Understanding what happens when $\delta > 0$

$$\exp\left\{ -x\,u_t(\theta) - \int_0^t \phi \circ u_{t-s}(\theta)\, ds \right\}$$

If $\phi(\theta) = \delta\theta + \chi(\theta)$

$$= \exp\left\{ \underbrace{-x\,u_t(\theta) - \int_0^t \chi \circ u_{t-s}(\theta)\, ds} - \underbrace{\int_0^t \delta\, u_{t-s}(\theta)\, ds} \right\}$$

Lemma Suppose (Y, P_x) is a CSBP with br. mech. ψ (only SPLP case, $\psi'(0^+) \geq 0$, non-explosion). Then (Y, P_x^ϕ) has the same law as a CSBP with br. mech. ψ and immigration mech ϕ where

$$\phi(\theta) = \psi'(\theta) - \rho \quad : \theta \geq 0$$

$$\boxed{\rho = \psi'(0^+)}$$

Pf. Obviously (Y, P_x^ϕ) is Markovian as it is under P_x, also cadlag for same reason.

$$E_x^\phi\left(e^{-\theta Y_t}\right) = E_x\left(\frac{Y_t}{x} e^{\rho t} e^{-\theta Y_t}\right)$$

Recall $P_x^\phi(A) = E_x\left(\frac{Y_t}{x} e^{\rho t} \mathbf{1}_{(A, t<\zeta)}\right)$

$$= -\frac{e^{\rho t}}{x} \frac{\partial}{\partial \theta} \boxed{E_x\left(e^{-\theta Y_t}\right)} \quad e^{-x u_t(\theta)}$$

$$= + \frac{e^{\rho t}}{\cancel{x}} e^{-x u_t(\theta)} \cancel{x} \frac{\partial u_t}{\partial \theta}(\theta)$$

glance back thro' lecture notes an note we showed $\frac{\partial u_t(\theta)}{\partial \theta} = e^{-\int_0^t \psi'(u_s(\theta))\,ds}$

$$= e^{\rho t} e^{-x u_t(\theta) - \int_0^t \psi'(u_s(\theta))\,ds}$$

$$= e^{-x u_t(\theta) - \int_0^t (\psi'(u_{t-s}(\theta)) - \rho)\,ds}$$

$$= e^{-x u_t(\theta) - \int_0^t \phi \circ u_{t-s}(\theta)\,ds}$$

when $\phi(\theta) = \psi'(\theta) - \rho$.

Note: strictly speaking we should check that $\psi'(\theta) - \rho$ is a Laplace exponent of a subordinator!

Recall $\psi(\theta) = a\theta + \frac{1}{2}\theta^2\sigma^2 + \int_{(0,\infty)} (e^{-\theta x} - 1 + \theta x \mathbf{1}_{(x<1)}) \Pi(dx)$

exercise $\Longrightarrow$ using $\int_{(0,\infty)} 1 \wedge x^2 \, \Pi(dx) < \infty$ to pass $\frac{d}{d\theta}$ thro' $\int$

$$\psi'(\theta) = \boxed{\sigma^2\theta + \int_{(0,\infty)} (1 - e^{-\theta x}) \underbrace{x\Pi(dx)}_{\Pi_s(dx)}} + \underbrace{\psi'(0^+)}_{\rho}$$

Note: $\int_{(0,\infty)} 1 \wedge x \, \Pi_s(dx)$

$$= \int_{(0,\infty)} (x \wedge x^2) \Pi(dx)$$

$$= \int_{(0,1)} x^2 \Pi(dx) + \int_{(1,\infty)} x \Pi(dx)$$

$< \infty$

because Π is a Lévy meas. (for 1st integral) and because $E(|X_1|) < \infty$ (which is a consequence of $\psi'(0^+) \geq 0$ $\overset{\text{convexity}}{\Longrightarrow}$ $\psi'(0^+) < \infty$)

$\int_{(1,\infty)} x \Pi(dx) < \infty \iff$

Lemma Suppose ψ is the br. mech. for a CSBP Y with same conditions as previous Lemma however we impose further restriction $\psi'(0+)=0$

Write X for the SPLP corresponding to ψ

(1) If $\Theta_t = \inf\left\{ s>0 : \int_0^s \frac{1}{X_u}\,du > t \right\}$

then $\{ X_{\Theta_t} : t \geq 0 \}$ under $\mathbb{P}_x^{\uparrow}$

has the same law as $(Y, P_x^{\uparrow})$

(2) If $\varphi_t = \inf\left\{ s>0 : \int_0^s Y_u\,du > t \right\}$

Then $\{ Y_{\varphi_t} : t \geq 0 \}$ under $P_x^{\uparrow}$

is equal in law to $(X, \mathbb{P}_x^{\uparrow})$

$$\boxed{X_t^{(+)} := X_t \mathbf{1}_{(t<\tau_0^-)}}$$

$$\begin{tikzcd}
(X^{(+)}, \mathbb{P}_x) \arrow[r, shift left, "\Theta"] \arrow[d, "\text{Doob h-transform } h(x)=x"] & (Y, P_x) \arrow[l, shift left, "\varphi"] \arrow[d] \\
(X, \mathbb{P}_x^{\uparrow}) \arrow[r, shift left, "\Theta"] & (Y, P_x^{\uparrow}) \arrow[l, shift left, "\varphi", "\text{Lemma}"']
\end{tikzcd}$$

Pf (1) Easy to show that Θ_t is a stopping time wrt the natural filtration of X we can implement the change of measure density $\frac{X_t}{x}$ also at stopping times:

$$\mathbb{E}^{\uparrow}_x\left(F(X_{\Theta_s}, s \leq t)\,\mathbb{1}(\Theta_t < \infty)\right)$$

where F is a "path functional"

$$= \mathbb{E}_x\left(\frac{X_{\Theta_t}}{x} F(X_{\Theta_s} : s \leq t)\,\mathbb{1}(\Theta_t < \tau_0^-)\right)$$

Note that $\{\Theta_t < \tau_0^-\} = \{t < \zeta\}$

$$= \mathbb{E}_x\left(\frac{Y_t}{x} F(Y_s : s \leq t)\,\mathbb{1}(t < \zeta)\right)$$

$$= \mathbb{E}^{\triangle}_x\left(F(Y_s : s \leq t)\right)$$

Pf of (2) is very similar. ∎