Lévy processes and continuous-state branching processes: part III

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9 Continuous-State Branching Processes

Originating in part from the concerns of the Victorian British upper classes that aristocratic surnames were becoming extinct, the theory of branching processes now forms a cornerstone of classical applied probability. Some of the earliest work on branching processes dates back to Galton and Watson in 1874, [25]. However, approximately 100 years later, it was discovered in [13] that the less well-exposed work of I.J. Bienaymé, dated around 1845, contained many aspects of the later work of Galton and Watson. The *Bienaymé–Galton–Watson* process, as it is now known, is a discrete time Markov chain with state space $\{0, 1, 2, ...\}$ described by the sequence $\{Z_n : n = 0, 1, 2, ...\}$ satisfying the recursion $Z_0 > 0$ and

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_i^{(n)} \tag{9.1}$$

for $n = 1, 2, ...$ where $\{\xi_i^{(n)} : i = 1, 2, ...\}$ are independent and identically distributed on $\{0, 1, 2, ...\}$. We use the usual notation $\sum_{i=1}^{0}$ to represent the empty sum. The basic idea behind this model is that Z_n is the population count in the nth generation and from an initial population Z_0 (which may be randomly distributed) individuals reproduce asexually and independently with the same distribution of numbers of offspring. The latter reproductive properties are referred to as the *branching property*. Note that as soon as $Z_n = 0$ it follows from the given construction that $Z_{n+k} = 0$ for all $k = 1, 2, ...$ A particular consequence of the branching property is that if $Z_0 = a + b$ then Z_n is equal in distribution to $Z_n^{(1)} + Z_n^{(2)}$ where $Z_n^{(1)}$ and $Z_n^{(2)}$ are independent with the same distribution as an nth generation Bienaymé–Galton–Watson process initiated from population sizes a and b, respectively.

A mild modification of the Bienaymé–Galton–Watson process is to set it into continuous time by assigning life lengths to each individual which are independent and exponentially distributed with parameter $\lambda > 0$. Individuals reproduce at their moment of death in the same way as described previously for the Bienaymé-Galton-Watson process. If $Y = \{Y_t : t \geq 0\}$ is the $\{0, 1, 2, ...\}$-valued process describing the population size then it is straightforward to see that the lack of memory property of the exponential distribution implies that for all $0 \leq s \leq t$,

$$Y_t = \sum_{i=1}^{Y_s} Y_{t-s}^{(i)},$$

where given $\{Y_u : u \leq s\}$ the variables $\{Y_{t-s}^{(i)} : i = 1, ..., Y_s\}$ are independent with the same distribution as Y_{t-s} conditional on $Y_0 = 1$. In that case, we may talk of Y as a continuous-time Markov chain on $\{0, 1, 2, ...\}$, with probabilities, say, $\{P_y : y = 0, 1, 2, ...\}$ where P_y is the law of Y under the assumption that $Y_0 = y$. As before, the state 0 is absorbing in the sense that if $Y_t = 0$ then $Y_{t+u} = 0$ for all $u > 0$. The process Y is called the *continuous time Markov branching process.* The branching property for Y may now be formulated as follows.

Definition 9.1 (Branching property) *For any $t \geq 0$ and y_1, y_2 in the state space of Y, Y_t under $P_{y_1+y_2}$ is equal in law to the independent sum $Y_t^{(1)} + Y_t^{(2)}$ where the distribution of $Y_t^{(i)}$ is equal to that of Y_t under P_{y_i} for $i = 1, 2$.*

So far there appears to be little connection with Lévy processes. However a remarkable time transformation shows that the path of Y is intimately linked to the path of a compound Poisson process with jumps whose distribution is supported in $\{-1, 0, 1, 2, ...\}$, stopped at the first instant that it hits zero. To explain this in more detail let us introduce the probabilities $\{\pi_i : i = -1, 0, 1, 2, ...\}$, where $\pi_i = P(\xi = i + 1)$ and ξ has the same distribution as the typical family size in the Bienaymé–Galton–Watson process. To avoid complications let us assume that $\pi_0 = 0$ so that a transition in the state of Y always occurs when an individual dies. When jumps of Y occur, they are independent and always distributed according to $\{\pi_i : i = -1, 0, 1, ...\}$. The idea now is to adjust time accordingly with the evolution of Y in such a way that these jumps are spaced out with inter-arrival times that are independent and exponentially distributed. Crucial to the following exposition is the simple and well-known fact that the minimum of $n \in \{1, 2, ...\}$ independent and exponentially distributed random variables, each with parameter λ, is exponentially distributed with parameter λn. Further, that if $\mathbf{e}_\alpha$ is exponentially distributed with parameter $\alpha > 0$ then for $\beta > 0$, $\beta \mathbf{e}_\alpha$ is equal in distribution to $\mathbf{e}_{\alpha/\beta}$.

Write for $t \geq 0$,

$$J_t = \int_0^t Y_u \mathrm{d}u$$

set

$$\varphi_t = \inf\{s \geq 0 : J_s > t\}$$

with the usual convention that $\inf \emptyset = \infty$ and define

$$X_t = Y_{\varphi_t} \tag{9.2}$$

with the understanding that when $\varphi_t = \infty$ we set $X_t = 0$. Now observe that when $Y_0 = y \in \{1, 2, ...\}$ the first jump of Y occurs at a time, say T_1 (the minimum of y independent exponential random variables, each with parameter $\lambda > 0$) which is exponentially distributed with parameter λy and the size of the jump is distributed according to $\{\pi_i : i = -1, 0, 1, 2, ...\}$. However note that, on account of the fact that $\varphi_{J_{T_1}} = T_1$, the quantity $J_{T_1} = \int_0^{T_1} y \mathrm{d}u =$

yT_1 is the first time that the process $X = \{X_t : t \geq 0\}$ jumps. The latter time is exponentially distributed with parameter λ. The jump at this time is independent and distributed according to $\{\pi_i : i = -1, 0, 1, 2, ...\}$.

Given the information $\mathcal{G}_1 = \sigma(Y_t : t \leq T_1)$, the lack of memory property implies that the continuation $\{Y_{T_1+t} : t \geq 0\}$ has the same law as Y under P_y with $y = Y_{T_1}$. Hence if T_2 is the time of the second jump of Y then conditional on $\mathcal{G}_1$ we have that $T_2 - T_1$ is exponentially distributed with parameter λY_{T_1} and $J_{T_2} - J_{T_1} = Y_{T_1}(T_2 - T_1)$ which is again exponentially distributed with parameter λ and further, is independent of $\mathcal{G}_1$. Note that J_{T_2} is the time of the second jump of X and the size of the second jump is again independent and distributed according to $\{\pi_i : i = -1, 0, 1, ...\}$. Iterating in this way it becomes clear that X is nothing more than a compound Poisson process with arrival rate λ and jump distribution

$$F(\mathrm{d}x) = \sum_{i=-1}^{\infty} \pi_i \delta_i(\mathrm{d}x) \tag{9.3}$$

stopped on first hitting the origin.

A converse to this construction is also possible. Suppose now that $X = \{X_t : t \geq 0\}$ is a compound Poisson process with arrival rate $\lambda > 0$ and jump distribution $F(\mathrm{d}x) = \sum_{i=-1}^{\infty} \pi_i \delta_i(\mathrm{d}x)$. Write

$$I_t = \int_0^t X_u^{-1} \mathrm{d}u$$

and set

$$\theta_t = \inf\{s \geq 0 : I_s > t\}. \tag{9.4}$$

again with the understanding that $\inf \emptyset = \infty$. Define

$$Y_t = X_{\theta_t \wedge \tau_0^-}$$

where $\tau_0^- = \inf\{t > 0 : X_t < 0\}$. By analysing the behaviour of $Y = \{Y_t : t \geq 0\}$ at the jump times of X in a similar way to above one readily shows that the process Y is a continuous time Markov branching process. The details are left as an exercise to the reader.

The relationship between compound Poisson processes and continuous time Markov branching processes described above turns out to have a much more general setting. In the foundational work of Lamperti [20, 21] it is shown that there exists a correspondence between a class of branching processes called continuous-state branching processes and Lévy processes with no negative jumps (that is to say spectrally negative Lévy processes or processes which are the negative of a subordinator). We investigate this relation in more detail in the remainder of this text.

