6. Duality Lemma

For each fixed $t \geq 0$, define the reversed process

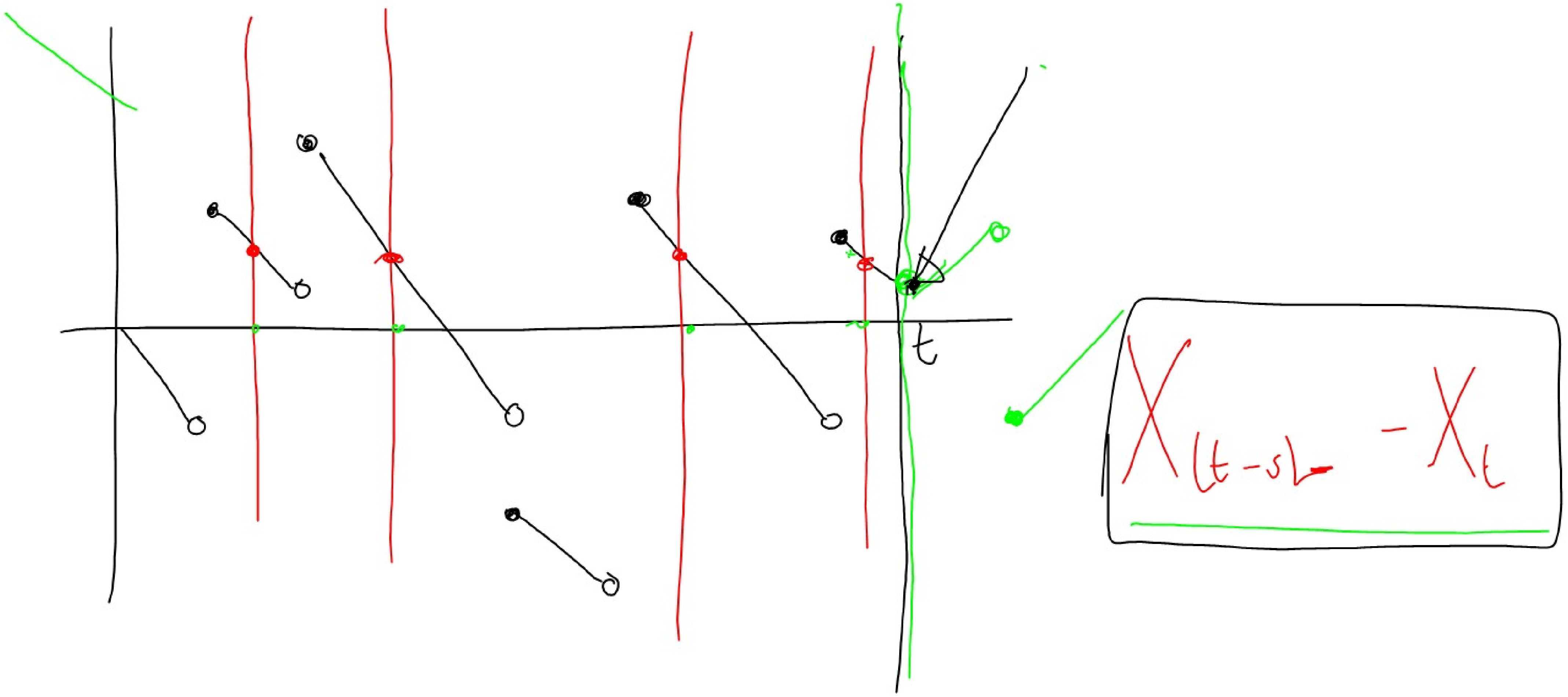
$$\{ X_{(t-s)-} - X_t \; : \; 0 \leq s \leq t \}$$

where X is any L.P. Define in addition the dual process

$$\{ -X_s \; : 0 \leq s \leq t \}$$

Then they are equal in law.

Pf



$$X_{(t-s)-} - X_t \overset{d}{=} -X_s$$

obvious by st. incr. of X and $(t-s)$ cannot be a jump time w.p. 1.

Now introduce $\overline{X}_t := \sup_{s \le t} X_s$

and $\underline{X}_t := \inf_{s \le t} X_s$

<u>Lemma 6.2</u> For each fixed $t > 0$.

