7.1 Subordinators

If X is a subordinator then

$$\Psi(\theta) = -i\delta\theta + \int_{(0,\infty)} (1 - e^{i\theta x})\,\Pi(dx), \quad \theta \in \mathbb{R}$$

where necessarily $\delta \geq 0$, $\int_{(0,\infty)} (1 \wedge x)\,\Pi(dx) < \infty$.

Moreover, we can write

$$X_t = \delta t + \sum_{s \leq t} \Delta X_s$$

where $\Delta X_s = X_s - X_{s-}$

According to theorem 7.1

hence $\mathbb{E}\left(e^{-\lambda X_t}\right) = \mathbb{E}\left(e^{\lambda(-X_t)}\right)$

where $\lambda \geq 0$ and jump measure associated with $-X$ assignes zero mass to $(0,\infty)$

Hence "$\int_{|x|\geq 1} e^{\lambda x} \Pi_{(-X)}(dx) < \infty$"

and thus $\mathbb{E}\left(e^{-\lambda X_t}\right) < \infty \quad \forall\, t \geq 0.$

By analytic extension we can also write

$$\mathbb{E}\left(e^{-\lambda X_t}\right) = \exp\left\{-\left(\lambda\delta + \int_{(0,\infty)} \left(1-e^{-\lambda x}\right) \Pi(dx)\right)t\right\}$$

In the future it will be more convenient to talk about (killed) subordinators.

~~Defn~~ A process $\{X_t : t \geq 0\}$ is a (killed) subordinator if

$$X_t = \begin{cases} Y_t & t < e_q \\ \partial & o/w \end{cases}$$

where $\{Y_t : t \geq 0\}$ a subordinator, $e_q \perp\!\!\!\perp Y$ and exponentially distributed with parameter $q \geq 0$

[We understand $e_0 := \infty$ w.p. 1] and ∂ is a cemetery state.

(Killed) Subordinator X has Laplace exponent

"$\partial = \infty$"

$$\mathbb{E}(e^{-\lambda X_t}) = \exp\left\{-\left(q + \delta\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\Pi(dx)\right)t\right\}$$

$\delta \geq 0$, $\int_{(0,\infty)} (1 \wedge x)\,\Pi(dx) < \infty$

$$\begin{aligned} &\| \\ &\mathbb{E}(e^{-\lambda Y_t} ; t < e_q) \\ &\| \\ &e^{-qt}\,\mathbb{E}(e^{-\lambda Y_t}) \end{aligned}$$

7.2 Spectrally negative Lévy processes

Recall a SNLP is a L.P which has $\Pi(0,\infty) = 0$ and does not have monotone paths. i.e. we exclude from the def^n $X_t = \delta t$ with $\delta \geq 0$ and $X_t = -Y_t$ where Y is a subordinator.

Again from Th 7.1 "$\int_{|x|\geq 1} e^{\lambda x}\,\Pi(dx) < \infty$" $\boxed{\forall \lambda \geq 0}$

Hence $\mathbb{E}(e^{\lambda X_t}) < \infty$ $\forall t \geq 0$.

A similar argument using analytic extension allows us to deduce that

$$\mathbb{E}(e^{\lambda X_t}) = \exp\Big\{ t\underbrace{\Big(-a\lambda + \tfrac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} \big(e^{\lambda x} - 1 - \lambda x \mathbb{1}_{|x|<1}\big)\Pi(dx)\Big)}_{\psi(\lambda)}\Big\}$$

$\lambda = i(-i\lambda)$

Recall that if X is a SNLP of BV then necessarily

$$X_t = \delta t - \underbrace{\sum_{s \le t} (-\Delta X_s)}_{\text{pure jump subordinator with jump measure is } \nu}$$

$$\nu(x, \infty) = \Pi(-\infty, -x) \qquad x > 0$$

$$\psi(\lambda) = \delta\lambda + \int_{(-\infty, 0)} (e^{\lambda x} - 1)\,\Pi(dx)$$

$$= \delta\lambda - \int_{(0,\infty)} (1 - e^{-\lambda x})\,\nu(dx)$$

$$\mathbb{E}(e^{\lambda X_t}) = e^{\psi(\lambda) t} \qquad e^{\delta \lambda t - t\,(\text{laplace exp. of sub})}$$

