Lévy processes and continuous-state branching processes: part II

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6 The Duality Lemma

In this section we discuss a simple feature of all Lévy processes which follows as a direct consequence of stationary independent increments. That is, when the path of a Lévy process over a finite time horizon is time reversed (in an appropriate sense) the new path is equal in law to the process reflected about the origin. This property will prove to be of crucial importance in a number of fluctuation calculations later on.

Lemma 6.1 (Duality Lemma) *For each fixed $t > 0$, define the reversed process*

$$\{X_{(t-s)-} - X_t : 0 \leq s \leq t\}$$

and the dual process,

$$\{-X_s : 0 \leq s \leq t\}.$$

Then the two processes have the same law under $\mathbb{P}$.

Proof. Define the time reversed process $Y_s = X_{(t-s)-} - X_t$ for $0 \leq s \leq t$ and note that under $\mathbb{P}$ we have $Y_0 = 0$ almost surely as t is a jump time with probability zero. (For the last statement recall that jumps appear as the countable superposition of compensated Poisson point processes). As can be seen from Fig. 7 (which is to be understood symbolically), the paths of Y are obtained from those of X by a reflection about the vertical axis with an

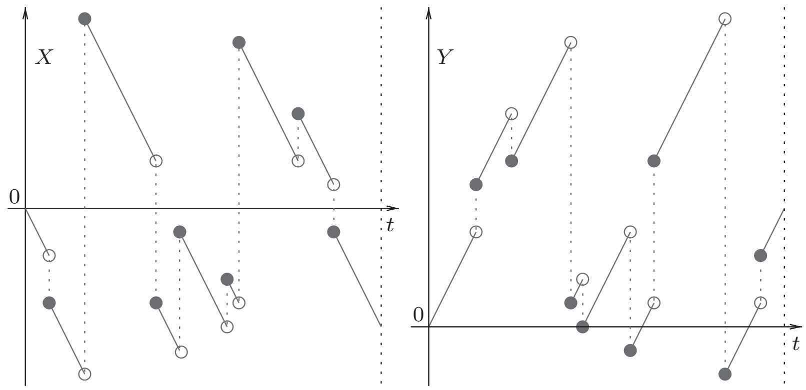

Figure 7: Duality of the processes $X = \{X_s : s \le t\}$ and $Y = \{X_{(t-s)-} - X_t : s \le t\}$. The path of Y is a reflection of the path of X with an adjustment of continuity at jump times.

adjustment of the continuity at the jump times so that its paths are almost surely right continuous with left limits. Further, the stationary independent increments of X imply directly the same as is true of Y. Further, for each $0 \le s \le t$, the distribution of $X_{(t-s)-} - X_t$ is identical to that of $-X_s$ and hence, since the finite time distributions of Y determine its law, the proof is complete. ■

The Duality Lemma is also well known for (and in fact originates from) random walks, the discrete time analogue of Lévy processes, and is justified using an identical proof. See for example [1].

One interesting feature that follows as a consequence of the Duality Lemma is the relationship between the running supremum, the running infimum, the process reflected in its supremum and the process reflected in its infimum. The last four objects are, respectively,

$$\overline{X}_t = \sup_{0\le s\le t} X_s, \qquad \underline{X}_t = \inf_{0\le s\le t} X_s$$

$$\{\overline{X}_t - X_t : t \ge 0\} \quad \text{and} \quad \{X_t - \underline{X}_t : t \ge 0\}.$$

Lemma 6.2 *For each fixed* $t > 0$, *the pairs* $(\overline{X}_t, \overline{X}_t - X_t)$ *and* $(X_t - \underline{X}_t, -\underline{X}_t)$ *have the same distribution under* $\mathbb{P}$.

Proof. Define $\widetilde{X}_s = X_t - X_{(t-s)}$ for $0 \le s \le t$ and write $\underline{\widetilde{X}}_t = \inf_{0\le s\le t} \widetilde{X}_s$. Using right continuity and left limits of paths we may deduce that

$$(\overline{X}_t, \overline{X}_t - X_t) = (\widetilde{X}_t - \underline{\widetilde{X}}_t, -\underline{\widetilde{X}}_t)$$

almost surely. One may visualise this in Fig. 8. By rotating the picture about by 180° one sees the almost sure equality of the pairs $(\overline{X}_t, \overline{X}_t - X_t)$ and $(\widetilde{X}_t -$

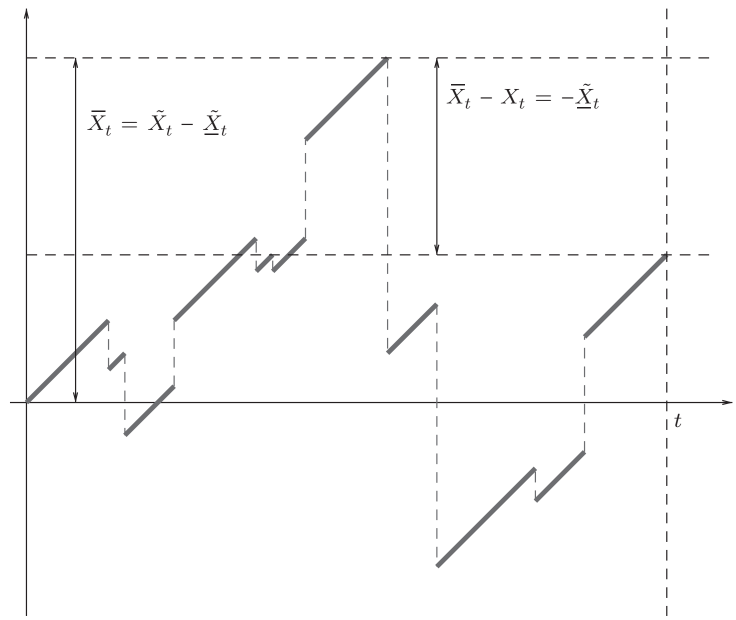

Figure 8: Duality of the pairs $(\overline{X}_t, \overline{X}_t - X_t)$ and $(X_t - \underline{X}_t, -\underline{X}_t)$.

