Proof of the Lévy–Itô decomposition

For given (a, σ, Π) — we already know how to deal with a, σ and $\Pi|_{|x|\geq 1}$

for $\Pi|_{|x|<1}$, we just define

$$\lambda_n = \Pi\left(\left\{x : 2^{-(n+1)} \leq |x| < 2^{-n}\right\}\right)$$

$$F_n = \underbrace{\Pi}_{\lambda_n}\Big|_{\{x : 2^{-(n+1)} \leq |x| < 2^{-n}\}}$$

Apply directly Theorem 4.1.

Think of $\lambda_n F_n(dx) = \Pi|_{\{\dots\}}(dx)$.

The rate at which jumps arrive of size $2^{-(n+1)} \leq |X| < 2^{-n}$.

Note also that the Lévy-Itô decomposition is "overkill" if, for example, $\Pi(\mathbb{R}) < \infty$

Because:

$$\int \left(1 - e^{i\theta x} + i\theta x \mathbb{1}_{|x|<1}\right) \Pi(dx)$$

$$= i\theta \int_{|x|<1} x \, \Pi(dx) + \Pi(\mathbb{R}) \int \left(1 - e^{i\theta x}\right) \frac{\Pi(dx)}{\Pi(\mathbb{R})}$$

Recall

$$\Psi(\theta) = i\theta a + \tfrac{1}{2}\sigma^2\theta^2 + \int \big(\underbrace{1 - e^{i\theta x} + i\theta x}_{O(x^2)} \mathbb{1}_{|x|\le 1}\big)\,\Pi(dx)$$

$|x|$ small

$$= i\theta a' + \tfrac{1}{2}\sigma^2\theta^2 + \int \big(1 - e^{i\theta x} + i\theta x\, \mathbb{1}_{(-\alpha < x < \beta)}\big)\,\Pi(dx)$$

$$\int_B x\,\Pi(dx) < \infty$$

providing B bounded and $0 \in \bar{B}$

5 Path variation

BM has a.s. paths of UBV.

If $f : [a,b] \longrightarrow \mathbb{R}$

$$f^c(x) = f(x) - \sum_{y \le x} \Delta f(y) \quad : \text{ for BV } f.$$

$$V(f : [a,b]) = V(f^c : [a,b]) + \sum_{[a,b]} |\Delta f|$$

If a Lévy process X contains a BM (ie $\sigma \neq 0$)
then X has paths of UBV.

Revisit the proof of Lévy-Itô decomposition.

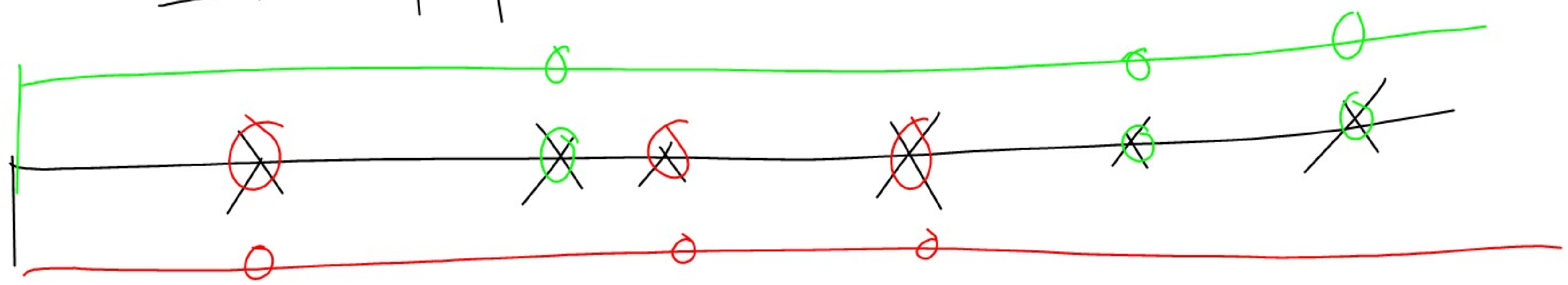
Recall $X_t^{(k)} = \sum_{n=1}^{k} M_t^{(n)} = \sum_{n=1}^{k} \left(M_t^{(n,+)} - M_t^{(n,-)} \right)$

Define for each n

$$M_t^{(n,\pm)} = \sum_{i=1}^{N_t^{(n)}} |\xi_i| \, \overset{(\xi_i<0)}{\mathbb{1}(\xi_i>0)} - t \int_{\substack{(0,\infty)\\(-\infty,0)}} |x| \lambda_n F_n(dx)$$

Note by Colouring Theorem of Poisson processes

$$M^{(n,+)} \perp\!\!\!\perp M^{(n,-)}$$



$$X_t^{(k,\pm)} = \sum_{n=1}^{k} M^{(n,\pm)}$$

So
$$X_t^{(k)} = X_t^{(k,+)} - X_t^{(k,-)}$$

$$C_t^{(k,\pm)} \overset{\otimes}{=} X_t^{(k,\pm)} + t \int_{\substack{(0,\infty)\\(-\infty,0)}} |x| \sum_{n=1}^{k} \lambda_n F_n(dx)$$

$$= \sum_{n=1}^{k} \sum_{i=1}^{N_t^{(n)}} |\xi_i| \, \mathbb{1}_{\substack{(\xi_i > 0)\\(\xi_i < 0)}}$$

Note that both $C_t^{(k,\pm)}$ are increasing in k and hence a.s. limits exist for both.

