Proof of Prop 4.1

Note by linearity of cond exp. then $\sum_{n=1}^{k} M^{(n)}$ is a mg since each $M^{(k)}$ is a mg.

Claim that for each k $\sum_{n=1}^{k} M^{(n)}$ belongs to a space of sq. int mg:

$$\mathbb{E}\left(\left(\sum_{n=1}^{k} M_t^{(n)}\right)^2\right) = \sum_{n=1}^{k} \mathbb{E}\left[\left(M_t^{(n)}\right)^2\right]$$

because by indep. cross terms involve $\mathbb{E}\left(M_t^{(i)} M^{(j)}\right) = \mathbb{E}\left(M_t^{(i)}\right)\mathbb{E}\left(M_t^{(j)}\right) = 0$

$$\overset{L4.1}{=} \sum_{n=1}^{k} \lambda_n t \int x^2 F_n(dx)$$

$< \infty$ by assumption.

$$X^{(k)} := \sum_{n=1}^{k} M^{(n)}$$

Claim that $\{X^{(k)} : k \geq 1\}$ is a Cauchy sequence.

need to check that $\|X^{(k)} - X^{(\ell)}\| \xrightarrow{k,\ell \uparrow \infty} 0$

But $\|X^{(k)} - X^{(\ell)}\| = \mathbb{E}\left(\left(X_T^{(k)} - X_T^{(\ell)}\right)^2\right)^{1/2}$

assume $\ell < k$

$$= \mathbb{E}\left(\left(\sum_{n=\ell+1}^{k} M_T^{(n)}\right)^2\right)^{1/2}$$

$$\overset{L4.1}{=} \left[T \sum_{n=\ell+1}^{k} \lambda_n \int x^2 F_n(dx)\right]^{1/2}$$

$$\xrightarrow{k,\ell \to \infty} 0 \quad \text{by assumption}$$

Hence we now have the existence of a limit pt

$$X = \{ X_t : t \leq T \}$$ which is then a right-cts square integrable mg.

Thanks to Doob's inequality

$$\lim_{k \uparrow \infty} \mathbb{E}\left(\sup_{t \leq T} \left(X_t - X_t^{(k)} \right)^2 \right) \leq \lim_{k\uparrow\infty} 4\, \mathbb{E}\left(\left(X_T - X_T^{(k)} \right)^2 \right)$$

$$= \lim_{k \uparrow \infty} 4 \| X - X^{(k)} \|^2$$

$$= 0$$

In particular $\Rightarrow$ $X_t^{(k)} \xrightarrow{d} X_t$ for each fixed $t \in [0,T]$

and also $\left(X_{t_1}^{(k)}, \ldots, X_{t_n}^{(k)} \right) \xrightarrow{d} \left(X_{t_1}, \ldots, X_{t_n} \right)$

hence considering

$$\mathbb{E}\left(e^{i\theta(X_t - X_s)}\right) = \lim_{k\uparrow\infty} \mathbb{E}\left(e^{i\theta(X_t^{(k)} - X_s^{(k)})}\right)$$

$T \geq t \geq s \geq 0$

$$= \lim_{k\uparrow\infty} \mathbb{E}\left(e^{i\theta X_{t-s}^{(k)}}\right)$$

$$= \mathbb{E}\left(e^{i\theta X_{t-s}}\right)$$

This shows that mg X is also has s.t. indep. inc. Still need to check that paths are $\boxed{\text{right cts}}$ with left limits.

because X is a limit pt. in space of right. cts mgs!

Easiest way to see left limits is to note that: If $D[0,T]$ the space of right cts, left-limit mappings from $[0,T] \longrightarrow \mathbb{R}$, then this space is complete w.r.t the metric $d(f,g) = \sup_{t \in [0,T]} |f(t) - g(t)|$

$f, g \in D[0,T]$.

Note that on a subsequence k_n

$$d(X, X^{(k_n)})^2 \longrightarrow 0 \quad \text{a.s. i.e. since } X^{(k_n)} \in D[0,T]$$

then so is X by closure w.r.t $d(\cdot,\cdot)$.

Now lets check the char. exponent of X

$$\mathbb{E}\left(e^{i\theta X_t}\right) = \lim_{k\uparrow\infty} \mathbb{E}\left(e^{i\theta X_t^{(k)}}\right) \quad t\in[0,T]$$

$$= \lim_{k\uparrow\infty} \exp\left\{-\int_{\mathbb{R}}\left(1-e^{i\theta x}+i\theta x\right)\sum_{n=1}^{k}\lambda_n F_n(dx)\right\}$$

$$= \exp\left\{-\int_{\mathbb{R}}\underbrace{\left(1-e^{i\theta x}+i\theta x\right)}_{O(x^2)}\sum_{n=1}^{\infty}\lambda_n F_n(dx)\right\}$$

In passing limit thro' integral, we used $\int x^2 \sum^{\infty}\lambda_n F_n(dx) < \infty$

Last part of pf is to show that limit is indep. of T. Suppose we index limit by T: X^T

$T_1 < T_2$

$$\mathbb{E}\left(\sup_{t \le T_1}\left(X_t^{T_1} - X_t^{T_2}\right)^2\right)^{1/2}$$

$$\le \mathbb{E}\left(\sup_{t \le T_1}\left(X_t^{T_1} - X_t^{(k)}\right)^2\right)^{1/2} \quad \forall k$$

$$+ \mathbb{E}\left(\sup_{t \le T_1}\left(X_t^{T_2} - X_t^{(k)}\right)^2\right)^{1/2} \longrightarrow 0$$

$$\Rightarrow \sup_{t \le T_1}\left(X_t^{T_1} - X_t^{T_2}\right)^2 = 0 \quad a.s.$$

$$\Rightarrow X^{T_1} \equiv X^{T_2}.$$

$\square$