$\underline{\widetilde{X}}_t, -\underline{\widetilde{X}})$. Now appealing to the Duality Lemma we have that $\{\widetilde{X}_s : 0 \leq s \leq t\}$ is equal in law to $\{X_s : 0 \leq s \leq t\}$ under $\mathbb{P}$. The result now follows. ■

7 Laplace exponents of processes with one sided jumps

In this section we are interested in looking at all Lévy processes which experience jumps in only one direction. It will be convenient in the forthcoming analysis to work with a Laplace exponent instead of the characteristic exponent described in the Lévy-Itô decomposition. We begin by addressing the issue of when such exponents exist. In the next theorem, we still assume that X is a general Lévy process.

Theorem 7.1 *Let $\beta \in \mathbb{R}$, then*

$$\mathbb{E}\left(\mathrm{e}^{\beta X_t}\right) < \infty \text{ for all } t \geq 0 \text{ if and only if } \int_{|x|\geq 1} \mathrm{e}^{\beta x}\Pi(\mathrm{d}x) < \infty.$$

Proof. First suppose that $\mathbb{E}\left(\mathrm{e}^{\beta X_t}\right) < \infty$ for some $t > 0$. Recall $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$ given in the Lévy–Itô decomposition. Note in particular that $X^{(2)}$ is a compound Poisson process with arrival rate $\lambda := \Pi(\mathbb{R}\backslash(-1,1))$ and jump distribution $F(\mathrm{d}x) := \mathbf{1}_{(|x|\geq 1)}\Pi(\mathrm{d}x)/\Pi(\mathbb{R}\backslash(-1,1))$ and $X^{(1)} + X^{(3)}$ is a Lévy process with Lévy measure $\mathbf{1}_{(|x|\leq 1)}\Pi(\mathrm{d}x)$. Since

$$\mathbb{E}\left(\mathrm{e}^{\beta X_t}\right) = \mathbb{E}\left(\mathrm{e}^{\beta X_t^{(2)}}\right)\mathbb{E}\left(\mathrm{e}^{\beta (X_t^{(1)}+X_t^{(3)})}\right),$$

it follows that

$$\mathbb{E}\left(\mathrm{e}^{\beta X_t^{(2)}}\right) < \infty, \tag{7.1}$$

and hence as $X^{(2)}$ is a compound Poisson process,

$$\begin{aligned}\mathbb{E}(\mathrm{e}^{\beta X_t^{(2)}}) &= \mathrm{e}^{-\lambda t}\sum_{k\geq 0}\frac{(\lambda t)^k}{k!}\int_{\mathbb{R}}\mathrm{e}^{\beta x}F^{*k}(\mathrm{d}x) \\ &= \mathrm{e}^{-\Pi(\mathbb{R}\backslash(-1,1))t}\sum_{k\geq 0}\frac{t^k}{k!}\int_{\mathbb{R}}\mathrm{e}^{\beta x}(\Pi|_{\mathbb{R}\backslash(-1,1)})^{*k}(\mathrm{d}x) < \infty, \quad (7.2)\end{aligned}$$

where F^{*n} and $(\Pi|_{\mathbb{R}\backslash(-1,1)})^{*n}$ are the n-fold convolution of F and $\Pi|_{\mathbb{R}\backslash(-1,1)}$, the restriction of Π to $\mathbb{R}\backslash(-1,1)$, respectively. In particular the summand corresponding to $k=1$ must be finite; that is

$$\int_{|x|\geq 1}\mathrm{e}^{\beta x}\Pi(\mathrm{d}x) < \infty.$$

Now suppose that $\int_{\mathbb{R}}\mathrm{e}^{\beta x}\mathbf{1}_{(|x|\geq 1)}\Pi(\mathrm{d}x) < \infty$ for some $\beta\in\mathbb{R}$. Since $(\Pi|_{\mathbb{R}\backslash(-1,1)})^{*n}(dx)$ is a finite measure, we have

$$\int_{\mathbb{R}}\mathrm{e}^{\beta x}(\Pi|_{\mathbb{R}\backslash(-1,1)})^{*n}(\mathrm{d}x) = \left(\int_{|x|\geq 1}\mathrm{e}^{\beta x}\Pi(\mathrm{d}x)\right)^n,$$

and hence (7.2) and (7.1) hold for all $t>0$. The proof is thus complete once we show that for all $t>0$,

$$\mathbb{E}\left(\mathrm{e}^{\beta (X_t^{(1)}+X_t^{(3)})}\right) < \infty. \tag{7.3}$$

However, since $X^{(1)}+X^{(3)}$ has a Lévy measure with bounded support, it follows that its characteristic exponent,

$$\begin{aligned}&-\frac{1}{t}\log\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\theta (X_t^{(1)}+X_t^{(3)})}\right) \\ &= \mathrm{i}a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-1,1)}(1-\mathrm{e}^{\mathrm{i}\theta x}+\mathrm{i}\theta x)\Pi(\mathrm{d}x), \ \theta\in\mathbb{R}, \quad (7.4)\end{aligned}$$

can be extended to an entire function (analytic on the whole of $\mathbb{C}$). To see this, note that

$$\int_{(-1,1)}(1-\mathrm{e}^{\mathrm{i}\theta x}+\mathrm{i}\theta x)\Pi(\mathrm{d}x) = -\int_{(-1,1)}\sum_{k\geq 0}\frac{(\mathrm{i}\theta x)^{k+2}}{(k+2)!}\Pi(\mathrm{d}x).$$

