6. Duality Lemma

For each fixed $t > 0$, define the reversed process

$$\{X_{(t-s)} - X_t : 0 \leq s \leq t\}$$

where $X$ is any L.P. Define in addition the dual process

$$\{-X_s : 0 \leq s \leq t\}$$

then they are equal in law.
\[ X_{(t-s)} - X_t = d - X_s \]

Obviously by st. mcr. of \( X \) and \( (t-s) \) cannot be a jump time d.p. 2.
Now introduce $\bar{X}_t := \sup_{s \leq t} X_s$

and $\underline{X}_t := \inf_{s \leq t} X_s$

Lemma 6.2 For each fixed $t > 0$

$$(\bar{X}_t, \bar{X}_t - X_t) \overset{d}{=} (X_t - \underline{X}_t, -\underline{X}_t)$$

From Duality Lemma

$\{X_s : 0 \leq s \leq t\}$ can be obtained in law by

due-reversing and reflecting about the time axis

(note last two transformations (time-reversal & reflection = rotating by $180^\circ$ !)
Stand on your head.
7. Laplace exponents of processes
(with one sided jumps)

We are interested in conditions for which

\[ E(e^{\beta X_t}) \text{ is well defined for } \beta \in \mathbb{R} \]

\[ \int_{\mathbb{R}} e^{\beta x} \mathbb{P}(X_t \in dx) \]

\[ E(e^{\gamma Y}) = \int e^{\gamma y} \mathbb{P}(Y \in dy) \]

\[ \sim e^{-\beta y} \quad dy \quad 0 < \gamma < \beta < \infty \]
Theorem 7.1. Let $\beta \in \mathbb{R}$

Then

$$E(e^{\beta X_t}) < \infty \quad \forall t \geq 0$$

$$\iff \int e^{\beta x} \Pi(dx) < \infty$$

where $X$ is a L.P. with jump measure $\Pi$. 

Proof (\rightarrow)

Recall Lévy–Itô's decomposition

\[ X = X^{(1)} + X^{(2)} + X^{(3)} \]

- Linear BM
- CPP with jump at rate \( \lambda \)
- \( \Pi = \prod \{ x \mid x \geq 1 \} \)
  and jump \( \text{d}x \) \( \text{d}x \)

\[ F(dx) = \prod_{x \geq 1} (dx) \]

\[ \Psi^{(3)}(\theta) = \int_{(-1,1)} (1 - e^{i\theta x} + i \theta x) \Pi(dx) \]
It suffices to check that $E(e^{i\beta X_t}) < \infty \forall t > 0$ for $i = 1, 2, 3,$ by independence $X(i)_s$

$E(e^{i\beta X_t}) < \infty \forall t > 0$ obvious!

$-\log E(e^{i\beta X_t}) = \int_{(-1,1)} \frac{1}{i\beta} x \Pi(dx)$

analytically extendable to $C$

\[\int_{(-1,1)} \Pi(dx) = \int_{(-1,1)} \sum_{k \geq 0} \frac{(i\beta x)^{k+2}}{(k+2)!} \Pi(dx)\]

Want to exchange sum & integral using Fubini\check{}:

\[\sum_{k \geq 0} \int_{(-1,1)} \frac{1}{(k+2)!} (i\beta x)^{k+2} \Pi(dx)\]

\[= \sum_{k \geq 0} \frac{1}{(k+2)!} \int_{(-1,1)} x^k \Pi(dx)\]

\[\leq \frac{1}{\beta} \int_{(-1,1)} x^2 \Pi(dx)\]

\[< \infty\]

Hence can interchange sum & integral.

Shaping $\Psi(t)$ is a C.Ag. process in 0 for $\forall \Theta \in \mathcal{F}$, i.e. $\Psi(t)$ is analytic in $C$.

Hence $E(e^{i(-\beta)X_t}) < \infty \forall t > 0$.

$E(e^{i\beta X_t})$
To finish note that for any $(A, F)$ CPE, when it exists
\[
E(e^{\beta x Y_t}) = e^{-2t} \sum_{k=0}^{\infty} \frac{(2t)^k}{k!}
\]

Note
\[
\int e^{\beta x} F^k(dx) ^k
= \left( \int e^{\beta x} F(dx) \right)^k
\]

then
\[
\int e^{\beta x} F(dx) < \infty \text{ is a SLLF condition.}
\]