Something v. important about $\psi(\lambda) := \frac{1}{t}\log \mathbb{E}(e^{\lambda X_t})$ for SNLP :

$\boxed{\lambda \geq 0}$

① Ψ is C^∞

② Ψ is strictly convex

③ $\psi(0) = 0$

④ $\psi(\infty) = \infty$

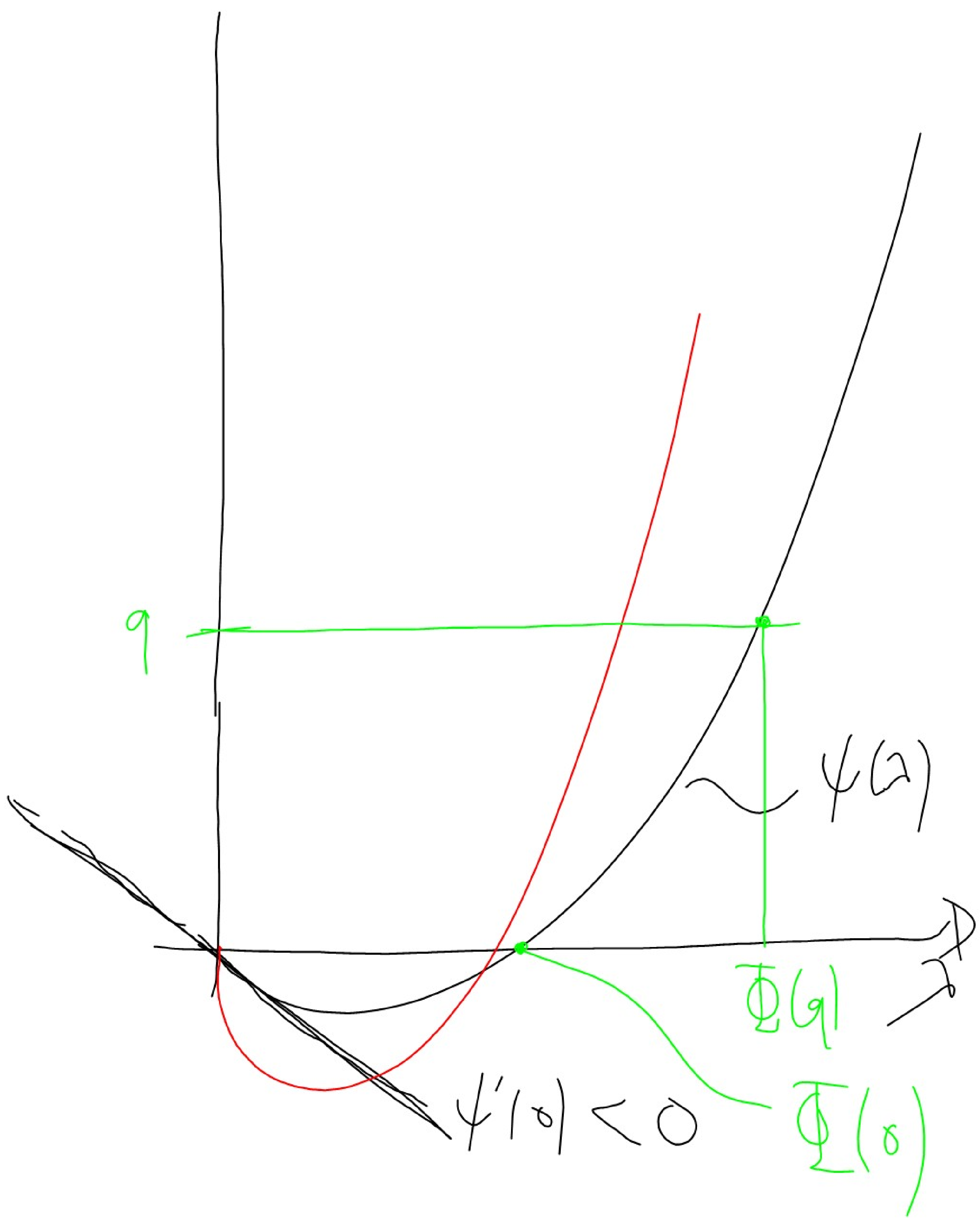
⑤ Note $\psi'(0) = \mathbb{E}(X_1) \in [-\infty, \infty)$