Note also that by the same L^2-limiting procedure as before we know that $X^{(k,\pm)} \xrightarrow{\|\cdot\|} X^{(\pm)}$

Considering ⊛ it follows that

$$\lim_{k\uparrow\infty} C_t^{(k,\pm)} < \infty \quad a.s.$$

$$\Longleftrightarrow \int_{\substack{(0,\infty) \\ (-\infty,0)}} |x| \sum_{n=0}^{\infty} \lambda_n F_n(dx) < \infty$$

Hence since $\displaystyle\sum_{n=1}^{k} \sum_{i=1}^{N_t^{(n)}} \xi_i = C_t^{(k,+)} - C_t^{(k,-)}$

There is absolute convergence as $k\uparrow\infty$ on LHS

$$\Longleftrightarrow \int_{\mathbb{R}} |x| \sum_{n\geq 1} \lambda_n F_n(dx) < \infty$$

$$\Longleftrightarrow \int_{(-1,1)} |x| \, \Pi(dx) < \infty$$

(remembering how λ_n, F_n are chosen in L-I decomp.)

Hence if $\int_{(-1,1)} |x| \, \Pi(dx) < \infty$

then $X_t = X_t^{(+)} - X_t^{(-)}$ exists

$$= \sum_{k=1}^{\infty} \sum_{i=1}^{N_t^{(n)}} \xi_i - \int_{(-1,1)} x \, \Pi(dx) t$$

But if $\int_{(-1,1)} |x| \, \Pi(dx) = \infty$

Then at least one of the $C^{(k,\pm)}$'s blows up in that case $\sum_{k=1}^{\infty} \sum_{i=1}^{N_t^{(n)}} \xi_i$ is not abs. cgt.

In conclusion in first case X has BV

in second case X has UBV

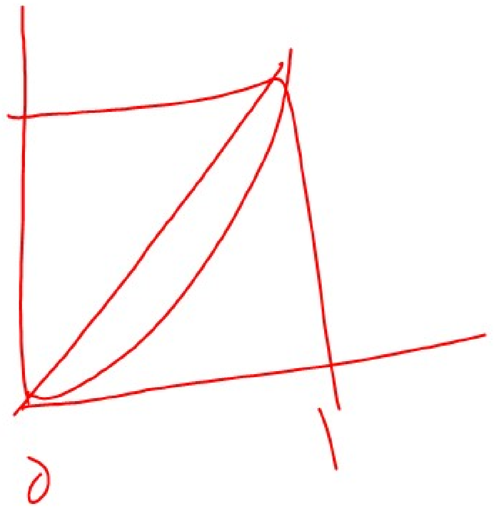
Lemma 5.1

Any Lévy process has paths of BV $\iff$

$$\sigma = 0 \quad \text{and} \quad \int_{(-1,1)} |x| \, \Pi(dx) < \infty$$

Note

$$\int_{(-1,1)} x^2 \, \Pi(dx) < \infty \;\overset{\text{necessarily}}{\not\Longrightarrow}\; \int_{(-1,1)} |x| \, \Pi(dx) < \infty$$



Moreover when $\int_{|x|<1} |x| \, \Pi(dx) < \infty$ and $\sigma = 0$

$$X_t = -at + \boxed{\sum_{s \le t} \Delta X_s} - \int_{(-1,1)} x \, \Pi(dx) \, t$$

$$\Psi(\theta) = a i \theta + \int \left(\boxed{1 - e^{i\theta x}} + i\theta \boxed{x \mathbb{1}_{|x|<1}} \right) \Pi(dx)$$

Within the class of BV Lévy processes are the subordinators i.e. Lévy processes with non-decreasing paths

i.e. $X_t = \sum_{s \le t} \Delta X_s + \delta t$ where $\Delta X_t > 0$, $\delta \ge 0$

Lemma 5.2

A Lévy process is a subordinator

$\Longleftrightarrow$ it has paths of BV $\not\Rightarrow \int_{(0,\infty)} (x \wedge 1)\, \Pi(dx) < \infty$

$\Pi(-\infty, 0) \equiv 0$

and

$$\to \Psi(\theta) = -i\delta\theta + \int_{(0,\infty)} \left(1 - e^{i\theta x}\right) \Pi(dx)$$

for some $\delta \geq 0$

Last remark:

For future reference we will always call a Lévy process, X, spectrally negative if $\Pi(0,\infty) = 0$ and $-X$ is not a subordinator.

Exercise: Prove that if X is a SNLP of BV then necessarily $X_t = ct - \sum_{s \le t} |\Delta X_s|$ for $c > 0$