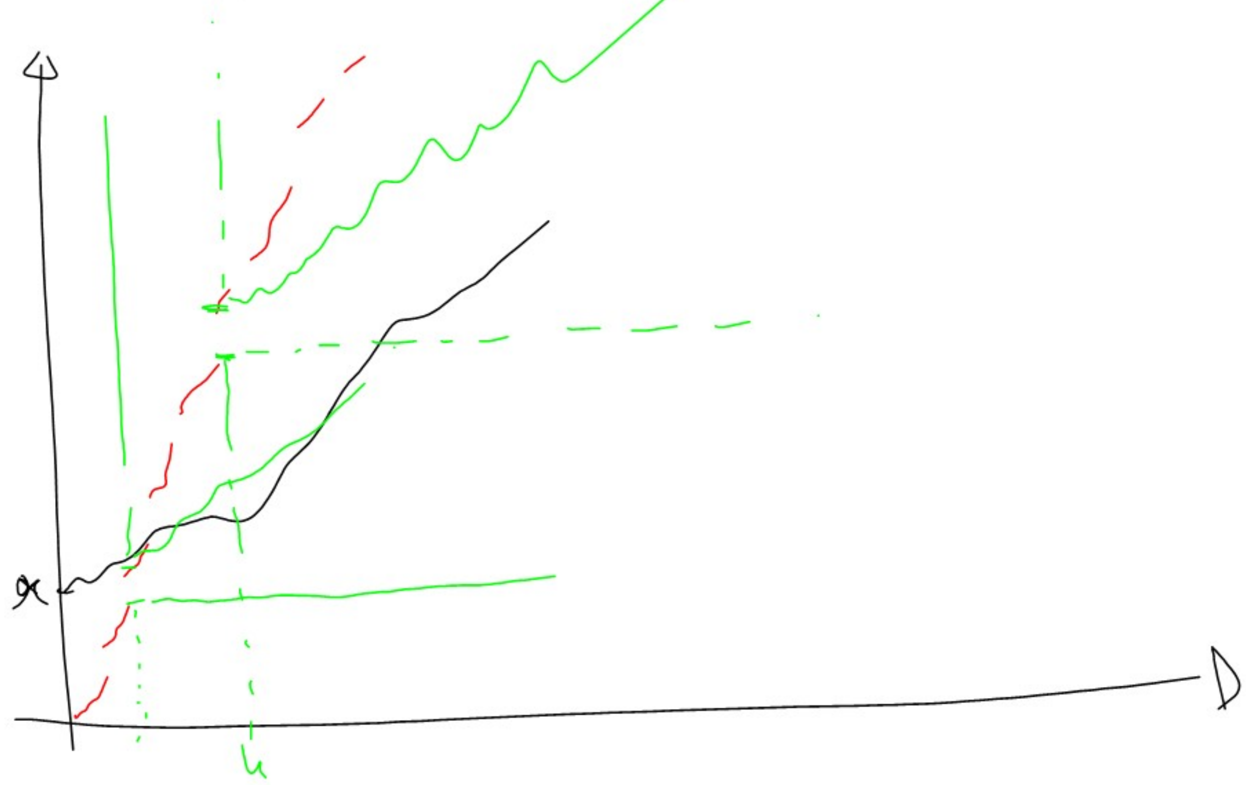
The basic idea: take a subordinator S whose Laplace exponent is ϕ (i.e. $E(e^{-\theta S_t}) = e^{-\phi(\theta)t}$).
& take

$$Y^*_t := Y_t + \sum_{\underbrace{u \leq t}} Y^{(\Delta S_u)}_{t-u} \underbrace{\mathbb{1}(\Delta S_u > 0)}_{\text{adding of jump times of } S \text{ in } [0,t]}$$

where $\{Y_t\}_{t\geq 0}$ is a CSBP(ψ) $\perp\!\!\!\perp$ of everything else starting from x
$\{Y^{(\Delta S_u)}_\tau\}_{\tau \geq 0}$ is another CSBP(ψ) $\perp\!\!\!\perp$ everything else started from initial value ΔS_u
where $\Delta S_u = S_u - S_{u-}$

For each $u: \Delta S_u > 0$, process $Y^{(\Delta S_u)}_\cdot$ is independent of everything else.

Claim: above Y^* is a CSBP(ψ,ϕ) according to def^n on previous slide.



To prove the claim, we return to CPP and the following

Lemma : Suppose that N is a PP rate λ and $\{\xi_i : i \geq 1\}$ are iid (positive) r.v. with common dist^n F

then
$$\mathbb{E}\left(e^{-\sum_{i=1}^{N_t} \xi_i \, \alpha(t-\tau_i)}\right) = \exp\left\{-\lambda \int_0^t \int_{(0,\infty)} \left(1 - e^{-x\alpha(t-s)}\right) F(dx)\, ds\right\}$$

where $\{\tau_i : i \geq 1\}$ are arrival times and $\alpha : [0,\infty) \to [0,\infty)$ is a cts function

Pf/
$$LHS = \mathbb{E}\,\mathbb{E}\left(e^{-\cdots} \,\middle|\, N_t, \{\tau_i : i \geq 1\}\right)$$

$$= \mathbb{E}\left(\prod_{i=1}^{N_t} \mathbb{E}\left(e^{-\xi_i \theta}\right)\Big|_{\theta = \alpha(t-\tau_i)}\right)$$

$$= \sum_{n \geq 0} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \mathbb{E}\left(\prod_{i=1}^{n} \mathbb{E}\left(e^{-\xi_i \theta}\right)\Big|_{\theta = \alpha(t-\tau_i)} \,\middle|\, N_t = n\right)$$