The sum and the integral may be exchanged in the latter using Fubini's Theorem and the estimate

$$\sum_{k\geq 0}\int_{(-1,1)}\frac{(|\theta x|)^{k+2}}{(k+2)!}\Pi\,(\mathrm{d}x)\leq\sum_{k\geq 0}\frac{(|\theta|)^{k+2}}{(k+2)!}\int_{(-1,1)}x^2\Pi\,(\mathrm{d}x)<\infty.$$

Hence the right–hand side of (7.4) can be written as a power series for all $\theta\in\mathbb{C}$ and is thus entire. By writing the exponential function as a power series, note that, with the help of Fubini's Theorem we can write $\mathbb{E}(\mathrm{e}^{\beta(X_t^{(1)}+X_t^{(3)})})$ as a power series in β which converges on account of the fact that, as a power series it is equal to the one corresponding to the analytical function $\exp\{-\Psi^{(1)}(\theta)-\Psi^{(3)}(\theta)\}$ with $\theta=-\mathrm{i}\beta$. ■

There are two particular families of Lévy processes that we would like to apply the above theorem to.

7.1 Subordinators

Note that if X is a subordinator then, by the above Theorem, since $-X$ has no positive jumps, $\mathbb{E}(\mathrm{e}^{-\lambda X_t})$ is finite for all $t\geq 0$ and $\lambda\geq 0$. It thus follows that $\mathbb{E}(\mathrm{e}^{i\theta X_t})$ is an analytic function on $\{\theta\in\mathbb{C}:\Im\theta\geq 0\}$. Reconsidering the characteristic exponent in Lemma 5.2 and noting that it may be analytically extended to $\{\theta\in\mathbb{C}:\Im\theta\geq 0\}$, it follows by the Identity Theorem of complex analysis that

$$\mathbb{E}(\mathrm{e}^{-\lambda X_t})=\exp\left\{-\left(\delta\lambda+\int_{(0,\infty)}(1-\mathrm{e}^{-\lambda x})\Pi(\mathrm{d}x)\right)t\right\}$$

for all $\lambda\geq 0$ where Π is the jump measure of X (which necessarily satisfies $\int_{(0,\infty)}(1\wedge x)\Pi(\mathrm{d}x)<\infty$) and δ is its drift.

We would like to introduce a slightly more general definition of a subordinator for future use at this point. It will be convenient to talk of *killed* subordinators. A process $X=\{X_t:t\geq 0\}$ is a killed subordinator if

$$X_t=\begin{cases}Y_t & \text{if } t<\mathbf{e}_q\\ \partial & \text{otherwise}\end{cases}$$

where $\{Y_t:t\geq 0\}$ is a subordinator, for $q>0$, $\mathbf{e}_q$ is an independent and exponentially distributed random variable and ∂ is some 'cemetery' state. In the case that $q=0$ we understand $\mathbf{e}_q=\infty$, that is to say there is no killing at all. We may also talk of the Laplace exponent of such a killed subordinator as for all $\lambda\geq 0$ we have[1]

$$\mathbb{E}(\mathrm{e}^{-\lambda X_t})=\mathbb{E}(\mathrm{e}^{-\lambda Y_t}\mathbf{1}_{(t<\mathbf{e}_q)})=\mathrm{e}^{-qt}\mathbf{E}(\mathrm{e}^{-\lambda Y_t})$$

[1] We are making an abuse of notation in the use of the measure $\mathbb{P}$. Strictly speaking we should work with the measure $\mathbb{P}\times\mathcal{P}$ where $\mathcal{P}$ is the probability measure on the space in which the random variable $\mathbf{e}_q$ is defined. This abuse of notation will be repeated for the sake of convenience at various points throughout this text.

and hence

$$\mathbb{E}(\mathrm{e}^{-\lambda X_t}) = \mathrm{e}^{-\Phi(\lambda)t}$$

where

$$\Phi(\lambda) = q + \delta\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x})\Pi(\mathrm{d}x)$$

where $\delta \geq 0$ and Π is a measure on $(0,\infty)$ satisfying $\int_{(0,\infty)}(1 \wedge x)\Pi(\mathrm{d}x) < \infty$.

7.2 Spectrally negative Lévy processes

A spectrally negative Lévy process is any Lévy process whose jump measure Π satisfies $\Pi(0,\infty) = 0$ and whose paths are not monotone. The latter proviso excludes the process $X_t = \delta t$, $t \geq 0$, for some $\delta \geq 0$ and $X_t = -Y_t$, $t \geq 0$, where $\{Y_t : t \geq 0\}$ is a subordinator. We may make a similar observation to the discussion above concerning subordinators and note that it is immediate from Theorem 7.1 that $\mathbb{E}(\mathrm{e}^{\lambda X_t})$ is finite for all $t \geq 0$ and $\lambda \geq 0$ and hence $\mathbb{E}(\mathrm{e}^{i\theta X_t})$ is an analytic function on $\{\theta \in \mathbb{C} : \Im\theta \geq 0\}$. Inspecting its characteristic exponent

$$\Psi(\theta) = \mathrm{i}a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} (1 - \mathrm{e}^{\mathrm{i}\theta x} + \mathrm{i}\theta x\mathbf{1}_{(|x|<1)})\Pi(\mathrm{d}x)$$

one sees that the above function can be analytically extended to $\{\theta \in \mathbb{C} : \Im\theta \geq 0\}$. Hence, once again, thanks to the Identity Theorem, we may deduce that for all $\lambda \geq 0$,

$$\mathbb{E}(\mathrm{e}^{\lambda X_t}) = \mathrm{e}^{\psi(\lambda)t}$$

where

$$\psi(\lambda) = -a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} (\mathrm{e}^{\lambda x} - 1 - \lambda x\mathbf{1}_{(|x|<1)})\Pi(\mathrm{d}x).$$