10 The Lamperti Transform

A $[0,\infty)$-valued strong Markov process $Y = \{Y_t : t \geq 0\}$ with probabilities $\{P_x : x \geq 0\}$ is called a continuous-state branching process if it has paths that are right continuous with left limits and its law observes the branching property given in Definition 9.1. Another way of phrasing the branching property is that for all $\theta \geq 0$ and $x, y \geq 0$,

$$E_{x+y}(\mathrm{e}^{-\theta Y_t}) = E_x(\mathrm{e}^{-\theta Y_t})E_y(\mathrm{e}^{-\theta Y_t}). \tag{10.1}$$

Note from the above equality that after an iteration we may always write for each $x > 0$,

$$E_x(\mathrm{e}^{-\theta Y_t}) = E_{x/n}(\mathrm{e}^{-\theta Y_t})^n \tag{10.2}$$

showing that Y_t is infinitely divisible for each $t > 0$. If we define for $\theta, t \geq 0$,

$$g(t, \theta, x) = -\log E_x(\mathrm{e}^{-\theta Y_t}),$$

then (10.2) implies that for any positive integer m,

$$g(t, \theta, m) = ng(t, \theta, m/n) \text{ and } g(t, \theta, m) = mg(t, \theta, 1)$$

showing that for $x \in \mathbb{Q} \cap [0,\infty)$,

$$g(t, \theta, x) = xu_t(\theta), \tag{10.3}$$

where $u_t(\theta) = g(t, \theta, 1) \geq 0$. From (10.1) we also see that for $0 \leq z < y$, $g(t, \theta, z) \leq g(t, \theta, y)$ which implies that $g(t, \theta, x-)$ exists as a left limit and is less than or equal to $g(t, \theta, x+)$ which exists as a right limit. Thanks to (10.3), both left and right limits are the same so that for all $x > 0$

$$E_x(\mathrm{e}^{-\theta Y_t}) = \mathrm{e}^{-xu_t(\theta)}. \tag{10.4}$$

The Markov property in conjunction with (10.4) implies that for all $x > 0$ and $t, s, \theta \geq 0$,

$$\mathrm{e}^{-xu_{t+s}(\theta)} = E_x\left(E(\mathrm{e}^{-\theta Y_{t+s}}|Y_t)\right) = E_x\left(\mathrm{e}^{-Y_t u_s(\theta)}\right) = \mathrm{e}^{-xu_t(u_s(\theta))}.$$

In other words the Laplace exponent of Y obeys the semi-group property

$$u_{t+s}(\theta) = u_t(u_s(\theta)).$$

The first significant glimpse one gets of Lévy processes in relation to the above definition of a continuous-state branching process comes with the following result for which we offer no proof on account of technicalities (see however Exercise 19 for intuitive motivation and Chap. II of Le Gall (1999) or Silverstein [24] for a proof).

Theorem 10.1 *For $t, \theta \geq 0$, suppose that $u_t(\theta)$ is the Laplace functional given by (10.4) of some continuous-state branching process. Then it is necessarily differentiable in t and satisfies*

$$\frac{\partial u_t}{\partial t}(\theta) + \psi(u_t(\theta)) = 0 \tag{10.5}$$

with initial condition $u_0(\theta) = \theta$ where for $\lambda \geq 0$,

$$\psi(\lambda) = -q - a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(0,\infty)} (\mathrm{e}^{-\lambda x} - 1 + \lambda x \mathbf{1}_{(x<1)})\Pi(\mathrm{d}x) \tag{10.6}$$

and in the above expression, $q \geq 0$, $a \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure supported in $(0, \infty)$ satisfying $\int_{(0,\infty)}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$.

Note that for $\lambda \geq 0$, $\psi(\lambda) = \log \mathbb{E}(\mathrm{e}^{-\lambda X_1})$ where X is either a spectrally positive Lévy process[1] or a subordinator, killed independently at rate $q \geq 0$.[2] Otherwise said, ψ is the Laplace exponent of a killed spectrally negative Lévy process or the negative of the Laplace exponent of a killed subordinator. From Section 7.2, we know for example that ψ is convex, infinitely differentiable on $(0, \infty)$, $\psi(0) = q$ and $\psi'(0+) \in [-\infty, \infty)$. Further, if X is a (killed) subordinator, then $\psi(\infty) - \psi(0) < 0$ and otherwise we have that $\psi(\infty) = \infty$.

For each $\theta > 0$ the solution to (10.5) can be uniquely identified by the relation

$$-\int_\theta^{u_t(\theta)} \frac{1}{\psi(\xi)}\mathrm{d}\xi = t. \tag{10.7}$$

(This is easily confirmed by elementary differentiation, note also that the lower delimiter implies that $u_0(\theta) = \theta$ by letting $t \downarrow 0$.)

From the discussion earlier we may deduce that *if a continuous-state branching process exists*, then it is associated with a particular function $\psi : [0, \infty) \mapsto \mathbb{R}$ given by (10.6). Formally speaking, we shall refer to all such ψ as *branching mechanisms*. We will now state without proof the Lamperti transform which, amongst other things, shows that for every branching mechanism ψ there exists an associated continuous-state branching process.

Theorem 10.2 *Let ψ be any given branching mechanism.*

(i) *Suppose that $X = \{X_t : t \geq 0\}$ is a Lévy process with no negative jumps, initial position $X_0 = x$, killed at an independent exponentially distributed time with parameter $q \geq 0$. Further, $\psi(\lambda) = \log \mathbb{E}_x(e^{-\lambda(X_1 - x)})$. Define for $t \geq 0$,*

$$Y_t = X_{\theta_t \wedge \tau_0^-},$$

[1] Obviously a spectrally positive processes is, by definition, the negative of a spectrally negative Lévy process and thus excludes subordinators.

[2] As usual, we understand the process X killed at rate q to mean that it is killed after an independent and exponentially distributed time with parameter q. Further $q = 0$ means there is no killing.

where $\tau_0^- = \inf\{t > 0 : X_t < 0\}$ *and*

$$\theta_t = \inf\{s > 0 : \int_0^s \frac{\mathrm{d}u}{X_u} > t\}$$

then $Y = \{Y_t : t \geq 0\}$ *is a continuous-state branching process with branching mechanism* ψ *and initial value* $Y_0 = x$.

(ii) *Conversely suppose that* $Y = \{Y_t : t \geq 0\}$ *is a continuous-state branching process with branching mechanism* ψ, *such that* $Y_0 = x > 0$. *Define for* $t \geq 0$,

$$X_t = Y_{\varphi_t},$$

where

$$\varphi_t = \inf\{s > 0 : \int_0^s Y_u \mathrm{d}u > t\}.$$

Then $X = \{X_t : t \geq 0\}$ *is a Lévy process with no negative jumps, killed at the minimum of the time of the first entry into* $(-\infty, 0)$ *and an independent and exponentially distributed time with parameter* $q \geq 0$, *with initial position* $X_0 = x$ *and satisfying* $\psi(\lambda) = \log \mathbb{E}(\mathrm{e}^{-\lambda X_1})$.

It can be shown that a general continuous-state branching process appears as the result of an asymptotic re-scaling (in time and space) of the continuous time Bienaymé–Galton–Watson process discussed in Sect. 9; see [14]. Roughly speaking the Lamperti transform for continuous-state branching processes then follows as a consequence of the analogous construction being valid for the continuous time Bienaymé–Galton–Watson process; recall the discussion in Sect. 9.

11 Long-term Behaviour

Recalling the definition of $Z = \{Z_n : n = 0, 1, 2, ...\}$, the Bienaymé–Galton–Watson process, without specifying anything further about the common distribution of the offspring there are two events which are of immediate concern for the Markov chain Z; explosion and absorption. In the first case it is not clear whether or not the event $\{Z_n = \infty\}$ has positive probability for some $n \geq 1$ (the latter could happen if, for example, the offspring distribution has no moments). When $P_x(Z_n < \infty) = 1$ for all $n \geq 1$ we say the process is conservative (in other words there is no explosion). In the second case, we note from the definition of Z that if $Z_n = 0$ for some $n \geq 1$ then $Z_{n+m} = 0$ for all $m \geq 0$ which makes 0 an absorbing state. As Z_n is to be thought of as the size of the nth generation of some asexually reproducing population, the event $\{Z_n = 0 \text{ for some } n > 0\}$ is referred to as extinction.

In this section we consider the analogues of conservative behaviour and extinction within the setting of continuous-state branching processes. In addition we shall examine the laws of the supremum and total progeny process of continuous-state branching processes. These are the analogues of

$$\sup_{n\geq 0} Z_n \text{ and } \{\sum_{0\leq k\leq n} Z_k : n \geq 0\}$$

for the Bienaymé–Galton–Watson process. Note in the latter case, total progeny is interpreted as the total number of offspring to date.

11.1 Conservative Processes

A continuous-state branching process $Y = \{Y_t : t \geq 0\}$ is said to be *conservative* if for all $t > 0$, $P(Y_t < \infty) = 1$. The following result is taken from Grey [12].

Theorem 11.1 *A continuous-state branching process with branching mechanism ψ is conservative if and only if*

$$\int_{0+} \frac{1}{|\psi(\xi)|} \mathrm{d}\xi = \infty.$$

A necessary condition is, therefore, $\psi(0) = 0$ and a sufficient condition is $\psi(0) = 0$ and $|\psi'(0+)| < \infty$ (equivalently $q = 0$ and $\mathbb{E}|X_1| < \infty$).

