Proof of Lemma 1

Note by linearity of $\sum$ and expectation $E$, then $\sum_{n=1}^{k} M(n)$ is a mg. Since each $M(k)$ is a mg, then $\sum_{n=1}^{k} M(n)$ is a mg.

Claim that for each $k$, $\sum_{n=1}^{k} E[M_{t}^{(n)}] = \sum_{n=1}^{k} E[M_{t}^{(n)}]$

because by independence cross terms involve $E[M_{t}^{(i)} M_{t}^{(j)}] = E(M_{t}^{(i)}) E(M_{t}^{(j)}) = 0$

$\sum_{n=1}^{k} \lambda_{t} \int x^{2} F_{n}(dx) < \infty$ by assumption.
Claim that \( \{X^{(k)} : k \geq 1\} \) is a Cauchy sequence.

need to check that \( \|X^{(k)} - X^{(l)}\| \xrightarrow{k, l \to \infty} 0 \)

But \( \|X^{(k)} - X^{(l)}\| = \mathbb{E} \left( \left( X^{(k)}_T - X^{(l)}_T \right)^2 \right)^{1/2} \)

Assume \( l < k \)

\[
= \mathbb{E} \left( \left( \sum_{n = l+1}^{k} M_{n}^{(n)} \right)^2 \right)^{1/2}
\]

\[l.4.1\]
\[
= \left[ T \sum_{n = l+1}^{k} \lambda_n \int x^2 F_n (dx) \right]^{1/2}
\]

\(k, l \to \infty \xrightarrow{} 0\) by assumption
Hence we now have the existence of a limit \( \mathbf{X} = \{ X_t : t \leq T \} \) which is then a right-continuous square integrable mg.

Thanks to Doob's inequality,

\[
\lim_{k \to \infty} \mathbb{E} \left( \sup_{t \leq T} (X_t - X_t^{(k)})^2 \right) \leq \lim_{k \to \infty} 4 \mathbb{E} \left( (X_T - X_T^{(k)})^2 \right) = \lim_{k \to \infty} 4 \| X - X^{(k)} \|_2^2 = 0
\]

In particular \( \Rightarrow X_t^{(k)} \to_d X_t \) for each fixed \( t \in [0, T] \) and also \( (X_t^{(k)}_{t_1}, \ldots, X_t^{(k)}_{t_n}) \to_d (X_{t_1}, \ldots, X_{t_n}) \)
hence considering

\[
E(e^{i \theta (X_t - X_s)}) = \lim_{k \to \infty} E(e^{i \theta (X_t^{(k)} - X_s^{(k)})})
\]

\[
= \lim_{k \to \infty} E(e^{i \theta X_t^{(k)}})
= E(e^{i \theta X_t})
\]

This shows that my $X$ is also has S.t. indep.

Incr. Still need to check that paths are right

It's with left limits.

Because $X$ is a limit pt. in space of right. At mgs!
Easiest way to see left limits is to note that:

$\mathds{A} = \mathcal{D}[0, T]$, the space of right-continuous, left-limit

rappings from $[0, T] \rightarrow \mathbb{R}$, then this space is complete

w.r.t. the metric $d(f, g) = \sup_{t \in [0, T]} |f(t) - g(t)|$

for $f, g \in \mathcal{D}[0, T]$.

Note that on a subsequence $k_n$

\[ d\left( X, X^{(k_n)} \right)^2 \rightarrow 0 \quad \text{a.s. i.e. since } X^{(k_n)} \in \mathcal{D}[0, T] \]

then so is $X$ by closure of $\mathcal{D}[0, T]$.

Now let's check the char. exponent of $X$. 

\[ E(e^{i \theta X_t}) = \lim_{k \to \infty} E(e^{i \theta X_t^k}) \quad t \in [0, T] \]

\[ = \lim_{k \to \infty} \exp \left\{ - \int_{\mathbb{R}} (1-e^{i \theta x} + i \theta x) \sum_{n=1}^{k} \lambda_n F_n(dx) \right\} \]

\[ = \exp \left\{ - \int_{\mathbb{R}} (1-e^{i \theta x} + i \theta x) \sum_{n=1}^{\infty} \lambda_n F_n(dx) \right\} \]

\[ = \exp \left\{ - \int_{\mathbb{R}} O(x^2) \sum_{n=1}^{\infty} \lambda_n F_n(dx) \right\} \]

In passing limits two integrals, we used \( \int x^2 \sum_{n=1}^{\infty} \lambda_n F_n(dx) < \infty \)

Last part of proof is to show that limit is independent of \( T \). Suppose we index limit by \( T \):

\[ X^T \]
\[ T_1 < T_2 \]
\[ E \left( \sup_{t \leq T_1} (X_t^{T_1} - X_t^{T_2})^2 \right)^{1/2} \leq E \left( \sup_{t \leq T_1} (X_t^{T_1} - X_t^{(k)})^2 \right)^{1/2} \forall k \]
\[ + E \left( \sup_{t \leq T_1} (X_t^{T_2} - X_t^{(k)})^2 \right)^{1/2} \rightarrow 0 \]

\[ \sup_{t \leq T_1} (X_t^{T_1} - X_t^{T_2})^2 = 0 \quad \text{a.s.} \]
\[ \Rightarrow X^{T_1} = X^{T_2} \quad \square \]