The reader may feel that there is a pedantic change of notation at this point as we have chosen to work with a negatively signed Laplace exponent in the case of a subordinator and positively signed Laplace exponent in the spectrally negative Lévy process. However, a little thought shows that in fact it is a consistent choice of notation. Indeed if we look back to the representation (5.2) of any bounded variation Lévy process, then it follows that a spectrally negative Lévy process of bounded variation *necessarily* takes the form

$$X_t = ct - \sum_{s\leq t}(-\Delta X_s) \qquad t \geq 0$$

where $c > 0$. That is to say, it is the difference of a linear drift and a pure jump subordinator and hence

$$\psi(\lambda) = c\lambda - \int_{(-\infty,0)} (1 - \mathrm{e}^{\lambda x})\Pi(\mathrm{d}x) = c\lambda - \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x})\nu(\mathrm{d}x)$$

where $\nu(x,\infty) = \Pi(-\infty,-x)$ and so ν should be thought of as the jump measure of the subordinator $\sum_{s\leq t}(-\Delta X_s)$, $t \geq 0$. Given the form of X we see then that

ψ does indeed respect the notational conventions for the Laplace exponent of a subordinator.

Before concluding this section let us make some notes about the analytical properties of the Laplace exponent $\psi(\lambda)$ and its right inverse function

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}.$$

Exercise 14 shows that on $[0, \infty)$, ψ is infinitely differentiable, strictly convex and that $\psi(0) = 0$ whilst $\psi(\infty) = \infty$. As a particular consequence of these facts, it follows that $\mathbb{E}(X_1) = \psi'(0+) \in [-\infty, \infty)$. In the case that $\mathbb{E}(X_1) \geq 0$, $\Phi(q)$ is the unique solution to $\psi(\theta) = q$ in $[0, \infty)$. When $\mathbb{E}(X_1) < 0$ the latter statement is true only when $q > 0$ and when $q = 0$ there are two roots to the equation $\psi(\theta) = 0$, one of them being $\theta = 0$ and the other being $\Phi(0) > 0$. See Fig. 9 for further clarification.

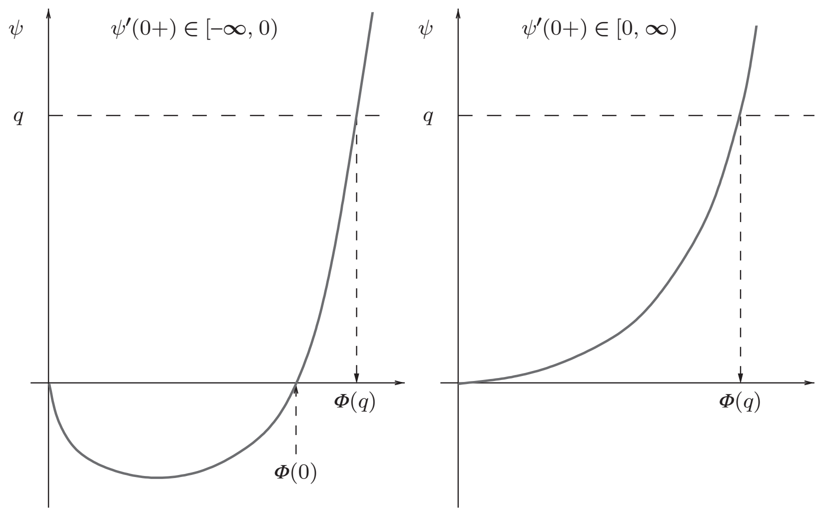

Figure 9: Two examples of ψ, the Laplace exponent of a spectrally negative Lévy process, and the relation to Φ.

8 Exponential martingales for spectrally negative Lévy processes and applications

In this section we assume that X is a spectrally negative Lévy process and we loot at two exponential type martingales which lead to further information about the paths of the process.

8.1 The first exponential martingale

Following the reasoning in Exercise 5, it is easy to show that

$$\mathcal{E}_t(\beta) := \mathrm{e}^{\beta X_t - \psi(\beta)t}, \qquad t \geq 0$$

is a martingale with respect to $\{\mathcal{F}_t : t \geq 0\}$ the natural filtration generated by X. When X is a Brownian motion, this martingale is the same as the exponential martingale that is used in the context of the Girsanov change of measure. Indeed Exercise 15 shows that within the current context of spectrally neagtive Lévy processes, the above martingale may still be used to perform an exponential change of measure resulting in a new process which is still a spectrally negative Lévy process.

Our first result here is to use the exponential martingale to characterise the law of the first passage times

$$\tau_x^+ := \inf\{t > 0 : X_t > x\}$$

for $x \geq 0$.

Theorem 8.1 *For any spectrally negative Lévy process, with $q \geq 0$,*

$$\mathbb{E}(\mathrm{e}^{-q\tau_x^+}\mathbf{1}_{(\tau_x^+<\infty)}) = \mathrm{e}^{-\Phi(q)x},$$

where $\Phi(q)$ is the largest root of the equation $\psi(\theta) = q$.

