

Understanding what happens when $\delta > 0$

$$\exp \left\{ -\kappa u_T(\theta) - \int_0^t \phi \circ u_{t-s}(\theta) ds \right\}$$

$$\mathbb{E} \phi(\theta) = \delta \Theta + \chi(\theta)$$

$$\Rightarrow = \exp \left\{ \underbrace{-\kappa u_T(\theta) - \int_0^t \chi \circ u_{t-s}(\theta) ds}_{\text{}} - \underbrace{\int_0^t \delta u_{t-s}(\theta) ds}_{\text{}} \right\}$$

Lemma Suppose (Y, P_x) is a CSBP with br. mech. ψ (only SPBP case) $\psi'(0^+) \geq 0$ non explosion. Then (Y, P_x^θ) has the same law as a CSBP with br. mech. ψ and immigration mech ϕ where

$$\phi(\theta) = \psi'(\theta) - \rho \quad ; \quad \theta \geq 0$$

~~PF~~ Obviously (Y, P_x^θ) is Markovian as $\psi = \psi'(0^+)$ it is under P_x , also cadlag for same reason.

$$E_x^\theta (e^{-\theta Y_t}) = E_x \left(\frac{Y_t e^{\rho t}}{x} \right)$$

[Recall $P_x^\theta(A) = E_x \left(\frac{Y_t}{x} e^{\rho t} 1_{A, t < \tau} \right)$]

$$= \frac{e^{\rho t}}{x} \frac{\partial}{\partial \theta} \left[E_x (e^{-\theta Y_t}) \right] \leftarrow e^{-x u_t(\theta)}$$

$$= + \frac{e^{\rho t}}{x} e^{-x u_t(\theta)} x \frac{\partial u_t(\theta)}{\partial \theta}$$

glance back thro' lecture notes on note we showed

$$\frac{\partial u_t(\theta)}{\partial \theta} = e^{-\int_0^t \psi'(u_{t-s}(\theta)) ds}$$

$$= e^{\rho t} e^{-x u_t(\theta)} - \int_0^t \psi'(u_{t-s}(\theta)) ds$$

$$= e^{-x u_t(\theta)} - \int_0^t (\psi'(u_{t-s}(\theta)) - \rho) ds$$

$$= e^{-x u_t(\theta)} - \int_0^t \phi \circ u_{t-s}(\theta) ds$$

when $\phi(\theta) = \psi'(\theta) - \rho$.

Note: strictly speaking we should check that $\psi'(\theta) - \rho$ is a Laplace exponent of a subordinator!

Recall $\psi(\theta) = a\theta + \frac{1}{2}\theta^2\sigma^2 + \int_{(0, \infty)} (e^{-\theta x} - 1 + \theta x 1_{(x < 1)}) \Pi(dx)$

exercise \Rightarrow using $\int_{(0, \infty)} x^2 \Pi(dx) < \infty$ to pass $\frac{d}{d\theta}$ thro'

$$\psi'(\theta) = \frac{x^2 \theta + \int_{(0, \infty)} (1 - e^{-\theta x}) x \Pi(dx)}{e^{-\theta x} + \psi'(0^+)}$$

Note: $\int_{(0, \infty)} x \Pi(dx) = \int_{(0, \infty)} (x \wedge x^2) \Pi(dx) = \int_{(0, 1)} x^2 \Pi(dx) + \int_{(1, \infty)} x \Pi(dx) < \infty$

because Π is a Levy meas. (for 1st integral) and because $E(\|X\|) < \infty$ (which is a consequence of $\psi'(0^+) \geq 0$) \Rightarrow a.s. $\psi'(0^+) < \infty$

$$\int_{(1, \infty)} x \Pi(dx) < \infty \iff$$

Lemma Suppose φ is a b.i. mech. for a CSBP

Y with same conditions as previous Lemma
 however we impose further restriction $\varphi'(0^+) = 0$
 Write X for the SPLP corresponding to φ

$$(1) \quad \mathcal{L} \left\{ \Theta_t = \inf \left\{ s > 0 : \int_0^s \frac{1}{X_u} du > t \right\} \right\}$$