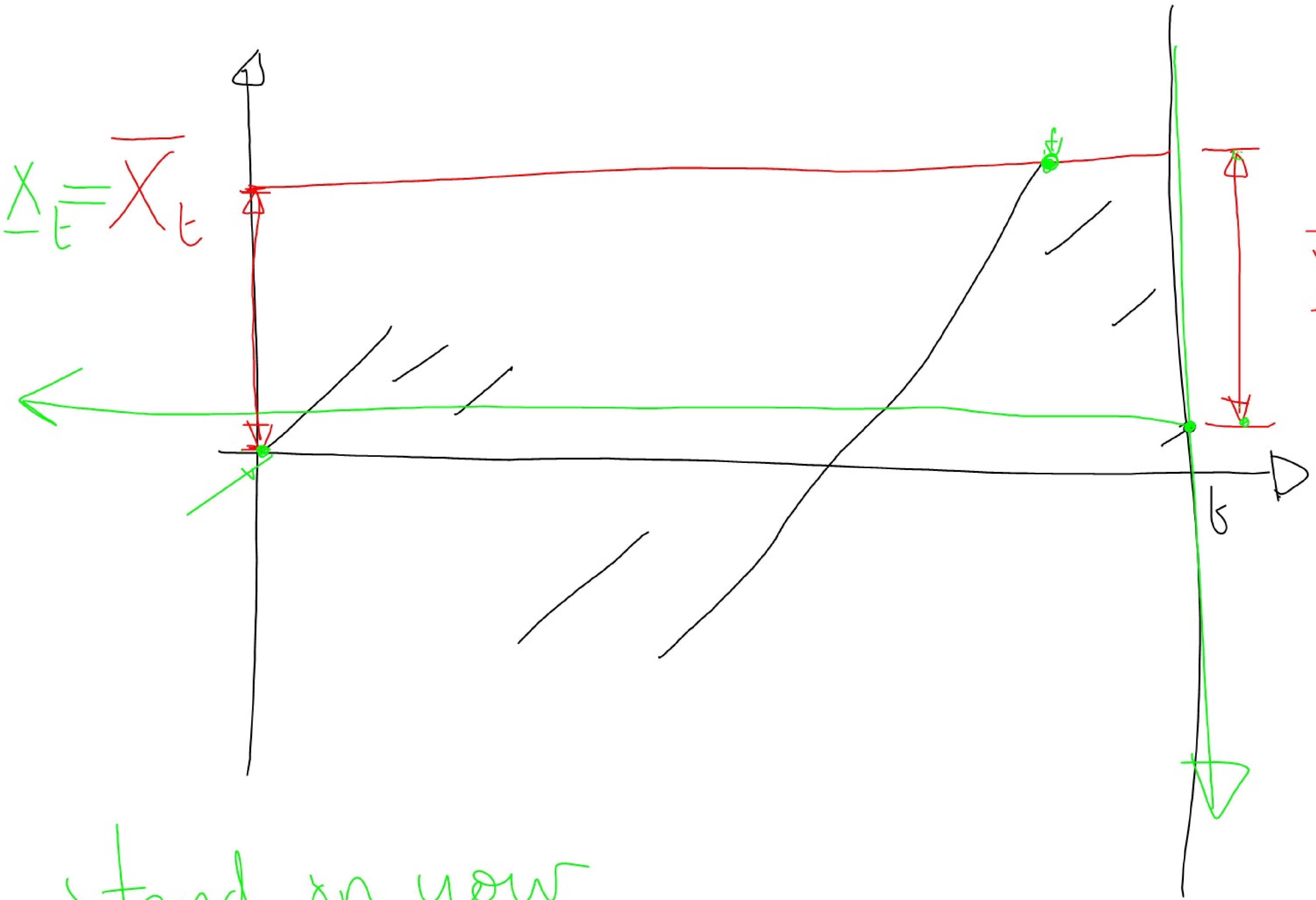
$$\left(\overline{X}_t , \overline{X}_t - X_t \right) \stackrel{d}{=} \left(X_t - \underline{X}_t , -\underline{X}_t \right)$$

From Duality Lemma

$\{ X_s : 0 \le s \le t \}$ can be obtained <u>in law</u> by time-reversing and reflecting about the time axis

(note last two transformations (time reversal & reflection = rotating by $180°$!))