Proof. Fix $q > 0$. Using spectral negativity to write $x = X_{\tau_x^+}$ on $\{\tau_x^+ < \infty\}$, note with the help of the Strong Markov Property that

$$\begin{aligned}
&\mathbb{E}(\mathrm{e}^{\Phi(q)X_t - qt}|\mathcal{F}_{\tau_x^+})\\
&= \mathbf{1}_{(\tau_x^+\geq t)}\mathrm{e}^{\Phi(q)X_t-qt} + \mathbf{1}_{(\tau_x^+<t)}\mathrm{e}^{\Phi(q)x-q\tau_x^+}\mathbb{E}(\mathrm{e}^{\Phi(q)(X_t-X_{\tau_x^+})-q(t-\tau_x^+)}|\mathcal{F}_{\tau_x^+}),\\
&= \mathrm{e}^{\Phi(q)X_{t\wedge\tau_x^+}-q(t\wedge\tau_x^+)}
\end{aligned}$$

where in the final equality we have used the fact that $\mathbb{E}(\mathcal{E}_t(\Phi(q))) = 1$ for all $t \geq 0$. Taking expectations again we have

$$\mathbb{E}(\mathrm{e}^{\Phi(q)X_{t\wedge\tau_x^+}-q(t\wedge\tau_x^+)}) = 1.$$

Noting that the expression in the latter expectation is bounded above by $\mathrm{e}^{\Phi(q)x}$, an application of dominated convergence yields

$$\mathbb{E}(\mathrm{e}^{\Phi(q)x-q\tau_x^+}\mathbf{1}_{(\tau_x^+<\infty)}) = 1$$

which is equivalent to the statement of the theorem. ■

The following two corollaries are worth recording for later. The first one hints at forthcoming remarks; that spectrally negative Lévy processes drift to $+\infty$ (resp. oscillate, resp. drift to $-\infty$) accordingly as $\psi'(0+) = \mathbb{E}(X_1) > 0$ (resp. $= 0$, resp. < 0).

Corollary 8.1 *From the previous theorem we have that* $\mathbb{P}(\tau_x^+ < \infty) = \mathrm{e}^{-\Phi(0)x}$ *which is one, if and only if* $\Phi(0) = 0$*, if and only if* $\psi'(0+) \geq 0$*, if and only if* $\mathbb{E}(X_1) \geq 0$*.*

Corollary 8.2 *If* $\mathbb{E}(X_1) \geq 0$ *then the process* $\{\tau_x^+ : x \geq 0\}$ *is a subordinator and otherwise it is equal in law to a subordinator killed at an independent exponential time with parameter* $\Phi(0)$*.*

Proof. First we claim that $\Phi(q) - \Phi(0)$ is the Laplace exponent of a non-negative infinitely divisible random variable. To see this, note that for all $x \geq 0$,

$$\mathbb{E}(\mathrm{e}^{-q\tau_x^+}|\tau_x^+ < \infty) = \mathrm{e}^{-(\Phi(q)-\Phi(0))x} = \mathbb{E}(\mathrm{e}^{-q\tau_1^+}|\tau_1^+ < \infty)^x,$$

and hence in particular

$$\mathbb{E}(\mathrm{e}^{-q\tau_1^+}|\tau_1^+ < \infty) = \mathbb{E}(\mathrm{e}^{-q\tau_{1/n}^+}|\tau_{1/n}^+ < \infty)^n$$

showing that $\mathbb{P}(\tau_1^+ \in \mathrm{d}z|\tau_1^+ < \infty)$ for $z \geq 0$ is the law of an infinitely divisible random variable. Next, using the Strong Markov Property, spatial homogeneity and again the special feature of spectral negativity that $\{X_{\tau_x^+} = x\}$ on the event $\{\tau_x^+ < \infty\}$, we have for $x, y \geq 0$ and $q \geq 0$,

$$\begin{aligned}
&\mathbb{E}(\mathrm{e}^{-q(\tau_{x+y}^+ - \tau_x^+)}\mathbf{1}_{(\tau_{x+y}^+<\infty)}|\mathcal{F}_{\tau_x^+})\mathbf{1}_{(\tau_x^+<\infty)} \\
&= \mathbb{E}(\mathrm{e}^{-q\tau_y^+}\mathbf{1}_{(\tau_y^+<\infty)})\mathbf{1}_{(\tau_x^+<\infty)} \\
&= \mathrm{e}^{-(\Phi(q)-\Phi(0))y}\mathrm{e}^{-\Phi(0)y}\mathbf{1}_{(\tau_x^+<\infty)}.
\end{aligned}$$

In the first equality we have used standard notation for Markov processes, $\mathbb{E}_x(\cdot) = \mathbb{E}(\cdot|X_0 = x)$. We see then that the increment $\tau_{x+y}^+ - \tau_x^+$ is independent of $\mathcal{F}_{\tau_x^+}$ on $\{\tau_x^+ < \infty\}$ and has the same law as the subordinator with Laplace exponent $\Phi(q) - \Phi(0)$ but killed at an independent exponential time with parameter $\Phi(0)$.

When $\mathbb{E}(X_1) \geq 0$ we have that $\Phi(0) = 0$ and hence the concluding statement of the previous paragraph indicates that $\{\tau_x^+ : x \geq 0\}$ is a subordinator (without killing). On the other hand, if $\mathbb{E}(X_1) < 0$, or equivalently $\Phi(0) > 0$ then the second statement of the corollary follows. ■

For the next corollary, recall that

$$\overline{X}_t := \sup_{s \leq t} X_s, \qquad t \geq 0,$$

Corollary 8.3 *Suppose that* $q > 0$ *and let* $\mathbf{e}_q$ *be an exponentially distributed random variable which is independent of the spectrally negative Lévy process* X*. Then* $\overline{X}_{\mathbf{e}_q}$ *is exponentially distributed with parameter* $\Phi(q)$*.*

Proof. The result is an easy consequence of the fact that

$$\mathbb{P}(\overline{X}_{\mathbf{e}_q} > x) = \mathbb{P}(\tau_x^+ < \mathbf{e}_q) = \mathbb{E}(e^{-q\tau_x^+}\mathbf{1}_{(\tau_x^+<\infty)})$$

together with the conclusion of Theorem 8.1. ■

8.2 The second exponential martingale

The second martingale of interest has finds its roots in queuing theory where one typically thinks of the process $\{\overline{X}_t \vee x - X_t : t \geq 0\}$ as the workload of a queue with initial workload $x \geq 0$. In particular when X is the difference of a linear drift and a compound Poisson process with negative jumps, then the aforementioned process is precisely the workload process of an $M/G/1$ queue. The martingale was introduced in Kella and Whitt (1992).

