$$\Psi(\theta) = \left\{ \underset{\sim}{BM} \right\} + \left\{ \begin{matrix} \underset{\sim}{CPP} \\ \text{large} \\ \text{jumps} \end{matrix} \right\}$$

$$+ \int_{\underline{0<|x|<1}} \left(\underline{1 - e^{i\theta x} + ix\theta} \right) \Pi(dx)$$

3rd integral

$$= \sum_{\underline{n=0}}^{\infty} \left\{ \overbrace{\lambda_n \int_{\underline{2^{-(n+1)} \le |x| < 2^{-n}}} \left(\underline{1-e^{i\theta x}} \right) F_n(dx)}^{\Pi(dx)} + \underline{i\theta} \left(\lambda_n \int_{\underline{2^{-(n+1)} \le |x| < 2^{-n}}} \underline{x}\, F_n(dx) \right) \right\}$$

$$\lambda_n = \Pi\left(\left\{ x : 2^{-(n+1)} \le |x| < 2^{-n} \right\} \right)$$

$$F_n(dx) = \frac{\Pi(dx)}{\lambda_n} \Bigg|_{\{x : 2^{-(n+1)} \le |x| < 2^{-n}\}}$$

$$\lambda_n < \infty \quad \text{and} \quad \int_{2^{-(n+1)} \le |x| < 2^{-n}} x \, F_n(dx) < \infty$$

because we are given a Π s.t.

$$\int_{(-1,1)} \cancel{1 \wedge} \, x^2 \, \Pi(dx) < \infty$$

§4

4.1 square integrable mgs.

$(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t^* : t \geq 0\}, \mathbb{P})$ a filtered pr. space and $\mathbb{F}$ is $\underbrace{\text{right cts}}$ & $\mathbb{P}$-complete

$$\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$$

~~Def^n~~ Fix $\infty > T > 0$, $\mathcal{M}_T^2 = \mathcal{M}_T^2(\Omega, \ldots, \mathbb{P})$ is the space of $\mathbb{R}$-valued zero mean right-cts, $\underbrace{\text{square integrable}}_{\mathbb{E} M_t^2 < \infty \ \forall t \leq T}$ mgs w.r.t $\mathbb{P}, \mathbb{F}$.

With the right inner product, $\mathcal{M}_T^2$ is a Hilbert space

$M, N \in \mathcal{M}^2_T$ (of course $M = \{M_t : t \le T\}$)
same for N.

$$\langle M, N \rangle := \mathbb{E}(M_T N_T)$$

Why is this an i.p.?

$$\langle aM_1 + bM_2, N \rangle \underset{\text{obvious!}}{=} a\langle M_1, N \rangle + b\langle M_2, N \rangle$$

$a, b \in \mathbb{R}$, $M_i, N \in \mathcal{M}^2_T$

$\langle M, M \rangle \geq 0$ obvious!

$$\langle M, N \rangle = \langle N, M \rangle$$

(*) $\langle M, M \rangle = 0 \implies M \equiv \underline{0} = \{M_t = 0 : t \le T\}$

(*) is because of Doob's maximal ineq.

$$\mathbb{E}\left(\sup_{s\le T} M_s^2\right) \le 4\,\mathbb{E}(M_T^2) \quad \text{for } M\in \mathcal{M}_T^2$$

hence $\langle M, M\rangle = 0$

$$\langle M, M\rangle = \mathbb{E} M_T^2$$

$$\overset{D}{\Longrightarrow} \mathbb{E}\left(\sup_{s\le T} M_s^2\right) = 0$$

$$\Longrightarrow \sup_{s\le T} M_s^2 = 0 \text{ a.s.}$$

$$\Longrightarrow M \equiv 0 \text{ a.s.}$$

Why is $\mathcal{M}_T^2$ a Hilbert space with $\langle \cdot, \cdot \rangle$?

need to check Cauchy sequences converge.

