Last week:

$$\tau_x^+ := \inf\{t>0 : X_t > x\}$$

X SNLP with Laplace exponent of ψ

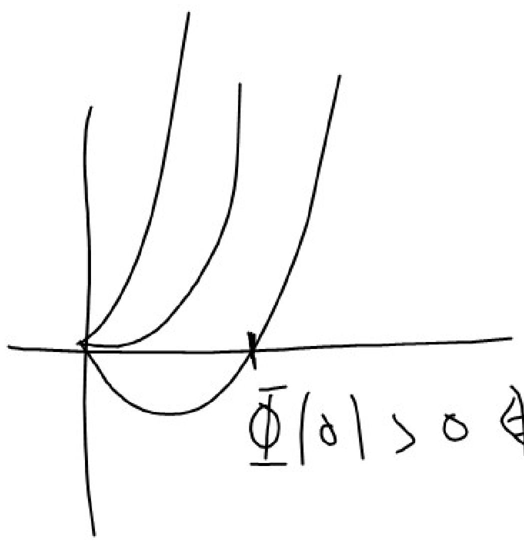
$$\circledast \quad \mathbb{E}\left(e^{-q\tau_x^+}\mathbb{1}(\tau_x^+<\infty)\right) = e^{-\Phi(q)x}$$

where Φ is right inverse of ψ

Corollary letting $q \downarrow 0$ in $\circledast$ gives

$$\mathbb{P}(\tau_x^+ < \infty) = e^{-\Phi(0)x}$$

hence $\mathbb{P}(\tau_x^+<\infty) = 1 \iff \mathbb{E}(X_1) \geq 0$



$\Phi(0)>0 \iff \mathbb{E}(X_1) = \psi'(0+) < 0$

Corollary If $\mathbb{E}(X_1) \geq 0$ then $\{\tau_x^+ : x \geq 0\}$ is a subordinator and o/w then it is a subordinator killed at rate $\Phi(0)$.

WARNING: This is only true for SNLP.

~~Pf~~ First we claim that $\Phi(q) - \Phi(0)$ is the Laplace exponent of a subordinator, equiv. it is the Laplace exp. of an inf. div +ve r.v.

To see this note

$$LHS = \mathbb{E}\left(e^{-q\tau_x^+} \,\middle|\, \tau_x^+ < \infty\right) = \boxed{e^{-(\Phi(q) - \Phi(0))x}} \quad \forall x \geq 0$$

in particular $\mathbb{E}\left(e^{-q\tau_1^+} \,\middle|\, \tau_1^+ < \infty\right) = e^{-(\Phi(q) - \Phi(0))}$

plug into red box

$$LHS = \mathbb{E}\left(e^{-q\tau_1^+} \,\middle|\, \tau_1^+ < \infty\right)^x \quad (*)$$