Proof. From the definition of $u_t(\theta)$, a continuous-state branching process is conservative if and only if $\lim_{\theta\downarrow 0} u_t(\theta) = 0$ since, for each $x > 0$,

$$P_x(Y_t < \infty) = \lim_{\theta\downarrow 0} E_x(\mathrm{e}^{-\theta Y_t}) = \exp\{-x \lim_{\theta\downarrow 0} u_t(\theta)\},$$

where the limits are justified by monotonicity. However, note from (10.7) that as $\theta \downarrow 0$,

$$t = -\int_\theta^\delta \frac{1}{\psi(\xi)}\mathrm{d}\xi + \int_{u_t(\theta)}^\delta \frac{1}{\psi(\xi)}\mathrm{d}\xi,$$

where $\delta > 0$ is sufficiently small. However, as the left-hand side is independent of θ we are forced to conclude that $\lim_{\theta\downarrow 0} u_t(\theta) = 0$ if and only if

$$\int_{0+} \frac{1}{|\psi(\xi)|} \mathrm{d}\xi = \infty.$$

Note that $\psi(\theta)$ may be negative in the neighbourhood of the origin and hence the absolute value is taken in the integral.

From this condition and the fact that ψ is a smooth function, one sees immediately that a *necessary* condition for a continuous-state branching process to be conservative is that $\psi(0) = 0$; in other words the "killing rate" $q = 0$. It is also apparent that a *sufficient* condition is that $q = 0$ and that $|\psi'(0+)| < \infty$ (so that ψ is locally linear passing through the origin). Due to the fact that $\psi(\lambda) = \log \mathbb{E}(\mathrm{e}^{-\lambda X_1})$ where X is a Lévy process with no negative jumps, it follows that the latter condition is equivalent to $\mathbb{E}|X_1| < \infty$ where X is the Lévy processes with no negative jumps associated with ψ. ■

Henceforth we shall assume that there always conservativeness.

11.2 Extinction Probabilities

Thanks to the representation of continuous-state branching processes given in Theorem 10.2 (i), it is clear that the latter processes observe the fundamental property that if $Y_t = 0$ for some $t > 0$, then $Y_{t+s} = 0$ for $s \geq 0$. Let $\zeta = \inf\{t > 0 : Y_t = 0\}$. The event $\{\zeta < \infty\} = \{Y_t = 0 \text{ for some } t > 0\}$ is thus referred to as *extinction* in line with terminology used for the Bienaymé–Galton–Watson process.

This can also be seen from the branching property (10.1). By taking $y = 0$ there we see that P_0 must be the measure that assigns probability one to the processes which is identically zero. Hence by the Markov property, once in state zero, the process remains in state zero.

Note from (10.4) that $u_t(\theta)$ is continuously differentiable in $\theta > 0$ (since by dominated convergence, the same is true of the left-hand side of the aforementioned equality). Differentiating (10.4) in $\theta > 0$ we find that for each $x, t > 0$,

$$E_x(Y_t \mathrm{e}^{-\theta Y_t}) = x \frac{\partial u_t}{\partial \theta}(\theta) \mathrm{e}^{-x u_t(\theta)} \tag{11.1}$$

and hence taking limits as $\theta \downarrow 0$ we obtain

$$E_x(Y_t) = x \frac{\partial u_t}{\partial \theta}(0+) \tag{11.2}$$

so that both sides of the equality are infinite at the same time. Differentiating (10.5) in $\theta > 0$ we also find that

$$\frac{\partial}{\partial t} \frac{\partial u_t}{\partial \theta}(\theta) + \psi'(u_t(\theta)) \frac{\partial u_t}{\partial \theta}(\theta) = 0.$$

Standard techniques for first-order differential equations then imply that

$$\frac{\partial u_t}{\partial \theta}(\theta) = c \mathrm{e}^{-\int_0^t \psi'(u_s(\theta)) \mathrm{d}s} \tag{11.3}$$

where $c > 0$ is a constant. Inspecting (11.1) as $t \downarrow 0$ we see that $c = 1$. Now taking limits as $\theta \downarrow 0$ and recalling that for each fixed $s > 0$, $u_s(\theta) \downarrow 0$ (thanks to the assumption of conservativeness) it is straightforward to deduce from (11.2) and (11.3) that

$$E_x(Y_t) = x\mathrm{e}^{-\psi'(0+)t}, \tag{11.4}$$

where we understand the left-hand side to be infinite whenever $\psi'(0+) = -\infty$. Note that from the definition $\psi(\theta) = \log \mathbb{E}(\mathrm{e}^{-\theta X_1})$ where X is a Lévy process with no negative jumps, we know that ψ is convex and $\psi(0+) \in [-\infty, \infty)$ (cf. Section 7.2). Hence in particular to obtain (11.4), we have used dominated convergence in the integral in (11.3) when $|\psi'(0+)| < \infty$ and monotone convergence when $\psi'(0+) = -\infty$.

This leads to the following classification of continuous-state branching processes.

Definition 11.1 *A continuous-state branching process with branching mechanism ψ is called*

(*i*) *subcritical, if $\psi'(0+) > 0$,*

(*ii*) *critical, if $\psi'(0+) = 0$ and*

(*iii*) *supercritical, if $\psi'(0+) < 0$.*

The use of the terminology "criticality" refers then to whether the process will, on average, decrease, remain constant or increase. The same terminology is employed for Bienaymé–Galton–Watson processes where now the three cases in Definition 11.1 correspond to the mean of the offspring distribution being strictly less than, equal to and strictly greater than unity, respectively. The classic result due to the scientists after which the latter process is named states that there is extinction with probability 1 if and only if the mean offspring size is less than or equal to unity (see Chap. I of Athreya and Ney (1972) for example). The analogous result for continuous-state branching processes might therefore read that there is extinction with probability one if and only if $\psi'(0+) \geq 0$. However, here we encounter a subtle difference for continuous-state branching processes as the following simple example shows: In the representation given by Theorem 10.2, take $X_t = 1 - t$ corresponding to $Y_t = \mathrm{e}^{-t}$. Clearly $\psi(\lambda) = \lambda$ so that $\psi'(0+) = 1 > 0$ and yet $Y_t > 0$ for all $t > 0$.

Extinction is characterised by the following result due to Grey [12]; see also Bingham [4].

Theorem 11.2 *Suppose that Y is a continuous-state branching process with branching mechanism ψ. Let $p(x) = P_x(\zeta < \infty)$.*

(i) *If $\psi(\infty) < 0$, then for all $x > 0$, $p(x) = 0$*

(ii) *Otherwise, when $\psi(\infty) = \infty$, $p(x) > 0$ for some (and then for all) $x > 0$ if and only if*

$$\int^{\infty} \frac{1}{\psi(\xi)} \mathrm{d}\xi < \infty$$

in which case $p(x) = \mathrm{e}^{-\Phi(0)x}$ where $\Phi(0) = \sup\{\lambda \geq 0 : \psi(\lambda) = 0\}$.

Proof. (i) If $\psi(\lambda) = \log \mathbb{E}(\mathrm{e}^{-\lambda X_1})$ where X is a subordinator, then clearly from the path representation given in Theorem 10.2 (i), extinction occurs with probability zero. From the discussion following Theorem 10.1, the case that X is a subordinator is equivalent to $\psi(\lambda) < 0$ for all $\lambda > 0$.

(ii) Since for $s, t > 0$, $\{Y_t = 0\} \subseteq \{Y_{t+s} = 0\}$ we have by monotonicity that for each $x > 0$,

$$P_x(Y_t = 0) \uparrow p(x) \tag{11.5}$$

as $t \uparrow \infty$. Hence $p(x) > 0$ if and only if $P_x(Y_t = 0) > 0$ for some $t > 0$. Since $P_x(Y_t = 0) = \mathrm{e}^{-xu_t(\infty)}$, we see that $p(x) > 0$ for some (and then all) $x > 0$ if and only if $u_t(\infty) < \infty$ for some $t > 0$.

Fix $t > 0$. Taking limits in (10.7) as $\theta \uparrow \infty$ we see that if $u_t(\infty) < \infty$, then it follows that

$$\int^{\infty} \frac{1}{\psi(\xi)} \mathrm{d}\xi < \infty. \tag{11.6}$$

Conversely, if the above integral holds, then again taking limits in (10.7) as $\theta \uparrow \infty$ it must necessarily hold that $u_t(\infty) < \infty$.