[Recall dist^n of $\{\tau_1, \ldots, \tau_n\}$ given $\{N_t = n\}$ is that of ordered i.i.d uniform distributed r.v.s on $[0,t]$]

$$= \sum_{n \geq 0} e^{-\lambda t} \frac{(\lambda \cancel{t})^n}{n!} \left[\frac{1}{\cancel{t}} \int_0^t \int e^{-x\alpha(t-\tau)} F(dx)\, d\tau\right]^n$$

$$= \sum_{n \geq 0} e^{-\lambda t} \frac{\left[\lambda \int_0^t \int_{(0,\infty)} e^{-x\alpha(t-\tau)} F(dx)\, d\tau\right]^n}{n!}$$

$$= \exp\left\{-\lambda \int_0^t 1 - \int_{(0,\infty)} e^{-x\alpha(t-\tau)} F(dx)\; d\tau\right\}$$

$$= \exp\left\{-\lambda \int_0^t \int_{(0,\infty)} \left(1 - e^{-x\alpha(t-\tau)}\right) F(dx)\, d\tau\right\}$$

Pf of claim: $(Y_\cdot, \mathcal{P}) \overset{law}{=} (Y_\cdot, P_x)$

$$Y^*_t := Y_t + \sum_{u \le t} Y^{(\Delta S_u)}_{t-u}(t)$$

First note that since sum is empty at $t=0$ then $Y^*_0 = Y_0 = x$. Call $\mathcal{P}$ intrinsic pr meas. assoc. with (t) and $\mathbb{E}_e$ is expectation wrt $\mathcal{P}$

$$\mathbb{E}_e\left(e^{-\theta Y^*_t}\right) = E_x\left(e^{-\theta Y_t}\right) \mathbb{E}_e\left(e^{-\theta \sum_{u\le t} Y^{(\Delta S_u)}_{t-u}}\right)$$

$$= e^{-x u_t(\theta)} \mathbb{E}_e\left(e^{-\theta \sum_{u\le t} Y^{(\Delta S_u)}_{t-u}}\right)$$

Hence need to show that

1. $$\mathbb{E}_e\left(e^{-\theta \sum_{u\le t} Y^{(\Delta S_u)}_{t-u}}\right) = e^{-\int_0^t \phi \circ u_{t-s}(\theta)\, ds}$$

2. (Also need to show that Y^* is Markovian !!!)

1. $$\mathcal{E}\left(e^{-\theta \sum_{u \le t} Y^{(\Delta S_u)}_{t-u}}\right) = \mathcal{E}\left(\lim_{\varepsilon \downarrow 0} \mathcal{E}\left(\lim_{\varepsilon\downarrow 0} e^{-\theta \sum_{u\le t} Y^{(\Delta S_u)}_{t-u} \mathbb{1}_{(\Delta S_u > \varepsilon)}} \,\Big|\, S\right)\right)$$

(DCT)

$$= \mathcal{E}_e\left(\lim_{\varepsilon\downarrow 0} \prod_{u \le t} \mathcal{E}\left(e^{-\theta Y^{(\Delta S_u)}_{t-u} \mathbb{1}_{(\Delta S_u > \varepsilon)}} \,\Big|\, S\right)\right)$$

$$\overset{MCT}{=} \mathcal{E}\left(\prod_{u\le t} \mathcal{E}\left(e^{-\theta Y^{(\Delta S_u)}_{t-u}} \,\Big|\, S\right)\right)$$

$$= \mathcal{E}\left(\prod_{u \le t} e^{-\Delta S_u u_{t-u}(\theta)}\right)$$

$$= \mathcal{E}\left(\lim_{\varepsilon\downarrow 0} \prod_{u\le t} e^{-\Delta S_u u_{t-s}(\theta) \mathbb{1}_{(\Delta S_u > \varepsilon)}}\right)$$

$$= \mathcal{E}\left(\lim_{\varepsilon \downarrow 0} e^{-\sum_{u \le t} \Delta S_u \mathbb{1}_{(\Delta S_u > \varepsilon)} u_{t-s}(\theta)}\right)$$

$$\overset{DCT}{=} \lim_{\varepsilon\downarrow 0} \mathcal{E}\left(e^{-\sum_{u\le t} \overbrace{\Delta S_u}^{\xi_i} \mathbb{1}_{(\Delta S_u > \varepsilon)} \underbrace{u_{t-u}(\theta)}_{\alpha(t-u)}}\right)$$

PP arrival rate $\Lambda(\varepsilon,\infty)$

jump dist $\Lambda(dx)|_{(\varepsilon,\infty)} / \Lambda(\varepsilon,\infty)$

jumps of size $> \varepsilon$ arrive at rate $\Lambda(\varepsilon,\infty)$

$$\overset{\text{Lemma}}{=} \lim_{\varepsilon \downarrow 0} e^{-\Lambda(\varepsilon,\infty) \int_0^t \int_{(\varepsilon,\infty)} \left(1 - e^{-x u_{t-s}(\theta)}\right) \frac{\Lambda(dx)}{\Lambda(\varepsilon,\infty)}\, ds}$$

$$= e^{-\int_0^t \int_{(0,\infty)} \left(1 - e^{-x u_{t-s}(\theta)}\right) \Lambda(dx)\, ds}$$

$$= e^{-\int_0^t \phi \circ u_{t-s}(\theta)\, ds}$$

Markovian property of Y^* is an exercise!