Theorem 8.2 *For* $\lambda > 0$ *and* $x \geq 0$,

$$M_t^x := \psi(\lambda) \int_0^t \mathrm{e}^{-\lambda(\overline{X}_s \vee x - X_s)}\, \mathrm{d}s + 1 - \mathrm{e}^{-\lambda(\overline{X}_t \vee x - X_t)} - \lambda(\overline{X}_t \vee x), \qquad t \geq 0 \quad (8.1)$$

is a martingale with respect to $\{\mathcal{F}_t : t \geq 0\}$.

Proof. We give a proof, which on the one hand can be understood as rigorous for the student proficient in stochastic calculus of semi-martingales, and otherwise may be understood heuristically. Firstly recall that $\mathcal{E}_t(\lambda) = \exp\{\lambda X_t - \psi(\lambda)t\}$ is a martingale and hence loosely speaking, $\mathrm{d}\mathcal{E}_t(\lambda)$ is a martingale increment. Next, taking on face value the integration by parts formula

$$\begin{aligned} \mathrm{d}[e^{-\lambda(\overline{X}_t \vee x) + \psi(\lambda)t}\mathcal{E}_t(\lambda)] &= e^{-\lambda(\overline{X}_t \vee x) + \psi(\lambda)t}\mathrm{d}\mathcal{E}_t(\lambda) \\ &\quad - e^{-\lambda(\overline{X}_t \vee x) + \psi(\lambda)t}\mathcal{E}_t(\lambda)[\lambda \mathrm{d}(\overline{X}_t \vee x) - \psi(\lambda)\mathrm{d}t] \end{aligned}$$

where the first integral on the right hand side is understood as a stochastic integral and the second integral on the right hand side is understood as a Lebesgue-Stieltjes integral, we have

$$\begin{aligned} \mathrm{d}M_t^x &= \psi(\lambda)e^{-\lambda(\overline{X}_t \vee x) + \psi(\lambda)t}\mathcal{E}_t(\lambda)\mathrm{d}t - \mathrm{d}[e^{-\lambda(\overline{X}_t \vee x) + \psi(\lambda)t}\mathcal{E}_t(\lambda)] \\ &\quad - \lambda \mathrm{d}(\overline{X}_t \vee x) \\ &= \psi(\lambda)e^{-\lambda(\overline{X}_t \vee x) + \psi(\lambda)t}\mathcal{E}_t(\lambda)\mathrm{d}t - e^{-\lambda(\overline{X}_t \vee x) + \psi(\lambda)t}\mathrm{d}\mathcal{E}_t(\lambda) \\ &\quad + e^{-\lambda(\overline{X}_t \vee x) + \psi(\lambda)t}\mathcal{E}_t(\lambda)[\lambda \mathrm{d}(\overline{X}_t \vee x) - \psi(\lambda)\mathrm{d}t] \\ &\quad - \lambda \mathrm{d}(\overline{X}_t \vee x). \end{aligned}$$

Next note that the process $\overline{X}_t \vee x$ does not increment until $t > \tau_x^+$ in which case it follows that $\overline{X}_t \vee x - X_t = \overline{X}_t - X_t$. Moreover, the latter difference is precisely equal to zero whenever t is in the support of the measure $\mathrm{d}\overline{X}_t$. It follows that

$$\mathrm{d}M_t^x = -e^{-\lambda(\overline{X}_t \vee x) + \psi(\lambda)t}\mathrm{d}\mathcal{E}_t(\lambda)$$

showing that M_t^x is a local martingale since $\mathrm{d}\mathcal{E}_t(\lambda)$ is a martingale increment. To complete the proof rigorously one needs to show that M is a proper martingale (as opposed to strictly a local martingale). It suffices to show that for each $t > 0$,

$$\mathbb{E}\left(\sup_{s \leq t} |M_s^x|\right) < \infty.$$

To this end recall from Corollary 8.3 that $\overline{X}_{\mathbf{e}_q}$ is exponentially distributed with parameter $\Phi(q)$ and hence

$$\mathbb{E}\big(\overline{X}_{\mathbf{e}_q}\big) = \int_0^\infty q\mathrm{e}^{-qt}\,\mathbb{E}\big(\overline{X}_t\big)\,\mathrm{d}t = \frac{1}{\Phi(q)} < \infty.$$

Since $\overline{X}_t$ is a continuous (on account of no positive jumps) and increasing process, we have $\mathbb{E}\big(\overline{X}_t\big) < \infty$ for all t. Now note by the positivity of the process Z and again since $\overline{X}$ increases,

$$\mathbb{E}\left(\sup_{s\leq t}|M^x_s|\right) \leq \psi(\lambda)t + 2 + \lambda\,\mathbb{E}\big(\overline{X}_t \vee x\big) < \infty$$

for each finite $t > 0$. ■

Remark 8.1 For the remainder of this section we will use this martingale in the case $x = 0$ in which case we shall refer to it as M instead of M^0

With the first exponential martingale we were able to compute the law of the running maximum sampled at an independent and exponentially distributed time. With the second exponential martingale we are able to characterise the law of the running minimum sampled at an independent and exponentially distributed time as follows.