Finally, assuming (11.6), we have learnt that

$$\int_{u_t(\infty)}^{\infty} \frac{1}{\psi(\xi)} \mathrm{d}\xi = t. \tag{11.7}$$

From (11.5) and the fact that $u_t(\infty) = -x^{-1} \log P_x(Y_t = 0)$ we see that $u_t(\infty)$ decreases as $t \uparrow \infty$ to the largest constant $c \geq 0$ such that $\int_c^\infty 1/\psi(\xi)\mathrm{d}\xi$ becomes infinite. Appealing to the convexity and smoothness of ψ, the constant c must necessarily correspond to a root of ψ in $[0, \infty)$, at which point it ψ behave linearly and thus cause $\int_c^\infty 1/\psi(\xi)\mathrm{d}\xi$ to blow up. There are at most two such points, and the largest of these is described precisely by $c = \Phi(0) \in [0, \infty)$ (see Section 7.2). In conclusion,

$$p(x) = \lim_{t \uparrow \infty} \mathrm{e}^{-xu_t(\infty)} = \mathrm{e}^{-\Phi(0)x}$$

as required. ■

On account of the convexity of ψ we also recover the following corollary to part (ii) of the above theorem.

Corollary 11.1 *For a continuous-state branching process with branching mechanism ψ satisfying $\psi(\infty) = \infty$ and*

$$\int^{\infty} \frac{1}{\psi(\xi)} \mathrm{d}\xi < \infty,$$

we have $p(x) < 1$ for some (and then for all) $x > 0$ if and only if $\psi'(0+) < 0$.

To summarise the conclusions of Theorem 11.2 and Corollary 11.1, we have the following cases for the extinction probability $p(x)$:

Condition	$p(x)$
$\psi(\infty) < 0$	0
$\psi(\infty) = \infty,\ \int^\infty \psi(\xi)^{-1}\mathrm{d}\xi = \infty$	0
$\psi(\infty) = \infty,\ \psi'(0+) < 0,\ \int^\infty \psi(\xi)^{-1}\mathrm{d}\xi < \infty$	$\mathrm{e}^{-\Phi(0)x} \in (0,1)$
$\psi(\infty) = \infty,\ \psi'(0+) \geq 0,\ \int^\infty \psi(\xi)^{-1}\mathrm{d}\xi < \infty$	1

Note that rather interestingly, assuming further that $\psi(0+) < 0$ in the second case above, we have an instance where the path of the underlying Lévy process can be made to have a very strong tendency to move towards $-\infty$ and yet, despite this, after the Lamperti transformation has been applied, the corresponding continuous state branching process never becomes extinct. What is happening in such cases is that even though $\zeta = \infty$ almost surely, it is also the case that $Y_t \to 0$ almost surely as $t \uparrow \infty$.

11.3 Total Progeny and the Supremum

Thinking of a continuous-state branching process, $\{Y_t : t \geq 0\}$ as the continuous time, continuous-state analogue of the Bienaymé–Galton–Watson process, it is reasonable to refer to

$$J_t := \int_0^t Y_u \mathrm{d}u$$

as the total progeny until time $t \geq 0$. In this section our main goal, given in the theorem below, is to provide distributional identities for

$$J_\zeta = \int_0^\zeta Y_u \mathrm{d}u \text{ and } \sup_{s \leq \zeta} Y_s.$$

Let us first recall the following notation. As noted above, for any branching mechanism ψ, when $\psi(\infty) = \infty$ (in other words when the Lévy process associated with ψ is not a subordinator) we have that ψ is the Laplace exponent of a spectrally negative Lévy process. Let $\Phi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}$ (cf. Section 7.2). The following Lemma is due to Bingham [4].

Lemma 11.1 *Suppose that Y is a continuous-state branching process with branching mechanism ψ which satisfies $\psi(\infty) = \infty$. Then*

$$E_x(\mathrm{e}^{-q\int_0^\zeta Y_s \mathrm{d}s}) = \mathrm{e}^{-\Phi(q)x}$$

and in particular for $x > 0$,

$$P_x(\sup_{s \leq \infty} Y_s < \infty) = \mathrm{e}^{-\Phi(0)x}.$$

The right-hand side is equal to unity if and only if Y is not supercritical.

Proof. Suppose now that X is the Lévy process mentioned in Theorem 10.2 (ii). Write in the usual way $\tau_0^- = \inf\{t > 0 : X_t < 0\}$. Then a little thought shows that

$$\tau_0^- = \int_0^\zeta Y_s \mathrm{d}s.$$

The proof of the first part is now completed by invoking Theorem 8.1. Note that X is a spectrally positive Lévy process and hence to implement the aforementioned result, which applies to spectrally negative processes, one must consider the problem of first passage to level x of $-X$ when $X_0 = 0$.

For the proof of the second part, note that $\sup_{s\leq\infty} Y_s < \infty$ if and only if $\int_0^\zeta Y_s \mathrm{d}s < \infty$. The result follows by taking limits as $q \downarrow 0$ in the first part noting that this gives the probability of the latter event. ■

12 Conditioned Processes and Immigration

In the classical theory of Bienaymé–Galton–Watson processes where the offspring distribution is assumed to have finite mean, it is well understood that by taking a critical or subcritical process (for which extinction occurs with probability one) and conditioning it in the long term to remain positive uncovers a beautiful relationship between a martingale change of measure and processes with immigration; cf. Athreya and Ney [1] and Lyons et al. [23]. Let us be a little more specific.

A Bienaymé–Galton–Watson process with immigration is defined as the Markov chain $Z^* = \{Z_n^* : n = 0, 1, ...\}$ where $Z_0^* = z \in \{0, 1, 2, ...\}$ and for $n = 1, 2, ...,$

$$Z_n^* = Z_n + \sum_{k=1}^{n} Z_{n-k}^{(k)}, \tag{12.1}$$

where now $Z = \{Z_n : n \geq 0\}$ has law P_z and for each $k = 1, 2, ..., n$, $Z_{n-k}^{(k)}$ is independent and equal in distribution to numbers in the $(n-k)$th generation of (Z, P_{η_k}) where it is assumed that the initial numbers, η_k, are, independently of everything else, randomly distributed according to the probabilities $\{p_k^* : k = 0, 1, 2, ...\}$. Intuitively speaking one may see the process Z^* as a variant of the Bienaymé–Galton–Watson process, Z, in which, from the first and subsequent generations, there is a generational stream of immigrants $\{\eta_1, \eta_2, ...\}$ each of whom initiates an independent copy of (Z, P_1).

Suppose now that Z is a Bienaymé–Galton–Watson process with probabilities $\{P_x : x = 1, 2, ...\}$ as described above. For any event A which belongs to the sigma algebra generated by the first n generations, it turns out that for each $x = 0, 1, 2, ...$

$$P_x^*(A) := \lim_{m\uparrow\infty} P_x(A|Z_k > 0 \text{ for } k = 0, 1, ..., n+m)$$

is well defined and further,

$$P_x^*(A) = E_x(\mathbf{1}_A M_n),$$

where $M_n = m^{-n}Z_n/Z_0$ and $m = E_1(Z_1)$ which is assumed to be finite. It is not difficult to show that $E_x(Z_n) = xm^n$ and that $\{M_n : n \geq 0\}$ is a martingale using the iteration (9.1). What is perhaps more intriguing is that the new process (Z, P_x^*) can be identified in two different ways:

1. The process $\{Z_n - 1 : n \geq 0\}$ under P_x^* can be shown to have the same law as a Bienaymé–Galton–Watson process with immigration having $x-1$ initial ancestors. The immigration probabilities satisfy $p_k^* = (k+1)p_{k+1}/m$ for $k = 0, 1, 2, ...$ where $\{p_k : k = 0, 1, 2, ...\}$ is the offspring distribution of the original Bienaymé–Galton–Watson process and immigrants initiate independent copies of Z.

2. The process Z under P_x^* has the same law as $x - 1$ initial individuals each initiating independently a Bienaymé–Galton–Watson process under P_1 together with one individual initiating an independent immortal genealogical line of descent, *the spine*, along which individuals reproduce with the tilted distribution $\{kp_k/m : k = 1, 2, ...\}$. The offspring of individuals on the spine who are not themselves part of the spine initiate copies of a Bienaymé–Galton–Watson process under P_1. By subtracting off individuals on the spine from the aggregate population, one observes a Bienaymé–Galton–Watson process with immigration described in (1).

Effectively, taking the second interpretation above to hand, the change of measure has adjusted the statistics on just one genealogical line of descent to ensure that it, and hence the whole process itself, is immortal. See Fig. 10.

Our aim in this section is to establish the analogue of these ideas for critical or subcritical continuous-state branching processes. This is done in Sect. 12.2. However, we first address the issue of how to condition a spectrally positive Lévy process to stay positive. Apart from as being a useful comparison for the case of conditioning a continuous-state branching process, there are reasons to believe that the two classes of conditioned processes might be connected through a Lamperti-type transform on account of the relationship given in Theorem 10.2. This is the very last point we address in Sect 12.2.