then $\{X_{\Theta_t} : t \geq 0\}$ under \mathbb{P}_x^φ

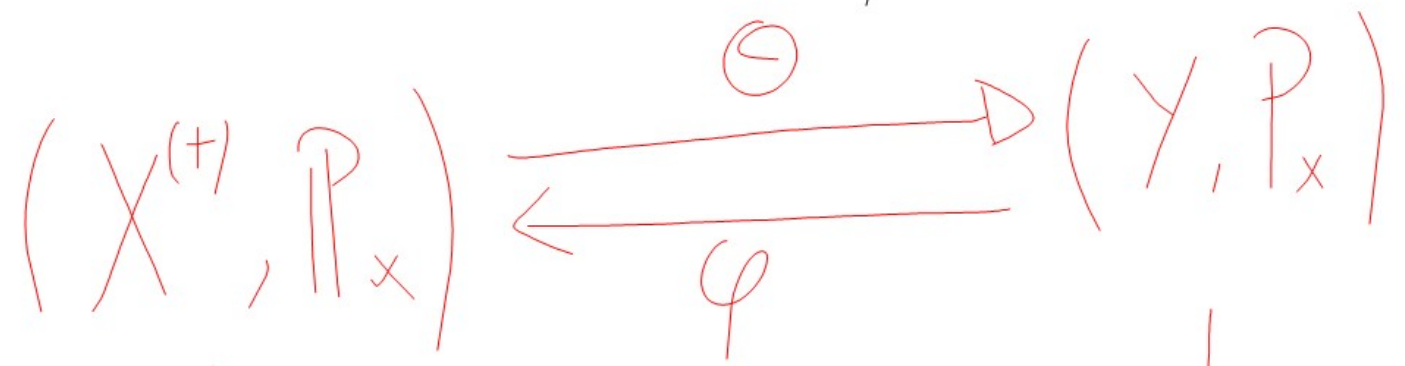
has the same law as $(Y, \mathbb{P}_x^\varphi)$

$$(2) \quad \mathcal{L} \left\{ \varphi_t = \inf \left\{ s > 0 : \int_0^s Y_u du > t \right\} \right\}$$

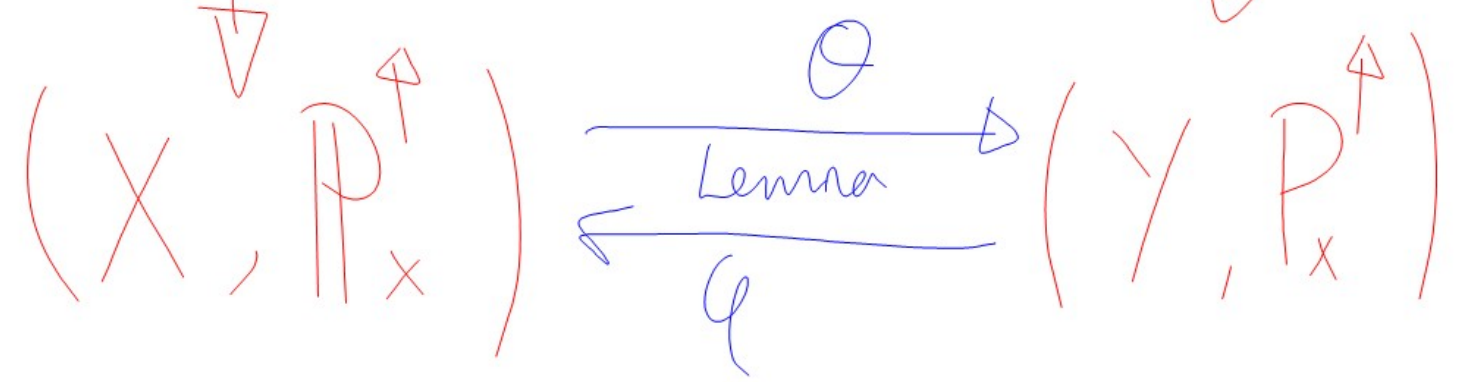
then $\{Y_{\varphi_t} : t \geq 0\}$ under \mathbb{P}_x^φ

is equal in law to $(X, \mathbb{P}_x^\uparrow)$

$$X_t^{(+)} = X_t \mathbb{1}_{(t < \tau;]}$$



\Downarrow h-transform
 $h(x) = x$



Pf (1) Easy to show that Θ_t is a stopping time w.r.t the natural filtration of X we can implement the change of measure density $\frac{X_t}{x}$ also at stopping times:

$$\mathbb{E}_x \left(F(X_{\Theta_s} : s \leq t) \mathbb{1}_{(\Theta_t < \infty)} \right)$$

where F is a "path functional"

$$= \mathbb{E}_x \left(\frac{X_{\Theta_t}}{x} F(X_{\Theta_s} : s \leq t) \mathbb{1}_{(\Theta_t < t_0)} \right)$$

Note that $\{\Theta_t < t_0\} = \{t < \tau\}$

$$= \mathbb{E}_x \left(\frac{Y_t}{x} F(Y_s : s \leq t) \mathbb{1}_{(t < \tau)} \right)$$

$$= \mathbb{E}_x^\uparrow (F(Y_s : s \leq t))$$

Pf of (2) is very similar. 