Theorem 8.3 *Let* $\underline{X}_t = \inf_{0\leq u\leq t} X_u$ *and suppose that* $\mathbf{e}_q$ *is an exponentially distributed random variable with parameter* $q > 0$ *independent of the process* X. *Then for* $\alpha > 0$,

$$\mathbb{E}\big(\mathrm{e}^{\alpha\underline{X}_{\mathbf{e}_q}}\big) = \frac{q(\alpha - \Phi(q))}{\Phi(q)(\psi(\alpha) - q)}, \tag{8.2}$$

where the right hand side is understood in the asymptotic sense when $\alpha = \Phi(q)$, *i.e.* $q/\Phi(q)\psi'(\Phi(q))$

Proof. We begin by noting some facts which will be used in conjunction with the martingale (8.1). Recall that $\mathbf{e}_q$ is an exponentially distributed random variable with parameter $q > 0$ independent of the process X.

Let $Z_t = \overline{X}_t - X_t$. Note that by an application of Fubini's theorem together with Lemma 6.2,

$$\mathbb{E}\int_0^{\mathbf{e}_q} \mathrm{e}^{-\alpha Z_s}\,\mathrm{d}s = \int_0^\infty \mathrm{e}^{-qs}\,\mathbb{E}\big(\mathrm{e}^{-\alpha Z_s}\big)\,\mathrm{d}s = \frac{1}{q}\,\mathbb{E}\big(\mathrm{e}^{-\alpha Z_{\mathbf{e}_q}}\big) = \frac{1}{q}\,\mathbb{E}\big(\mathrm{e}^{\alpha\underline{X}_{\mathbf{e}_q}}\big).$$

From Theorem 8.2 we have that $\mathbb{E}\big(M_{\mathbf{e}_q}\big) = \mathbb{E}(M_0) = 0$ and hence using the last observation we obtain

$$\frac{\psi(\alpha) - q}{q}\,\mathbb{E}\big(\mathrm{e}^{\alpha\underline{X}_{\mathbf{e}_q}}\big) = \alpha\,\mathbb{E}\big(\overline{X}_{\mathbf{e}_q}\big) - 1.$$

Recall from Corollary 8.3 that $\overline{X}_{\mathbf{e}_q}$ is exponentially distributed with parameter $\Phi(q)$ and hence $\mathbb{E}(\overline{X}_{\mathbf{e}_q}) = 1/\Phi(q)$. It follows that

$$\frac{\psi(\alpha) - q}{q} \mathbb{E}(\mathrm{e}^{\alpha \underline{X}_{\mathbf{e}_q}}) = \frac{\alpha - \Phi(q)}{\Phi(q)} \tag{8.3}$$

and the theorem is now proved. ■

Remark 8.2 *Another identity that we shall shortly use which follows from the fact that $\overline{X}_{\mathbf{e}_q}$ is exponentially distributed with parameter $\Phi(q)$ is that for $\alpha > 0$*

$$\mathbb{E}\left(\mathrm{e}^{-\alpha \overline{X}_{\mathbf{e}_q}}\right) = \frac{\Phi(q)}{\Phi(q) + \alpha}. \tag{8.4}$$

The final result of this section uses the results we have obtained from our two exponential martingales to prove what otherwise is intuitively obvious regarding the long term behaviour of a spectrally negative Lévy process. In reading the statement of the next Lemma it is important to recall that $\psi'(0+) = \mathbb{E}(X_1)$.

Lemma 8.1 *We have that*

(i) $\overline{X}_\infty$ *and* $-\underline{X}_\infty$ *are either infinite almost surely or finite almost surely,*

(ii) $\overline{X}_\infty = \infty$ *if and only if* $\psi'(0^+) \geq 0$,

(iii) $\underline{X}_\infty = -\infty$ *if and only if* $\psi'(0^+) \leq 0$.

Proof. On account of the strict convexity ψ it follows that $\Phi(0) > 0$ if and only if $\psi'(0^+) < 0$ and hence

$$\lim_{q \downarrow 0} \frac{q}{\Phi(q)} = \begin{cases} 0 & \text{if } \psi'(0^+) \leq 0 \\ \psi'(0^+) & \text{if } \psi'(0^+) > 0. \end{cases}$$

By taking q to zero in the identity (8.2) we now have that

$$\mathbb{E}(\mathrm{e}^{\alpha \underline{X}_\infty}) = \begin{cases} 0 & \text{if } \psi'(0^+) \leq 0 \\ \psi'(0^+)\alpha/\psi(\alpha) & \text{if } \psi'(0^+) > 0. \end{cases}$$

Next, recall from (8.4) that for $\alpha > 0$

$$\mathbb{E}\left(\mathrm{e}^{-\alpha \overline{X}_{\mathbf{e}_q}}\right) = \frac{\Phi(q)}{\Phi(q) + \alpha}$$

and hence by taking the limit of both sides as q tends to zero,

$$\mathbb{E}\left(\mathrm{e}^{-\alpha \overline{X}_\infty}\right) = \begin{cases} (\alpha/\Phi(0) + 1)^{-1} & \text{if } \psi'(0^+) < 0 \\ 0 & \text{if } \psi'(0^+) \geq 0. \end{cases}$$

Parts (i)–(iii) follow immidiately from the previous two identities by considering their limits as $\alpha \downarrow 0$. ■

Remark 8.3 In fact an even stronger statement that of the the last Lemma can be proved. Namely that

(i) if $\psi'(0+) = 0$ then $\limsup_{t\uparrow\infty} X_t = -\liminf_{t\uparrow\infty} X_t = \infty$ (i.e. the process oscillates),

(ii) if $\psi'(0^+) > 0$ then $\lim_{t\uparrow\infty} X_t = \infty$ (i.e. the process drift to infinity) and

(iii) if $\psi'(0^+) > 0$ then $\lim_{t\uparrow\infty} X_t = -\infty$ (i.e. the process drift to minus infinity).

This is clear, at least when the limits are taken along lattice times, by the strong law of large numbers and stationary independent increments of X.