12.1 Conditioning a Spectrally Positive Lévy Process to Stay Positive

It is possible to talk of conditioning any Lévy process to stay positive and this is now a well understood and well documented phenomenon; also for the case of random walks. See [2, 3, 5, 6, 7] to name but some of the most recent additions to the literature; see also [17] who considers conditioning a spectrally negative Lévy process to stay in a strip. We restrict our attention to the case of spectrally positive Lévy processes; in part because this is what is required for the forthcoming discussion and in part because this facilitates the mathematics.

Suppose that $X = \{X_t : t \geq 0\}$ is a spectrally positive Lévy process with $\psi(\lambda) = \log \mathbb{E}(\mathrm{e}^{-\lambda X_1})$ for all $\lambda \geq 0$. (So as before, ψ is the Laplace exponent of

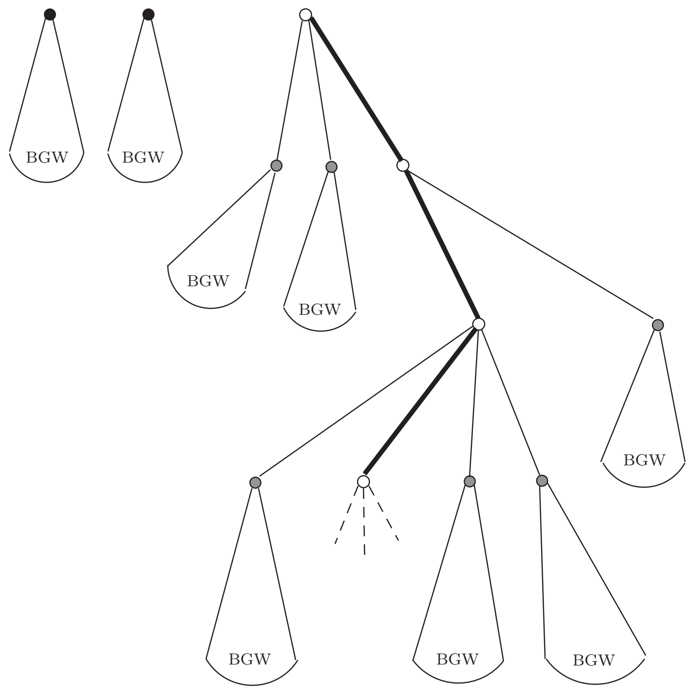

Figure 10: Nodes shaded in *black* initiate Bienaymé–Galton–Watson processes under P_1. Nodes in *white* are individuals belonging to the immortal genealogical line of descent known as the spine. Nodes shaded in *grey* represent the offspring of individuals on the spine who are not themselves members of the spine. These individuals may also be considered as “immigrants”.

the spectrally negative process $-X$). First recall from Theorem 8.1 that for all $x > 0$,

$$\mathbb{E}_x(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{(\tau_0^-<\infty)}) = \mathrm{e}^{-\Phi(q)x}, \tag{12.2}$$

where, as usual, $\tau_0^- = \inf\{t > 0 : X_t < 0\}$ and Φ is the right inverse of ψ. In particular when $\psi'(0+) < 0$, so that $\lim_{t\uparrow\infty} X_t = \infty$, we have that $\Phi(0) > 0$ and $\mathbb{P}(\tau_0^- = \infty) = 1 - \mathrm{e}^{-\Phi(0)x}$. In that case, for any $A \in \mathcal{F}_t$, we may simply apply Bayes' formula and the Markov property, respectively, to deduce that for all $x > 0$,

$$\begin{aligned} \mathbb{P}_x^{\uparrow}(A) &:= \mathbb{P}_x(A|\tau_0^- = \infty) \\ &= \frac{\mathbb{E}_x\left(\mathbf{1}_{(A,t<\tau_0^-)}\mathbb{P}(\tau_0^- = \infty|\mathcal{F}_t)\right)}{\mathbb{P}_x(\tau_0^- = \infty)} \\ &= \mathbb{E}_x\left(\mathbf{1}_{(A,t<\tau_0^-)}\frac{1 - \mathrm{e}^{-\Phi(0)X_t}}{1 - \mathrm{e}^{-\Phi(0)x}}\right) \end{aligned}$$

thus giving sense to "conditioning X to stay positive". If however $\psi'(0+) \geq 0$, in other words, $\liminf_{t\uparrow\infty} X_t = -\infty$, then the above calculation is not possible as $\Phi(0) = 0$ and it is less clear what it means to condition the process to stay positive. The sense in which this may be understood is given in [5].

Theorem 12.1 *Suppose that* $\mathbf{e}_q$ *is an exponentially distributed random variable with parameter* q *independent of* X*. Suppose that* $\psi'(0+) \geq 0$*. For all* $x, t > 0$ *and* $A \in \mathcal{F}_t$*,*

$$\mathbb{P}_x^{\uparrow}(A) := \lim_{q\downarrow 0} \mathbb{P}_x(A, t < \mathbf{e}_q | \tau_0^- > \mathbf{e}_q)$$

exists and satisfies

$$\mathbb{P}_x^{\uparrow}(A) = \mathbb{E}_x(\mathbf{1}_{(A, t<\tau_0^-)} \frac{X_t}{x}).$$

Proof. Again appealing to Bayes' formula followed by the Markov property in conjunction with the lack of memory property and (12.2), we have

$$\begin{aligned}
\mathbb{E}_x(A, t < \mathbf{e}_q | \tau_0^- > \mathbf{e}_q) &= \frac{\mathbb{P}_x(A,\ t < \mathbf{e}_q,\ \tau_0^- > \mathbf{e}_q)}{\mathbb{P}_x(\tau_0^- > \mathbf{e}_q)} \\
&= \frac{\mathbb{E}_x(\mathbf{1}_{(A, t<\mathbf{e}_q \wedge \tau_0^-)} \mathbb{E}(\tau_0^- > \mathbf{e}_q | \mathcal{F}_t))}{\mathbb{E}_x(1 - \mathrm{e}^{-q\tau_0^-})} \\
&= \mathbb{E}_x\left(\mathbf{1}_{(A, t<\tau_0^-)} \mathrm{e}^{-qt} \frac{1 - \mathrm{e}^{-\Phi(q)X_t}}{1 - \mathrm{e}^{-\Phi(q)x}}\right). \qquad (12.3)
\end{aligned}$$

Under the assumption $\psi'(0+) \geq 0$, we know that $\Phi(0) = 0$ and hence by l'Hôpital's rule

$$\lim_{q\downarrow 0} \frac{1 - \mathrm{e}^{-\Phi(q)X_t}}{1 - \mathrm{e}^{-\Phi(q)x}} = \frac{X_t}{x}. \qquad (12.4)$$

Noting also that for all q sufficiently small,

$$\frac{1 - \mathrm{e}^{-\Phi(q)X_t}}{1 - \mathrm{e}^{-\Phi(q)x}} \leq \frac{\Phi(q)X_t}{1 - \mathrm{e}^{-\Phi(q)x}} \leq C\frac{X_t}{x},$$

where $C > 1$ is a constant. The condition $\psi'(0+) \geq 0$ also implies that for all $t > 0$, $\mathbb{E}(|X_t|) < \infty$ and hence by dominated convergence we may take limits in (12.3) as $q \downarrow 0$ and apply (12.4) to deduce the result. ■

It is interesting to note that, whilst $\mathbb{P}_x^{\uparrow}$ is a probability measure for each $x > 0$, when $\psi'(0+) < 0$, this is not necessarily the case when $\psi'(0+) \geq 0$. The following lemma gives a precise account.

Lemma 12.1 *Fix* $x > 0$*. When* $\psi'(0+) = 0$*,* $\mathbb{P}_x^{\uparrow}$ *is a probability measure and when* $\psi'(0+) > 0$*,* $\mathbb{P}_x^{\uparrow}$ *is a sub-probability measure.*

Proof. All that is required to be shown is that for each $t > 0$, $\mathbb{E}_x(\mathbf{1}_{(t<\tau_0^-)}X_t) = x$ for $\mathbb{P}_x^\uparrow$ to be a probability measure and $\mathbb{E}_x(\mathbf{1}_{(t<\tau_0^-)}X_t) < x$ for a sub-probability measure. To this end, recall from the proof of Theorem 12.1 that

$$\begin{aligned}
\mathbb{E}_x(\mathbf{1}_{(t<\tau_0^-)}\frac{X_t}{x}) &= \lim_{q\downarrow 0}\mathbb{P}_x(\ t < \mathbf{e}_q|\tau_0^- > \mathbf{e}_q) \\
&= 1 - \lim_{q\downarrow 0}\mathbb{P}_x(\mathbf{e}_q \leq t|\tau_0^- > \mathbf{e}_q) \\
&= 1 - \lim_{q\downarrow 0}\int_0^t \frac{q\mathrm{e}^{-qu}}{1-\mathrm{e}^{-\Phi(q)x}}\mathbb{P}_x(\tau_0^- > u)\mathrm{d}u \\
&= 1 - \lim_{q\downarrow 0}\frac{q}{\Phi(q)x}\int_0^t \mathrm{e}^{-qu}\mathbb{P}_x(\tau_0^- > u)\mathrm{d}u \\
&= 1 - \lim_{q\downarrow 0}\frac{\psi'(0+)}{x}\int_0^t \mathrm{e}^{-qu}\mathbb{P}_x(\tau_0^- > u)\mathrm{d}u.
\end{aligned}$$

It is now clear that when $\psi'(0+) = 0$ the right-hand side above is equal to unity and otherwise is strictly less than unity thus distinguishing the case of a probability measure from a sub-probability measure. ■

Note that when $\mathbb{E}_x(\mathbf{1}_{(t<\tau_0^-)}X_t) = x$, an easy application of the Markov property implies that $\{\mathbf{1}_{(t<\tau_0^-)}X_t/x : t \geq 0\}$ is a unit mean $\mathbb{P}_x$-martingale so that $\mathbb{P}_x^\uparrow$ is obtained by a martingale change of measure. Similarly when $\mathbb{E}_x(\mathbf{1}_{(t<\tau_0^-)}X_t) \leq x$, the latter process is a supermartingale.

