$$\Psi_t(\theta) = -\log \mathbb{E}(e^{i\theta X_t}) \qquad \forall t \geq 0, \theta \in \mathbb{R}$$

$$X_t = (X_{t/n} - X_0) \oplus \cdots \oplus (X_t - X_{t\frac{(n-1)}{n}})$$

$$\overset{d}{=} X_{t/n} \quad \cdots$$

$$e^{-\Psi_t(\theta)} = \mathbb{E}(e^{i\theta X_t}) = \mathbb{E}(e^{i\theta X_{t/n}})^n = e^{-\Psi_{t/n}(\theta) n}$$

$$\Longrightarrow \quad \Psi_t(\theta) = n \Psi_{t/n}(\theta) \quad \nabla$$

Now take $m, n \in \{1, 2, \ldots\}$

$m = m \times 1$, $m = \frac{m}{n} \times n$

$$\Psi_m(\theta) \begin{cases} = m \Psi_1(\theta) \\ = n \Psi_{m/n}(\theta) \end{cases} \Longrightarrow \Psi_{\frac{m}{n}}(\theta) = \frac{m}{n} \Psi_1(\theta)$$

$$\Psi_t(\theta) = t \Psi_1(\theta) \quad \forall t \in \mathbb{Q}_+$$

$$\mathbb{E}\left(e^{i\theta X_t}\right) = e^{-t\Psi_1(\theta)} \quad \forall t \in \mathbb{Q}_+$$

For $t \geq 0$ & $t_n \downarrow t$: $t_n \in \mathbb{Q}_+$

Right continuity of paths of X & DCT

$$\Rightarrow e^{-\lim t_n \Psi_1(\theta)} \overset{DCT}{=} \mathbb{E}\left(\lim_n e^{i\theta X_{t_n}}\right)$$

$$\overset{RCP}{=} \mathbb{E}\left(e^{i\theta X_t}\right)$$

$$= e^{-\Psi_t(\theta)}$$

$$\Rightarrow \Psi_t(\theta) = t\Psi_1(\theta) \quad \forall t \geq 0$$

<u>Theorem 1.2</u> (Lévy-Khinchine for L-P. leading to <u>Lévy-Itô decomp.</u>)

Given any (a, σ, Π) s.t. $a, \sigma \in \mathbb{R}$, Π is a measure concentrated on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge x^2) \Pi(dx) < \infty$

Then $\exists$ a Lévy process s.t. its dist^n at time one has char. exp

$\Psi_1(\theta) = - - - - - - - - -$ L-K formula

2.1 Poisson process (λ)

$$\mu_t(\{k\}) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$$

$$\mathbb{E}(e^{i\theta N_t}) = \int e^{i\theta x}\mu_t(dx) = \sum_{k\geq 0} e^{-\lambda t} e^{i\theta k}\frac{(\lambda t)^k}{k!}$$

$$= e^{-\lambda t(1-e^{i\theta})}$$

$$= e^{-[\lambda(1-e^{i\theta})]t}$$

$a=0$, $\sigma=0$ $\quad \Pi(dx) = \lambda\,\delta_1(dx)$

Compound Poisson Process (λ, F)

$$\mathbb{E}\left(e^{i\theta\sum_{i=1}^{N_t}\xi_i}\right) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\mathbb{E}\left(e^{i\theta\xi_1}\right)^n$$

$(\xi_i \text{ i.i.d. } F)$

$$= \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\left(\int_{\mathbb{R}} e^{i\theta x}F(dx)\right)^n$$

$$= e^{-\lambda t\left(1-\int_{\mathbb{R}} e^{i\theta x}F(dx)\right)}$$

$$= e^{\boxed{-\lambda t\int_{\mathbb{R}}\left(1-e^{i\theta x}\right)F(dx)}}$$

$\sigma = 0$, $\Pi(dx) = \lambda F(dx)$, a = exercise.

CPP + drift

$$X_t = \sum_{i=1}^{N_t} \xi_i + ct$$

$$\mathbb{E}\left(e^{i\theta X_t}\right) = e^{-\left(-i\theta c + \lambda \int_{\mathbb{R}} (1 - e^{i\theta x}) F(dx)\right) t}$$

take $c = -\lambda \int_{\mathbb{R}} x F(dx)$ $\quad$ (assuming $\mathbb{E}|\xi_1| < \infty$)

$$-ct = \mathbb{E}\left(\sum_{i=1}^{N_t} \xi_i\right)$$

$$\mathbb{E}\left(e^{i\theta X_t}\right) = \exp\left\{-t \boxed{\int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x) \lambda F(dx)}\right\}$$

$$X_t = \sigma B_t + \mu t$$

$$\mathbb{E}\left(e^{i\theta X_t}\right) = e^{i\theta\mu t - \frac{1}{2}\sigma^2\theta^2 t}$$

'cos $X_t \sim N(\mu t, \sigma^2 t)$

The last examples hint at a general Lévy process consisting of linear BM + independent "CPP + drift-type" process

to be made precise

3. Lévy Itô decomposition intuitively

$$\Psi(\theta) = \left\{ i\theta a + \frac{1}{2}\sigma^2\theta^2 \right\}^{(1)}$$

$$+ \left\{ \underbrace{\Pi(\mathbb{R}\setminus(-1,1))}_{\lambda} \int_{|x|\geq 1} \left(1 - e^{i\theta x}\right) \underbrace{\frac{\Pi(dx)}{\Pi(\mathbb{R}\setminus(-1,1))}}_{F} \right\}^{(2)}$$

$$+ \left\{ \int_{|x|<1} \left(1 - e^{i\theta x} + i\theta x\right) \Pi(dx) \right\}^{(3)}$$