in particular $x = \frac{1}{n}$

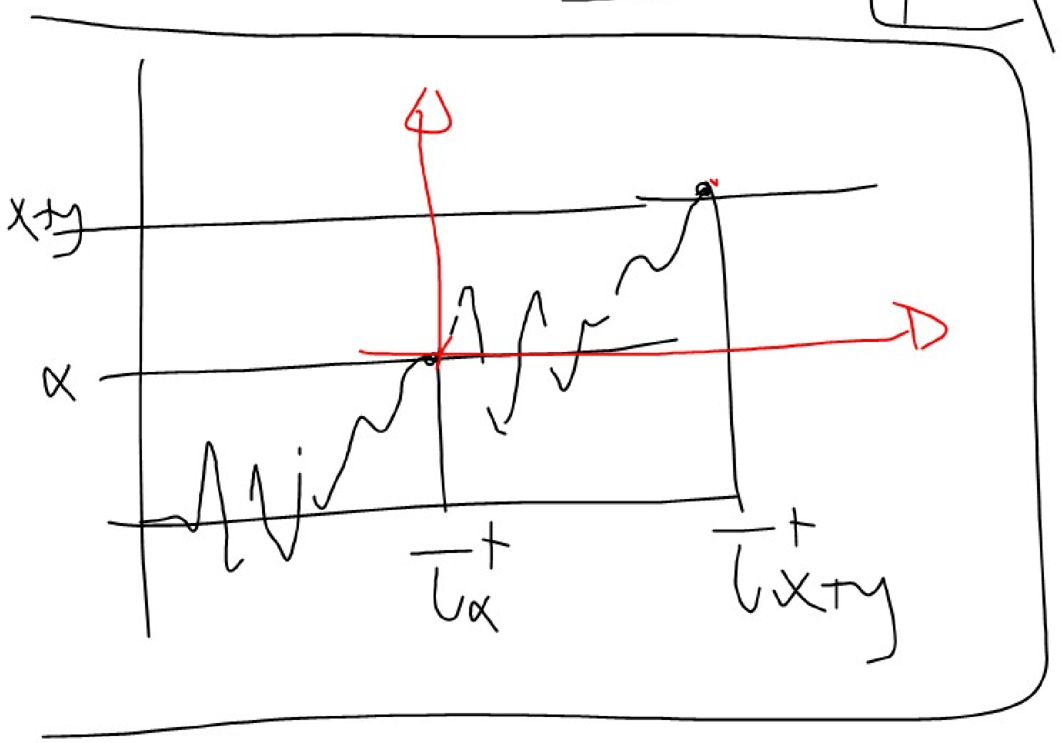
$$\mathbb{E}\left(e^{-q\tau_1^+} / \tau_1^+ < \infty\right) = \mathbb{E}\left(e^{-q\tau_{1/n}^+} \,\middle|\, \tau_{1/n}^+ < \infty\right)^n$$

Hence $\mathbb{P}\left(\tau_1^+ \in dt \,\middle|\, \tau_1^+ < \infty\right)$ is inf. div.

Next use spectral negativity ($X_{\tau_x^+} = x$ on $\tau_x^+ < \infty$)
and note that for $x, y \geq 0$

$$\mathbb{E}\left(e^{-q(\tau_{x+y}^+ - \tau_x^+)} \mathbb{1}_{(\tau_{x+y}^+ < \infty)} \,\middle|\, \mathcal{F}_{\tau_x^+} \right) \mathbb{1}_{(\tau_x^+ < \infty)}$$

$$\overset{\text{SMP}}{=} \mathbb{E}\left(e^{-q\tau_y^+} \mathbb{1}_{(\tau_y^+ < \infty)} \right) \mathbb{1}_{(\tau_x^+ < \infty)}$$



$$= \underbrace{e^{-(\Phi(q) - \Phi(0))y}}_{\text{inf divisible increment}} \underbrace{e^{-\Phi(0)y}}_{\text{surviving killing at rate } \Phi(0)} \mathbb{1}_{(\tau_x^+ < \infty)} \qquad \blacksquare$$

Note that $\mathbb{E}(X_1) \geq 0 \iff \Phi(0) = 0 \iff \tau_\infty^+$ has no killing

Introduce $\overline{X}_t := \sup_{s \le t} X_s$

<u>Corollary</u> for $q > 0$. $e_q \sim \exp(q) \perp\!\!\!\perp X$ (SNLP)

Then $\overline{X}_{e_q} \sim \exp(\Phi(q))$

~~Pf~~ naughty! $\mathbb{P}(\overline{X}_{e_q} > x) = \mathbb{P}(\tau_x^+ < e_q)$

$$\underset{\text{naughty}}{=} \mathbb{E}\left(e^{-q\tau_x^+} \mathbb{1}_{(\tau_x^+ < \infty)}\right)$$