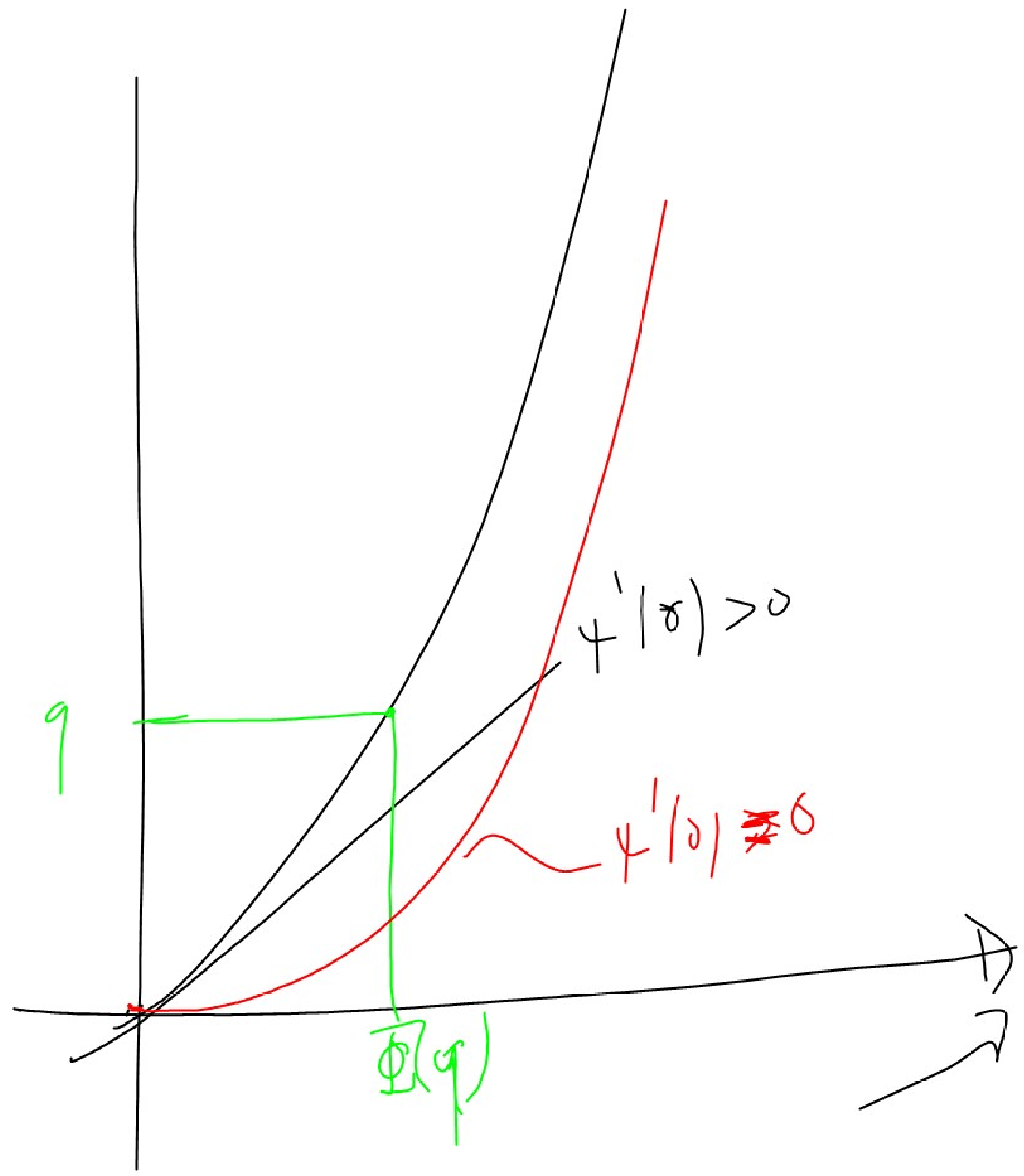
⑥ $\Phi(q) = \sup\{\lambda : \psi(\lambda) = q\}$

then [there are at most 2 roots to $\psi(\lambda) = q$]