On a final note, the reader may be curious as to how one characterises spectrally positive Lévy processes, and indeed a general Lévy process, to stay positive when the initial value $x = 0$. In general, this is a non-trivial issue, but possible by considering the weak limit of the measure $\mathbb{P}_x^\uparrow$ as measure on the space of paths that are right continuous with left limits. The interested reader should consult [7] for the most recent and up to date account.

12.2 Conditioning a (sub)Critical Continuous-State Branching Process to Stay Positive

Let us now progress to conditioning of continuous-state branching processes to stay positive, following closely Chap. 3 of [18]. We continue to adopt the notation of Sect. 10. Our interest is restricted to the case that there is extinction with probability one for all initial values $x > 0$. According to Corollary 11.1 this corresponds to $\psi(\infty) = \infty$, $\psi'(0+) \geq 0$ and

$$\int^\infty \frac{1}{\psi(\xi)}\mathrm{d}\xi < \infty$$

and henceforth we assume that these conditions are in force. For notational convenience we also set

$$\rho = \psi'(0+).$$

Theorem 12.2 *Suppose that $Y = \{Y_t : t \geq 0\}$ is a continuous-state branching process with branching mechanism ψ satisfying the above conditions. For each event $A \in \sigma(Y_s : s \leq t)$, and $x > 0$,*

$$P_x^{\uparrow}(A) := \lim_{s\uparrow\infty} P_x(A|\zeta > t+s)$$

is well defined as a probability measure and satisfies

$$P_x^{\uparrow}(A) = E_x(\mathbf{1}_A \mathrm{e}^{\rho t}\frac{Y_t}{x}).$$

In particular, $P_x^{\uparrow}(\zeta < \infty) = 0$ and $\{\mathrm{e}^{\rho t} Y_t : t \geq 0\}$ is a martingale.

Proof. From the proof of Theorem 11.2 we have seen that for $x > 0$,

$$P_x(\zeta \leq t) = P_x(Y_t = 0) = \mathrm{e}^{-xu_t(\infty)},$$

where $u_t(\theta)$ satisfies (11.7). Crucial to the proof will be the convergence

$$\lim_{s\uparrow\infty}\frac{u_s(\infty)}{u_{t+s}(\infty)} = \mathrm{e}^{\rho t} \tag{12.5}$$

for each $t > 0$ and hence we first show that this result holds.

To this end note from (11.7) that

$$\int_{u_{t+s}(\infty)}^{u_s(\infty)} \frac{1}{\psi(\xi)}\mathrm{d}\xi = t.$$

On the other hand, recall from the proof of Theorem 11.2 that $u_t(\theta)$ is decreasing to $\Phi(0) = 0$ as $t \downarrow 0$. Hence, since $\lim_{\xi\downarrow 0}\psi(\xi)/\xi = \psi'(0+) = \rho$, it follows that

$$\log\frac{u_s(\infty)}{u_{t+s}(\infty)} = \int_{u_{t+s}(\infty)}^{u_s(\infty)} \frac{1}{\xi}\mathrm{d}\xi = \int_{u_{t+s}(\infty)}^{u_s(\infty)} \frac{\psi(\xi)}{\xi}\frac{1}{\psi(\xi)}\mathrm{d}\xi \to \rho t,$$

as $s \uparrow \infty$ thus proving the claim.

With (12.5) in hand we may now proceed to note that

$$\lim_{s\uparrow\infty}\frac{1-\mathrm{e}^{-Y_t u_s(\infty)}}{1-\mathrm{e}^{-xu_{t+s}(\infty)}} = \frac{Y_t}{x}\mathrm{e}^{\rho t}.$$

In addition, for s sufficiently large,

$$\frac{1-\mathrm{e}^{-Y_t u_s(\infty)}}{1-\mathrm{e}^{-xu_{t+s}(\infty)}} \leq \frac{Y_t u_s(\infty)}{1-\mathrm{e}^{-xu_{t+s}(\infty)}} \leq C\frac{Y_t}{x}\mathrm{e}^{\rho t}$$

for some $C > 1$. Hence we may now apply the Markov property and then the Dominated Convergence Theorem to deduce that

$$\begin{aligned}
\lim_{s\uparrow\infty} P_x(A|\zeta > t+s) &= \lim_{s\uparrow\infty} E_x\left(\mathbf{1}_{(A,\zeta>t)}\frac{P_{Y_t}(\zeta > s)}{P_x(\zeta > t+s)}\right)\\
&= \lim_{s\uparrow\infty} E_x\left(\mathbf{1}_{(A,\zeta>t)}\frac{1-\mathrm{e}^{-Y_t u_s(\infty)}}{1-\mathrm{e}^{-xu_{t+s}(\infty)}}\right)\\
&= E_x(\mathbf{1}_{(A,\zeta>t)}\frac{Y_t}{x}\mathrm{e}^{\rho t}).
\end{aligned}$$

Note that we may remove the qualification $\{t < \zeta\}$ from the indicator on the right-hand side above as $Y_t = 0$ on $\{t \geq \zeta\}$. To show that $P_x^{\uparrow}$ is a probability measure it suffices to show that for each $x, t > 0$, $E_x(Y_t) = \mathrm{e}^{-\rho t}x$. However, the latter was already proved in (11.4). A direct consequence of this is that $P_x^{\uparrow}(\zeta > t) = 1$ for all $t \geq 0$ which implies that $P_x^{\uparrow}(\zeta < \infty) = 0$.

The fact that $\{\mathrm{e}^{\rho t}Y_t : t \geq 0\}$ is a martingale follows in the usual way from consistency of Radon–Nikodym densities. Alternatively, it follows directly from (11.4) by applying the Markov property as follows. For $0 \leq s \leq t$,

$$E_x(\mathrm{e}^{\rho t}Y_t|\sigma(Y_u : u \leq s)) = \mathrm{e}^{\rho s}E_{Y_s}(\mathrm{e}^{\rho(t-s)}Y_{t-s}) = \mathrm{e}^{\rho s}Y_s,$$

which establishes the martingale property. ■

Note that in older literature, the process $(Y, P_x^{\uparrow})$ is called the Q-process. See for example [1].

We have thus far seen that conditioning a (sub)critical continuous-state branching process to stay positive can be performed mathematically in a similar way to conditioning a spectrally positive Lévy processes to stay positive. Our next objective is to show that, in an analogous sense to what has been discussed for Bienaymé–Galton–Watson processes, the conditioned process has the same law as a continuous-state branching process with immigration. Let us spend a little time to give a mathematical description of the latter.

12.3 Continuous-state branching process with immigration

In general we define a Markov process $Y^* = \{Y_t^* : t \geq 0\}$ with probabilities $\{\mathbf{P}_x : x \geq 0\}$ to be a continuous-state branching process with branching mechanism ψ and immigration mechanism ϕ if it is $[0, \infty)$-valued and has paths that are right continuous with left limits and for all $x, t > 0$ and $\theta \geq 0$

$$\mathbf{E}_x(\mathrm{e}^{-\theta Y_t^*}) = \exp\{-xu_t(\theta) - \int_0^t \phi(u_{t-s}(\theta))\mathrm{d}s\}, \tag{12.6}$$

where $u_t(\theta)$ is the unique solution to (10.5) and ϕ is the Laplace exponent of any subordinator. Specifically, for $\theta \geq 0$,

$$\phi(\theta) = d\theta + \int_{(0,\infty)} (1 - \mathrm{e}^{-\theta x})\Lambda(\mathrm{d}x),$$

where Λ is a measure concentrated on $(0, \infty)$ satisfying $\int_{(0,\infty)}(1 \wedge x)\Lambda(\mathrm{d}x) < \infty$.