$$= e^{-\Phi(q) x}$$

Sub corr: $\mathbb{E}(X_1) < 0$, ie $\Phi(0) > 0$, letting $q \downarrow 0$

$$\{\tau_x^+ < \infty\}$$

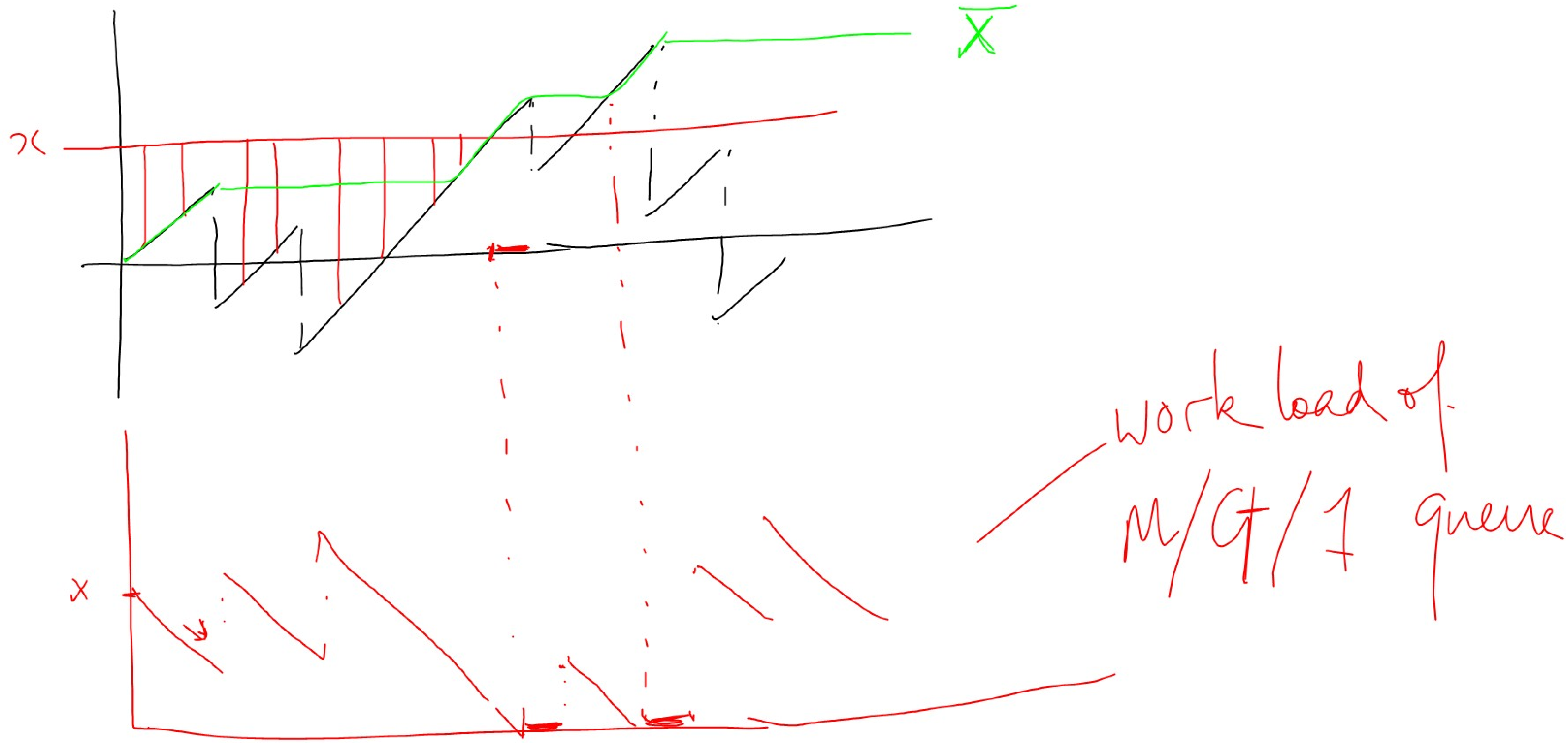
$$\mathbb{P}(\overline{X}_\infty > x) = e^{-\Phi(0) x} \quad \forall x \geq 0$$

above statement even makes sense when $\mathbb{E}(X_1) \geq 0$, i.e. $\Phi(0) = 0$

as it says $\overline{X}_\infty = \infty$ pr. 1

The second exponential mg

Introduce the reflected process $\{(\overline{X}_t \vee x) - X_t : t \geq 0\}$



Theorem For $\lambda \geq 0$, $x \geq 0$

$$M_t^x := \psi(\lambda)\int_0^t e^{-\lambda((\overline{X}_s \vee x) - X_s)}\,ds + 1 - e^{-\lambda((\overline{X}_t \vee x) - X_t)} - e^{-\lambda(\overline{X}_t \vee x)}$$

is a mg where X is a SNLP with Laplace exp ψ.

Kella-Whitt (1992)