Suppose $M^{(n)}$ is a Cauchy seq in $\mathcal{M}_T^2$

i.e. $\|M^{(n)} - M^{(k)}\| \xrightarrow{n,k\uparrow\infty} 0$ (**)

$\|\cdot\| := \langle \cdot , \cdot \rangle^{1/2}$

The fact that (**) holds $\Rightarrow$

$$\left(\mathbb{E}\left|M_T^{(n)} - M_T^{(k)}\right|^2\right)^{1/2} \xrightarrow{k,n\uparrow\infty} 0$$

i.e. $\|M_T^{(n)} - M_T^{(k)}\|_{L^2} \xrightarrow{\quad -\| - \quad} 0$

where $\|\cdot\|_{L^2}$ is the L^2-norm for r.v.s

$\therefore$ because space of L^2-r.v. is a H-space w.r.t. $\langle A,B\rangle_{L^2} = \mathbb{E}(AB)$ then $\exists$ some M_T s.t. $\mathbb{E}M_T^2 \underset{(\dagger)}{<} \infty$ and $\|M_T^{(n)} - M_T\|_{L^2} \xrightarrow{n\uparrow\infty} 0$

i.e. $\mathbb{E}(|M_T^{(n)} - M_T|) \longrightarrow 0$

Now define $M_t = \mathbb{E}(M_T | \mathcal{F}_t^*)$ right cts version

possible cts assumptions on $\mathbb{F}$

Note: $\mathbb{E}M_t^2 \underset{J}{=} \mathbb{E}\left(\mathbb{E}(M_T | \mathcal{F}_t^*)^2\right)$

$\leq \mathbb{E}\mathbb{E}(M_T^2 | \mathcal{F}_t^*)$

$= \mathbb{E}(M_T^2) \underset{(\dagger)}{<} \infty \Rightarrow M \in \mathcal{M}_T^2$ s.t. "$M_T = M_T$"

Hence we can now read the former statement

$$\|M_T^{(n)} - M_T\|_{L^2} \longrightarrow 0$$

as $\|M^{(n)} - M\| \longrightarrow 0$ i.e. M is the limit of C-seq $M^{(n)}$

Want to use the fact that $\mathcal{M}^2_T$ is a Hilbert space in a moment for certain mgs.
Which we describe in next Lemma

Lemma 4.1. Suppose F is a pr. meas s.t.
$\int_{\mathbb{R}} |x| F(dx) < \infty$

(I) $$M_t = \sum_{i=1}^{N_t} \xi_i - \lambda t \int_{\mathbb{R}} x F(dx)$$ where $\xi_i \overset{iid}{\sim} F$
is a mg w.r.t natural filtration $t \leq T$.

(II) If in addition $\int x^2 F(dx) < \infty$ then
M is square integrable mg satisfying $\mathbb{E} M_t^2 = \lambda t \int x^2 F(dx)$

Pf (I) note that M_t = CPP − drift i.e its a Lévy process

$$\mathbb{E}(M_t - M_s \mid \mathcal{Y}_s) = \mathbb{E}(M_{t-s})$$

natural filtration

and the RHS = 0 because

$$\mathbb{E} \sum_{i=1}^{N_t} \xi_i \overset{\text{exercise}}{=} \lambda \int x F(dx)\, t \quad \forall t \geq 0$$

(II)
$$\mathbb{E}(M_t^2) = \mathbb{E}\left(\left(\sum_{i=1}^{N_t} \xi_i\right)^2\right) - \lambda^2 t^2 \left(\int x F(dx)\right)^2$$

$$= \mathbb{E}\left(\sum_{i=1}^{N_t} \xi_i^2\right) + \mathbb{E}\left(\sum\sum \xi_i \xi_j\right) - \lambda^2 t^2 \left(\int x F(dx)\right)^2$$

$$= \lambda t \int x^2 F(dx) + \mathbb{E}(N_t^2 - N_t)\left(\int x F(dx)\right)^2 - \lambda^2 t^2 \left(\int x F(dx)\right)^2$$

$$\overset{\text{exercise}}{=} \lambda t \int x^2 F(dx)$$

Theorem 4.1 Suppose we have a sequence of <u>independent</u> CPP characterised by (λ_n, F_n), $n \geq 1$.

For the CPP corresponding to (λ_n, F_n) construct the associated mg $\{M_t^{(n)} : t \leq T\} =: M^{(n)}$

Δ note we need to assume that $\int |x| F_n(dx) < \infty$ th

If
$$\sum_{n \geq 1} \lambda_n \int x^2 F_n(dx) < \infty$$

then $\exists$ a Lévy process on the product space of all the pr. spaces of each CPP, called X, which is also a square integrable mg and whose char. exponent is given by

$$\forall \theta \in \mathbb{R} \quad \Psi(\theta) = \int_{\mathbb{R}} \left(1 - e^{i\theta x} + i\theta x\right) \left(\sum_{n \geq 1} \lambda_n F_n(dx) \right)$$

and moreover

$$\lim_{k \uparrow \infty} \mathbb{E}\left[\sup_{t \leq T} \left(X_t - \sum_{n=1}^{k} M_t^{(n)} \right)^2 \right] = 0$$