Proof of the Hévy-Skid decomposition

For given \((a, \sigma, \Pi)\) — we already know how to deal with \(a, \sigma\) and \(\Pi \mid x \mid \geq 1\) for \(\Pi \mid x \mid < 1\), we just define

\[
A_n = \Pi(\{x : 2^{-n} \leq |x| < 2^{-n+1}\})
\]

\[
F_n = \prod \{x : 2^{-m} |x| < 2^{-n}\}
\]

Apply directly Theorem 4.1.
Think of \( An F \), \( (dx) = \Pi (dx) \) \{..\} 

The rate at which jumps arrive of size 

\[ 2^{-\infty} \leq |x| < 2^{-n} \]

Note also that the Lévy–Doob decomposition is "overkill" if, for example, \( \Pi (\mathbb{R}) < \infty \).

Because:

\[
\int (1 - e^{i\Theta x} + i \Theta x \mathbb{1}_{|x| < 1}) \Pi (dx) \\
= i \Theta \int \mathbb{1}_{|x| < 1} \Pi (dx) + \Pi (\mathbb{R}) \int (1 - e^{i\Theta x}) \frac{\Pi (dx)}{\Pi (\mathbb{R})}
\]
Recall
\[ \Phi(x) = i \Theta \xi + \frac{1}{2} \sigma^2 \xi^2 + \int \left( 1 - e^{-i\Theta x} + i\Theta x \right) 1_{|x| \leq 1} \Pi(dx) \]
\[ \sim O(\sigma^2), \; |x| \text{ small} \]

\[ = i \Theta \xi + \frac{1}{2} \sigma^2 \xi^2 + \int \left( 1 - e^{-i\Theta x} + i\Theta x \right) 1_{(-\kappa < x < \kappa)} \Pi(dx) \]

\[ \int_{-\kappa}^{\kappa} \Pi(dx) < \infty \]

providing \( B \) bounded and \( D \in \overline{B} \)
5 Path Variation

BM has a.s. paths of UBV.

If $f : [a, b] \to \mathbb{R}$

$$f^c(x) = f(x) - \sum_{y \leq x} \Delta f(y) : \text{for BV f}$$

$$V(f : [a, b]) = V(f^c : [a, b]) + \sum_{[a, b]} |\Delta f|$$

If a Lévy process $X$ contains a BM (i.e. $\sigma \neq 0$)

then $X^1$ has paths of UBV.
Revisit the proof of Lévy-Itô decomposition.

Recall \( X_t = \sum_{n=1}^{k} M_t^{(n)} = \sum_{n=1}^{k} (M_t^{(n+1)} - M_t^{(n-1)}) \)

Define for each \( n \)

\[
M_{t}^{(n, \pm)} = \sum_{i=1}^{n} \begin{cases} W_t^{(n)}( \xi_i ) & (\xi_i < 0) \\ -t \int_{(0,\infty)} |x| \bar{F}_n(dx) & (\xi_i \geq 0) \end{cases}
\]

Note by Colouring Theorem of Poisson processes

\( M_{(n, +)} \perp M_{(n, -)} \)
\[
X_t^{(k, \pm)} = \sum_{n=1}^{K} W_n^{(a, \pm)} \\
\text{So } X_t^{(k)} = X_t^{(k, +)} - X_t^{(k, -)} \\
C_t^{(k, \pm)} = X_t^{(k, \pm)} + t \int _{\mathbb{R}} |x| \sum_{n=1}^{K} F_n (dx) \\
= \sum_{n=1}^{K} \sum_{i=1}^{M_n} I(\exists \xi_i > 0) + \sum_{n=1}^{K} \sum_{i=1}^{M_n} I(\exists \xi_i < 0) \\
\]

Note that both \( C_t^{(k, \pm)} \) are increasing in \( k \) and hence a.s. limit exist for both.

Note also that, by the same \( L^2 \)-limiting procedure as before, we know that \( X_t^{(k, \pm)} \to X_t^{(\pm)} \).
Considering \( \psi \) it follows that

\[
\lim_{k \to \infty} C_t^{(k, t)} < \infty \quad \text{a.s.}
\]

\[
\int_{\mathbb{R}} |x| \sum_{n=0}^{\infty} \lambda_n F_n (dx) < \infty
\]

\( (0, \infty) \)

\( (-\infty, 0) \)

Hence

\[
\sum_{n=1}^{k} \sum_{i=1}^{N_i^{(n)}} z_i = C_t^{(k, t+1)} - C_t^{(k_t-1)}
\]

since

There is absolute convergence as \( k \to \infty \) on LHS

\[
\int_{\mathbb{R}} |x| \sum_{n=1}^{\infty} \lambda_n F_n (dx) < \infty
\]

\[
\int_{\mathbb{R}} |x| \prod_{i=1}^{N_i} (dx) < \infty
\]

(rememvering that \( \lambda_n, F_n \) are chosen in L-1 decump.)
Hence if \( \int_{-1}^{1} \Pi(x) \, dx < \infty \)

\((-1,1)\)

Then \( X_t = X_t^{(+)} - X_t^{(-)} \) exists

\[
= \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} N_t^{(i)} \delta_i - \int_{-1}^{1} \Pi(x) \, dx \]

But if \( \int_{-1}^{1} \Pi(x) \, dx = \infty \)

\((-1,1)\)

Then at least one of the \( C^{(k, \pm)} \) blows up in that case

\[
\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} N_t^{(i)} \delta_i \] is not abs. cgs.

In conclusion in first case \( X \) has BV

in second case \( X \) has UBV
Lemma 5.1

Any Lévy process has paths of $BV$ if

$$\int (x^2 \Pi(dx)) < \infty$$

and

$$\int |x| \Pi(dx) < \infty$$

Note

$$\int (x^2 \Pi(dx)) < \infty \implies \int |x| \Pi(dx) < \infty$$

(-1, 1)

(-1, 1)
Moreover then \( \int |x| \Pi(dx) < \infty \) and \( a = 0 \)

\[
X_t = -at + \sum_{s \leq t} \Delta X_s - \int_{(-1,1)} x \pi(dx) t
\]

\[
\Psi(\theta) = ai\theta + \int (1 - e^{ix} + i\theta x 1_{|x| < 1}) \pi(dx)
\]

Within the class of BV Lévy processes are the subordinates, i.e. Lévy processes with non-decreasing paths

i.e. \( X_t = \sum_{s \leq t} \Delta X_s + ft \) where \( \Delta X_t > 0 \) \( \forall s > 0 \)
Lemma 5.2

A Lévy process is a subordinator if it has paths of BV

\[ \Pi(-\infty, 0) = 0 \]

and

\[ \mathbb{E}(\theta) = -i \Theta + \int_{(0, \infty)} (1 - e^{i\theta x}) \Pi(dx) \]

for some \( \theta \)
Last remark:

For future reference we will always call a Lévy process $X$, spectrally negative if

$$\Pi(0, \infty) = 0$$

and $-X$ is not a subordinator.

**Exercise:** Prove that if $X$ is a SNLT of BV, then necessarily

$$X_t = ct - \sum_{\delta_t \leq t} |\Delta X_\delta|$$

for $c > 0$.