It is possible to see how the above definition plays an analogous role to (12.1) by considering the following sample calculations (which also show the existence of continuous-state branching processes with immigration). Suppose that $S = \{S_t : t \geq 0\}$ under $\mathbb{P}$ is a pure jump subordinator with Laplace exponent $\phi(\theta)$ (hence $d = 0$). Now define a process

$$Y_t^* = Y_t + \sum_{u \leq t} Y_{t-u}^{(\Delta S_u)},$$

where $\Delta S_u = S_u - S_{u-}$ so that $\Delta S_u = 0$ at all but a countable number of $u \in [0,t]$, Y_t is a continuous-state branching process and for each $(u, \Delta S_u)$ the quantity $Y^{(\Delta S_u)}_{t-u}$ is an independent copy of the process (Y, P_x) at time $t-u$ with $x = \Delta S_u$. We immediately see that $Y^* = \{Y^*_t : t \geq 0\}$ is a natural analogue of (12.1) where now the subordinator S_t plays the role of $\sum_{i=1}^n \eta_i$, the total number of immigrants in Z^* up to and including generation n. It is not difficult to see that it is also a Markov process. Let us proceed further to compute its Laplace exponent. To this end, let us establish briefly the following result.

Lemma 12.2 *Consider a compound Poisson process with rate $\lambda \geq 0$ and positive jumps with common distribution F, then for any continuous $\alpha : [0,\infty) \to [0,\infty)$ we have*

$$\mathbb{E}\left(e^{-\sum_{i=1}^{N_t} \xi_i \alpha(t-\tau_i)}\right) = \exp\left\{-\lambda \int_0^t \int_{(0,\infty)} (1 - e^{-x\alpha(t-s)}) F(\mathrm{d}x)\mathrm{d}s\right\},$$

where $\{\tau_i : i = 1, 2, \cdots\}$ are the arrival times in the underlying Poisson process $N = \{N_t : t \geq 0\}$.

Proof. We have

$$\begin{aligned}
\mathbb{E}\left(e^{-\sum_{i=1}^{N_t} \xi_i \alpha(t-\tau_i)}\right) &= \mathbb{E}\left(\prod_{i=1}^{N_t} \mathbb{E}(e^{-\alpha(t-\tau_i)\xi_1})\right) \\
&= \mathbb{E}\left(\prod_{i=1}^{N_t} \int_{(0,\infty)} e^{-\alpha(t-\tau_i)x} F(\mathrm{d}x)\right) \\
&= \sum_{n\geq 0} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \mathbb{E}\left(\prod_{i=1}^{n} \int_{(0,\infty)} e^{-\alpha(t-\tau_i)x} F(\mathrm{d}x) \,\middle|\, N_t = n\right) \\
&= \sum_{n\geq 0} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \left(\frac{1}{t}\int_0^t \int_{(0,\infty)} e^{-\alpha(t-s)x} F(\mathrm{d}x)\mathrm{d}s\right)^n \\
&= \exp\left\{-\lambda \int_0^t \int_{(0,\infty)} (1 - e^{-x\alpha(t-s)}) F(\mathrm{d}x)\mathrm{d}s\right\},
\end{aligned}$$

where in the fourth equality we have used the classical result that given $N_t = n$, the arrival times are have the joint distribution of n ranked i.i.d. uniformly distributed random variables on $[0,t]$. ■

Returning to the computation of the Laplace exponent of Y^*_t, suppose that $\mathbf{P}_x$ is the law of Y^* when $Y^*_0 = Y_0 = x$ then, with $\mathbf{E}_x$ as the associated expectation operator, for all $\theta \geq 0$,

$$\mathbf{E}_x(\mathrm{e}^{-\theta Y^*_t}) = \mathbf{E}_x\left(\mathrm{e}^{-\theta Y_t} \prod_{v\leq t} E(\mathrm{e}^{-\theta Y^{\Delta S_v}_{t-v}} | S)\right),$$

where the interchange of the product and the conditional expectation is a consequence of monotone convergence.[3] Continuing this calculation we have

$$
\begin{aligned}
\mathbf{E}_x(\mathrm{e}^{-\theta Y^*_t}) &= E_x(\mathrm{e}^{-\theta Y_t})\mathbb{E}\left(\prod_{v\le t} E_{\Delta S_v}(\mathrm{e}^{-\theta Y_{t-v}})\right)\\
&= \mathrm{e}^{-xu_t(\theta)}\mathbb{E}\left(\prod_{v\le t}\mathrm{e}^{-\Delta S_v u_{t-v}(\theta)}\right)\\
&= \mathrm{e}^{-xu_t(\theta)}\lim_{\varepsilon\downarrow 0}\mathbb{E}\left(\mathrm{e}^{-\sum_{v\le t}\Delta S_v \mathbf{1}_{\{\Delta S_v>\varepsilon\}}u_{t-v}(\theta)}\right)\\
&= \mathrm{e}^{-xu_t(\theta)}\lim_{\varepsilon\downarrow 0}\exp\{-\Lambda(\varepsilon,\infty)\int_{[0,t]}\int_{(\varepsilon,\infty)}(1-\mathrm{e}^{-xu_{t-s}(\theta)})\mathrm{d}s\frac{\Lambda(\mathrm{d}x)}{\Lambda(\varepsilon,\infty)}\}\\
&= \exp\{-xu_t(\theta)-\int_0^t\phi(u_{t-s}(\theta))\mathrm{d}s\},
\end{aligned}
$$

where the third equality follows from the previous lemma on account of the fact that $\sum_{u\le t}\Delta S_u \mathbf{1}_{(\Delta S_u>\varepsilon)}$ is a compound Poisson process with arrival rate $\Lambda(\varepsilon,\infty)$ and jump distribution $\Lambda(\varepsilon,\infty)^{-1}\Lambda(dx)|_{(\varepsilon,\infty)}$.

Allowing a drift component in ϕ introduces some lack of clarity with regard to a path-wise construction of Y^* in the manner shown above (and hence its existence). Intuitively speaking, if d is the drift of the underlying subordinator, then the term $d\int_0^t u_{t-s}(\theta)\mathrm{d}s$ which appears in the Laplace exponent of (12.6) may be thought of as due to a "continuum immigration" where, with rate d, in each $\mathrm{d}t$ an independent copy of $(Y, P.)$ immigrates with infinitesimally small initial value. The problem with this intuitive picture is that there are an uncountable number of immigrating processes which creates measurability problems when trying to construct the "aggregate integrated mass" that has immigrated up to time t. Nonetheless, [19] gives a path-wise construction with the help of excursion theory and Itô synthesis; a technique which goes beyond the scope of this text. Returning to the relationship between processes with immigration and conditioned processes, we see that the existence of a process Y^* with an immigration mechanism containing drift can otherwise be seen from the following lemma.

Lemma 12.3 *Fix $x>0$. Suppose that (Y,P_x) is a continuous-state branching process with branching mechanism ψ. Then $(Y,P^{\uparrow}_x)$ has the same law as a continuous-state branching process with branching mechanism ψ and immigration mechanism ϕ where for $\theta\ge 0$,*

$$\phi(\theta)=\psi'(\theta)-\rho.$$

[3] Note that for each $\varepsilon>0$, the Lévy–Itô decomposition tells us that

$$\mathbf{E}(1-\prod_{u\le t}\mathbf{1}_{(\Delta S_u>\varepsilon)}\mathrm{e}^{-\theta Y^{\Delta S_u}_{t-u}}|S)=1-\prod_{u\le t}\mathbf{1}_{(\Delta S_u>\varepsilon)}\mathbf{E}(\mathrm{e}^{-\theta Y^{\Delta S_u}_{t-u}}|S)$$

due to there being a finite number of independent jumps greater than ε. Now take limits as $\varepsilon\downarrow 0$ and apply monotone convergence.