Exercises

Exercise 14 Suppose that ψ is the Laplace exponent of a spectrally negative Lévy process. By considering explicitly the formula

$$\psi(\beta) = -a\beta + \frac{1}{2}\sigma^2\beta^2 + \int_{(-\infty,0)} (\mathrm{e}^{\beta x} - 1 - \beta x \mathbf{1}_{(x>-1)})\Pi(\mathrm{d}x)$$

show that on $[0,\infty)$, ψ is infinitely differentiable, strictly convex and that $\psi(0) = 0$ whilst $\psi(\infty) = \infty$.

Exercise 15 Suppose that X is a spectrally negative Lévy process with characteristic triple (a, σ, Π) and characteristic exponent ψ. Fix $\beta \geq 0$ and show that under the change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}^\beta}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = e^{\beta X_t - \psi(\beta)t}$$

the process X is still a spectrally negative Lévy process with characteristic triple (a^*, σ^*, Π^*) where

$$a^* = a - \beta\sigma^2 + \int_{(-1,0)} (1 - \mathrm{e}^{\beta x})x\Pi(\mathrm{d}x),$$

$\sigma^* = \sigma$ and $\Pi^*(\mathrm{d}x) = \mathrm{e}^{\beta x}\Pi(\mathrm{d}x)$ on $(-\infty, 0)$.

Exercise 16 Suppose that X is a spectrally negative Lévy process with Lévy–Khintchine exponent Ψ. Here we give another proof of the existence of a finite Laplace exponent for all spectrally negative Lévy processes.

(i) Use spectral negativity together with the lack of memory property to show that for $x, y > 0$,

$$\mathbb{P}(\overline{X}_{\mathbf{e}_q} > x + y) = \mathbb{P}(\overline{X}_{\mathbf{e}_q} > x)\mathbb{P}(\overline{X}_{\mathbf{e}_q} > y)$$

where $\mathbf{e}_q$ is an exponentially distributed random variable independent of X and $\overline{X}_t = \sup_{s\leq t} X_s$.

(ii) Deduce that $\overline{X}_{\mathbf{e}_q}$ is exponentially distributed and hence the Laplace exponent $\psi(\beta) = -\Psi(-\mathrm{i}\beta)$ exists and is finite for all $\beta \geq 0$.

(iii) By considering the Laplace transform of the first passage time τ_x^+, show that one may also deduce via a different route that $\overline{X}_{\mathbf{e}_q}$ is exponentially distributed with parameter $\Phi(q)$. In particular show that $\overline{X}_\infty$ is either infinite with probability one or is exponentially distributed accordingly as $\mathbb{E}(X_1) \geq 0$ or $\mathbb{E}(X_1) < 0$. [Hint: reconsider Exercise 14].

Exercise 17 For this exercise it will be useful to refer to Sect. 2.6. Suppose that X is a Stable Lévy process with index $\beta = 1$; that is to say $\Pi(-\infty, 0) = 0$.

(i) Show that if $\alpha \in (0,1)$ then X is a *driftless* subordinator with Laplace exponent satisfying

$$-\log\ \mathbb{E}(\mathrm{e}^{-\theta X_1}) = c\theta^{\alpha},\ \theta \geq 0$$

for some $c > 0$.

(ii) Show that if $\alpha \in (1,2)$, then X has a Laplace exponent satisfying

$$-\log \mathbb{E}(\mathrm{e}^{-\theta X_1}) = -C\theta^{\alpha},\ \theta \geq 0$$

for some $C > 0$. Confirm that X has no integer moments of order 2 and above as well as being a process of unbounded variation.

Exercise 18 Suppose that X is a spectrally negative Lévy process of bounded variation with characteristic exponent Ψ.

(i) Show that for each $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} M_t &= -\Psi(\alpha)\int_0^t \mathrm{e}^{\mathrm{i}\alpha(X_s-\overline{X}_s)+\mathrm{i}\beta\overline{X}_s}\mathrm{d}s + 1 - \mathrm{e}^{\mathrm{i}\alpha(X_t-\overline{X}_t)+\mathrm{i}\beta\overline{X}_t} \\ &\quad -\mathrm{i}(\alpha-\beta)\int_0^t \mathrm{e}^{\mathrm{i}\alpha(X_s-\overline{X}_s)+\mathrm{i}\beta\overline{X}_s} d\overline{X}_s,\ t \geq 0 \end{aligned}$$

is a martingale. Note, for the reader familiar with general stochastic calculus for semi-martingales, one may equally prove that the latter is a martingale for a general spectrally negative Lévy process.

(ii) Use the fact that $\mathbb{E}(M_{\mathbf{e}_q}) = 0$, where $\mathbf{e}_q$ is an independent exponentially distributed random variable with parameter q, to show that

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\alpha(X_{\mathbf{e}_q}-\overline{X}_{\mathbf{e}_q})+\mathrm{i}\beta\overline{X}_{\mathbf{e}_q}}) = \frac{q(\Phi(q)-\mathrm{i}\alpha)}{(\Psi(\alpha)+q)(\mathrm{i}\beta-\Phi(q))}, \tag{8.5}$$

where Φ is the right inverse of the Laplace exponent $\psi(\beta) = -\Psi(-\mathrm{i}\beta)$.

(iii) Deduce that $\overline{X}_{\mathbf{e}_q} - X_{\mathbf{e}_q}$ and $\overline{X}_{\mathbf{e}_q}$ are independent.

References

[1] Feller, W. (1971) *An Introduction to Probability Theory and its Applications. Vol II, 2nd edition.* Wiley, New York.

[2] Kella, O. and Whitt, W. (1992) Useful martingales for stochastic storage processes with Lévy input, *J. Appl. Probab.* **29**, 396–403.