$\Phi(0) = 0$ if $\mathbb{E}(X_1) \geq 0$ o/w $\Phi(0) > 0$

and $\Phi(q) > 0$ when $q > 0$


q
ψ(λ)
Φ(q)
ψ'(0) < 0
Φ(0)


ψ'(0) > 0
q
ψ'(0) = 0
Φ(q)

8 Exponential mgs for SNLP with applications

8.1 $\mathcal{E}_t(\lambda) = e^{\lambda X_t - \psi(\lambda) t}$ $\quad \lambda \geq 0$

$$\mathbb{E}(\mathcal{E}_{t+s}(\lambda) \mid \mathcal{F}_t) = e^{\lambda X_t - \psi(\lambda) t}\, \mathbb{E}\left(e^{\lambda (X_{t+s} - X_t) - \psi(\lambda) s} \mid \mathcal{F}_t\right)$$

$$= e^{\lambda X_t - \psi(\lambda) t}\, \underbrace{\mathbb{E}\left(e^{\lambda X_s}\right) e^{-\psi(\lambda) s}}_{= 1}$$

$$= \mathcal{E}_t(\lambda).$$

mean 1

$\mathcal{E}(\lambda)$ is a mg!

$\mathcal{E}(\lambda) = 1$

Going to use $\mathcal{E}(\lambda)$ to study

$$\tau_x^+ := \inf\{t > 0 : X_t > x\}$$

Theorem 8.1 For any SNLP and $q \geq 0$

$$\mathbb{E}\left(e^{-q\tau_x^+}\mathbb{1}_{(\tau_x^+<\infty)}\right) = e^{-\Phi(q)x}$$

where $\Phi(q)$ is the largest root of $\psi(\lambda)=q$ and ψ is the Laplace exponent of the SNLP.

~~Proof~~ Spectral negativity means that on $\{\tau_x^+<\infty\}$ $X_{\tau_x^+}=x$. Bearing this in mind

$$\mathbb{E}\Big(\underbrace{e^{\Phi(q)X_t - qt}}_{\mathcal{E}(\Phi(q))}\,\Big|\,\mathcal{F}_{\tau_x^+}\Big) = \mathbb{1}_{(\tau_x^+<t)}\, e^{\Phi(q)X_{\tau_x^+}-q\tau_x^+} \times \overbrace{\mathbb{E}\Big(e^{\overbrace{\Phi(q)(X_t - X_{\tau_x^+}) - q(t-\tau_x^+)}^{\mathcal{E}(\Phi(q))_{t-\tau_x^+}}}\,\Big|\,\mathcal{F}_{\tau_x^+}\Big)}^{1}$$

$[\Phi(q)\geq 0,\ q\geq 0]$ $\mathcal{E}(\Phi(q))$, using $\psi(\Phi(q))=q$

$$+\,\mathbb{1}_{(t\leq\tau_x^+)}\, e^{\Phi(q)X_t - qt}$$

Hence $\mathbb{E}\left(e^{\Phi(q)X_t - qt} \mid \mathcal{F}_{\tau_x^+}\right)$

$$= e^{\Phi(q)X_{t\wedge\tau_x^+} - q(t\wedge\tau_x^+)}$$

Take expectations again:

$$1 = \lim_{t\uparrow\infty} \mathbb{E}\left(e^{\Phi(q)X_{t\wedge\tau_x^+} - q(t\wedge\tau_x^+)}\right)$$

note $X_{t\wedge\tau_x^+} \leq x$ (again uses spectral negativity!)

Hence DCT to deduce

$$1 = \mathbb{E}\Big[\lim_{t\uparrow\infty} \underbrace{e^{\Phi(q)X_{t\wedge\tau_x^+} - q(t\wedge\tau_x^+)}}_{(\dots)\mathbb{1}_{(\tau_x^+<\infty)} + (\dots)\mathbb{1}_{(\tau_x^+=\infty)}}\Big]$$

$$(\dots)\mathbb{1}_{(\tau_x^+<\infty)} \xrightarrow{\text{use SN!}} e^{\Phi(q)x - q\tau_x^+}, \qquad (\dots)\mathbb{1}_{(\tau_x^+=\infty)} \leq e^{\Phi(q)x - qt} \to 0$$

$$1 = \mathbb{E}\left(e^{\Phi(q)x - q\tau_x^+}\mathbb{1}_{(\tau_x^+<\infty)}\right)$$