Proof. Fix $x > 0$. Clearly $(Y, P_x^{\uparrow})$ has paths that are right continuous with left limits as for each $t > 0$, when restricted to $\sigma(Y_s : s \leq t)$ we have $P_x^{\uparrow} << P_x$. Next we compute the Laplace exponent of Y_t under $\mathbb{P}^{\uparrow}$ making use of (10.4),

$$\begin{aligned} E_x^{\uparrow}(\mathrm{e}^{-\theta Y_t}) &= E_x(\mathrm{e}^{\rho t}\frac{Y_t}{x}\mathrm{e}^{-\theta Y_t}) \\ &= -\frac{\mathrm{e}^{\rho t}}{x}\frac{\partial}{\partial\theta}\mathbb{E}_x(\mathrm{e}^{-\theta Y_t}) \\ &= -\frac{\mathrm{e}^{\rho t}}{x}\frac{\partial}{\partial\theta}\mathrm{e}^{-x u_t(\theta)} \\ &= \mathrm{e}^{\rho t}\mathrm{e}^{-x u_t(\theta)}\frac{\partial u_t}{\partial\theta}(\theta). \end{aligned} \tag{12.7}$$

Recall from (11.3) that

$$\frac{\partial u_t}{\partial\theta}(\theta) = \mathrm{e}^{-\int_0^t \psi'(u_s(\theta))\mathrm{d}s} = \mathrm{e}^{-\int_0^t \psi'(u_{t-s}(\theta))\mathrm{d}s},$$

in which case we may identify with the help of (10.6),

$$\begin{aligned} \phi(\theta) &= \psi'(\theta) - \rho \\ &= \sigma^2\theta + \int_{(0,\infty)}(1-\mathrm{e}^{-\theta x})x\Pi(\mathrm{d}x). \end{aligned}$$

The latter is the Laplace exponent of a subordinator with drift σ^2 and Lévy measure $x\Pi(\mathrm{d}x)$. ■

Looking again to the analogy with conditioned Bienaymé–Galton–Watson processes, it is natural to ask if there is any way to decompose the conditioned process in some way as to identify the analogue of the genealogical line of descent, earlier referred to as the spine, along which copies of the original process immigrate. This is possible, but again somewhat beyond the scope of this text. We refer the reader instead to [9] and [19].

Finally, as promised earlier, we show the connection between $(X, \mathbb{P}_x^{\uparrow})$ and $(Y, P_x^{\uparrow})$ for each $x > 0$. We are only able to make a statement for the case that $\psi'(0+) = 0$.

Lemma 12.4 *Suppose that $Y = \{Y_t : t \geq 0\}$ is a continuous-state branching process with branching mechanism ψ. Suppose further that $X = \{X_t : t \geq 0\}$ is a spectrally positive Lévy process with Laplace exponent $\psi(\theta)$ for $\theta \geq 0$. Fix $x > 0$. If $\psi'(0+) = 0$ and*

$$\int^{\infty} \frac{1}{\psi(\xi)}\mathrm{d}\xi < \infty,$$

then

(i) *the process $\{X_{\theta_t} : t \geq 0\}$ under $\mathbb{P}_x^{\uparrow}$ has the same law as $(Y, P_x^{\uparrow})$ where*

$$\theta_t = \inf\{s > 0 : \int_0^s \frac{1}{X_u}\mathrm{d}u > t\},$$

(ii) *the process* $\{Y_{\varphi_t} : t \geq 0\}$ *under* $P_x^{\uparrow}$ *has the same law as* $(X, \mathbb{P}_x^{\uparrow})$ *where*

$$\varphi_t = \inf\{s > 0 : \int_0^s Y_u \mathrm{d}u > t\}.$$

Proof. (i) It is easy to show that θ_t is a stopping time with respect to $\{\mathcal{F}_t : t \geq 0\}$ of X. Using Theorem 12.1 and the Lamperti transform we have that if $F(X_{\theta_s} : s \leq t)$ is a non-negative measurable functional of X, then for each $x > 0$,

$$\begin{aligned}
\mathbb{E}_x^{\uparrow}(F(X_{\theta_s} : s \leq t)\mathbf{1}_{(\theta_t < \infty)}) &= \mathbb{E}_x(\frac{X_{\theta_t}}{x} F(X_{\theta_s} : s \leq t)\mathbf{1}_{(\theta_t < \tau_0^-)}) \\
&= E_x(\frac{Y_t}{x} F(Y_s : s \leq t)\mathbf{1}_{(t < \zeta)}) \\
&= E_x^{\uparrow}(F(Y_s : s \leq t)).
\end{aligned}$$

(ii) The proof of the second part is a very similar argument and left to the reader. ■

13 Concluding Remarks

It would be impossible to complete this chapter without mentioning that the material presented above is but the tip of the iceberg of a much grander theory of continuous time branching processes. Suppose in the continuous time Bienaymé–Galton–Watson process we allowed individuals to independently move around according to some Markov process then we would have an example of a *spatial Markov branching particle process*. If continuous-state branching processes are the continuous-state analogue of continuous time Bienaymé–Galton–Watson process then what is the analogue of a spatial Markov branching particle process?

The answer to this question opens the door to the world of measure valued diffusions (or superprocesses) which, apart from its implicit probabilistic and mathematical interest, has many consequences from the point of view of mathematical biology, genetics and statistical physics. The interested reader is referred to the excellent monographs of Etheridge [11], Le Gall [22] and Duquesne and Le Gall [10] for an introduction.

Exercises

Exercise 19 In this exercise, we characterise the Laplace exponent of the continuous time Markov branching process Y described in Sect. 9.

(i) Show that for $\phi > 0$ and $t \geq 0$ there exists some function $u_t(\phi) > 0$ satisfying

$$\mathbb{E}_y(\mathrm{e}^{-\phi Y_t}) = \mathrm{e}^{-y u_t(\phi)},$$

where $y \in \{1, 2, ...\}$.

(ii) Show that for $s, t \geq 0$,

$$u_{t+s}(\phi) = u_s(u_t(\phi)).$$

(iii) Appealing to the infinitesimal behaviour of the Markov chain Y show that

$$\frac{\partial u_t(\phi)}{\partial t} = \psi(u_t(\phi))$$

and $u_0(\phi) = \phi$ where

$$\psi(q) = \lambda \int_{[-1,\infty)} (1 - \mathrm{e}^{-qx}) F(\mathrm{d}x)$$

and F is given in (9.3).

Exercise 20 This exercise is due to Prof. A.G. Pakes. Suppose that $Y = \{Y_t : t \geq 0\}$ is a continuous-state branching process with branching mechanism

$$\psi(\theta) = c\theta - \int_{(0,\infty)} (1 - \mathrm{e}^{\theta x}) \lambda F(\mathrm{d}x),$$

where $c, \lambda > 0$ and F is a probability distribution concentrated on $(0, \infty)$. Assume further that $\psi'(0+) > 0$ (hence Y is subcritical).

(i) Show that Y survives with probability one.

(ii) Show that for all t sufficiently large, $Y_t = \mathrm{e}^{-ct}\Delta$ where Δ is a positive random variable.

Exercise 21 This exercise is taken from [18]. Suppose that Y is a conservative continuous-state branching process with branching mechanism ψ (we shall adopt the same notation as the main text in this chapter). Suppose that $\psi'(\infty) = \infty$ (so that the underlying Lévy process is not a subordinator), $\int^{\infty} \psi(\xi)^{-1} \mathrm{d}\xi < \infty$ and $\rho := \psi'(0+) \geq 0$.

(i) Using (10.7) show that one may write for each $t, x > 0$ and $\theta \geq 0$,

$$E_x^{\uparrow}(\mathrm{e}^{-\theta Y_t}) = \mathrm{e}^{-x u_t(\theta) + \rho t} \frac{\psi(u_t(\theta))}{\psi(\theta)}$$

which is a slightly different representation to (12.6) used in the text.

(ii) Assume that $\rho = 0$. Show that for each $x > 0$

$$P_x^{\uparrow}(\lim_{t \uparrow \infty} Y_t = \infty) = 1.$$

(Hint: you may use the conclusion of Exercise **??** (iii)).

(iii) Now assume that $\rho > 0$. Recalling that the convexity of ψ implies that $\int_{(1,\infty)} x\Pi(\mathrm{d}x) < \infty$ (cf. Sect. **??**), show that

$$0 \leq \int_0^\theta \frac{\psi(\xi) - \rho\xi}{\xi^2}\mathrm{d}\xi = \frac{1}{2}\sigma^2\theta + \int_{(0,\infty)} x\Pi(\mathrm{d}x) \cdot \int_0^{\theta x} \left(\frac{\mathrm{e}^{-\lambda} - 1 + \lambda}{\lambda^2}\right) \mathrm{d}\lambda.$$

Hence using the fact that $\psi(\xi) \sim \rho\xi$ as $\xi \downarrow 0$ show that

$$\int^\infty x \log x \Pi(\mathrm{d}x) < \infty$$

if and only if

$$0 \leq \int_{0+} \left(\frac{1}{\rho\xi} - \frac{1}{\psi(\xi)}\right) \mathrm{d}\xi < \infty.$$

(iv) Keeping with the assumption that $\rho > 0$ and $x > 0$, show that

$$Y_t \overset{P_x^\uparrow}{\to} \infty$$

if $\int^\infty x \log x \Pi(\mathrm{d}x) = \infty$ and otherwise Y_t converges in distribution under $P_x^\uparrow$ as $t \uparrow \infty$ to a non-negative random variable Y_∞ with Laplace transform

$$E_x^\uparrow(\mathrm{e}^{-\theta Y_\infty}) = \frac{\rho\theta}{\psi(\theta)} \exp\left\{-\rho \int_0^\theta \left(\frac{1}{\rho\xi} - \frac{1}{\psi(\xi)}\right) \mathrm{d}\xi\right\}.$$

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