$X_t - \underline{X}_t = \overline{X}_t$



$\overline{X}_t - X_t \overset{d}{=} \underline{X}_t$

stand on your head.

7. Laplace exponents of processes (with one sided jumps)

We are interested in conditions for which

$$\mathbb{E}\left(e^{\beta X_t}\right) \text{ is well defined for } \beta \in \mathbb{R}$$

$$\parallel$$

$$\int_{\mathbb{R}} e^{\beta x}\, \mathbb{P}(X_t \in dx)$$

$$\mathbb{E}\left(e^{\beta Y}\right) = \int e^{\beta y} \underbrace{\mathbb{P}(Y \in dy)}_{\underset{y \uparrow \infty}{\sim} e^{-qy}\, dy \quad 0 < q < \beta < \infty}$$

Theorem 7.1. Let $\beta \in \mathbb{R}$

Then

$$\mathbb{E}\left(e^{\beta X_t}\right) < \infty \quad \forall t \geq 0$$

$$\Longleftrightarrow \quad \int_{|x| \geq 1} e^{\beta x} \, \Pi(dx) < \infty$$

where X is a L.P. with jump measure Π.

Proof ($\Longleftarrow$)

Recall Lévy-Itô decomposition

$$X = X^{(1)} + X^{(2)} + X^{(3)}$$

linear BM

CPP with jump at rate $\lambda := \Pi(\{x : |x| \geq 1\})$ and jump distⁿ $F(dx) := \dfrac{\Pi_{\{|x|\geq 1\}}(dx)}{\lambda}$

superposition of upto a countably infinite # of compensated CPPs.

$$\Psi^{(3)}(\theta) = \int_{(-1,1)} (1 - e^{i\theta x} + i\theta x)\, \Pi(dx)$$

It suffices to check that $\mathbb{E}(e^{\beta X_t^{(i)}}) < \infty \ \forall t \geq 0$ for $i = 1, 2, 3$, by indep of $X^{(i)}$s

$$\mathbb{E}(e^{\beta X_t^{(1)}}) < \infty \quad \forall t \geq 0 \quad \text{obvious!}$$

$$-\log \mathbb{E}(e^{i\theta X_t^{(3)}}) = \underbrace{\int_{(-1,1)} (1 - e^{i\theta x} + i\theta x) \Pi(dx)}_{\text{analytically extendable to } \mathbb{C}}$$

$$\int_{(-1,1)} (\ldots\ldots) \Pi(dx) = \int_{(-1,1)} \sum_{k \geq 0} \frac{(i\theta x)^{k+2}}{(k+2)!} \Pi(dx)$$

want to exchange sum & integral using Fubini

check:
$$\sum_{k \geq 0} \int_{(-1,1)} \frac{|\theta x|^{k+2}}{(k+2)!} \Pi(dx)$$

$$= \sum_{k \geq 0} \frac{|\theta|^{k+2}}{(k+2)!} \int_{(-1,1)} |x|^{k+2} \Pi(dx)$$

$$\leq \text{—|—} \int_{(-1,1)} x^2 \Pi(dx)$$

$$< \infty$$

Hence can interchange sum & integral showing $\Psi^{(3)}$ is a cgt. power series in θ for $\forall \theta \in \mathbb{C}$ i.e. $\Psi^{(3)}$ is analytic on $\mathbb{C}$

Hence $\mathbb{E}(e^{i(-i\beta) X_t}) < \infty \quad \forall t \geq 0$

$$= \mathbb{E}(e^{\beta X_t})$$

To finish note that for any (λ, F) CPP, Y_t, when it exists

$$\mathbb{E}\left(e^{\beta Y_t}\right) = e^{-\lambda t} \sum_{k \geq 0} \frac{(\lambda t)^k}{k!} \int e^{\beta x} F^{*k}(dx)$$

Note

$$\int e^{\beta x} F^{*k}(dx)$$

$$= \left(\int e^{\beta x} F(dx) \right)^k$$

hence $\int e^{\beta x} F(dx) < \infty$ is a suff condition.