LÉVY PROCESSES

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Abstract. We give a brief introduction to the class of stochastic processes known as Lévy processes, concentrating principally on their relation with infinitely divisible distributions and the Lévy-Itô decomposition.

Key words: Lévy process, infinite divisibility, Lévy-Itô decomposition.

1 Introduction

Lévy processes are a class of d-dimensional stochastic processes that may be thought of as the continuous-time analogue of a random walk. Mathematically speaking their definition is very straightforward as follows.

Definition 1 *An $\mathbb{R}^d$-valued stochastic process $X = \{X_t : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Lévy process if it possesses the following properties:*

(i) *The paths of X are $\mathbb{P}$-almost surely right continuous with left limits.*

(ii) $\mathbb{P}(X_0 = 0) = 1$.

(iii) *For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.*

(iv) *For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$.*

Historically they have always played a central role in the study of stochastic processes with some of the earliest work dating back to the early 1900s. The reason for this is that, mathematically speaking, they represent an extremely robust class of processes which exhibit many of the interesting phenomena that appear in, for example, the theories of stochastic and potential analysis. Moreover, this in turn, together with their elementary definition, has made Lévy processes an extremely attractive class of processes with which to model in a wide variety of physical, biological, engineering and economical scenarios. Indeed, the first appearance of particular examples of Lévy processes can be found in the foundational works of Bachelier [1, 2], concerning the use of Brownian motion within the context of financial mathematics, and Lundberg [9], concerning the use of Poisson processes within the context of insurance mathematics.

The term "Lévy process" honours the work of the French mathematician Paul Lévy who, although not alone in his contribution, played an instrumental role in bringing together an understanding and characterization of processes with stationary and independent increments. In earlier literature, Lévy processes can be found under a number of different names. In the 1940s, Lévy himself referred to them as a sub-class of *processus additifs* (additive processes), that is processes with independent increments. For the most part however, research literature through the 1960s and 1970s refers to Lévy processes simply as *processes with stationary and independent increments.* One sees a change in language through the 1980s and by the 1990s the use of the term "Lévy process" had become standard.

Judging by the volume of published mathematical research articles, the theory of Lévy processes can be said to have experienced a steady flow of interest from the time of the foundational works, for example of Lévy [8], Kolmogorov [7], Khintchine [6] and Itô [5]. However, it was arguably in the 1990s that a surge of interest in this field of research occurred, drastically accelerating the breadth and depth of understanding and application of the theory of Lévy processes. Whilst there are many who made prolific contributions during this period as well as thereafter, the general progression of this field of mathematics was enormously encouraged by the monographs of Bertoin [3] and Sato [10]. Somewhat romantically, it was also the growing research momentum in the field of financial and insurance mathematics which stimulated a great deal of the interest in Lévy processes in recent times; thus entwining the modern theory of Lévy processes ever more with its historical roots.

2 Lévy processes and infinite divisibility

The properties of stationary and independent increments implies that a Lévy process is a Markov process. One may show in addition that Lévy processes are strong Markov processes. From Definition 1 alone it is otherwise difficult to see just how rich a class of processes the class of Lévy processes forms. To get a better impression in this respect it is necessary to introduce the notion of an *infinitely divisible* distribution. Generally speaking, an $\mathbb{R}^d$-valued random variable Θ has an infinitely divisible distribution if for each $n = 1, 2, ...$ there exist a sequence of i.i.d. random variables $\Theta_{1,n}, ..., \Theta_{n,n}$ such that

$$\Theta \stackrel{d}{=} \Theta_{1,n} + \cdots + \Theta_{n,n}$$

where $\stackrel{d}{=}$ is equality in distribution. Alternatively, this relation can be expressed in terms of characteristic exponents. That is to say, if Θ has characteristic exponent $\Psi(u) := -\log \mathbb{E}(\mathrm{e}^{\mathrm{i}u\cdot\Theta})$, then Θ is infinitely divisible if and only if for all $n \geq 1$ there exists a characteristic exponent of a probability distribution, say Ψ_n, such that $\Psi(u) = n\Psi_n(u)$ for all $u \in \mathbb{R}^d$.

It turns out that Θ has an infinitely divisible distribution if and only if there exists a triple (a, Σ, Π), where $a \in \mathbb{R}^d$, Σ is a $d \times d$ matrix whose eigen values are all non-negative and Π is a measure concentrated on $\mathbb{R}^d \backslash \{0\}$ satisfying

$\int_{\mathbb{R}^d} \left(1 \wedge |x|^2\right) \Pi(\mathrm{d}x) < \infty$, such that

$$\Psi\left(u\right) = \mathrm{i}a \cdot u + \frac{1}{2} u \cdot \Sigma u + \int_{\mathbb{R}^d} (1 - \mathrm{e}^{\mathrm{i}u \cdot x} + \mathrm{i}u \cdot x \mathbf{1}_{(|x|<1)}) \Pi(\mathrm{d}x) \tag{1}$$

for every $\theta \in \mathbb{R}^d$. Here we use the notation $u \cdot x$ for the Euclidian inner product and $|x|$ for Euclidian distance. The measure Π is called the Lévy (characteristic) measure and it is is unique. The identity in (1) is known as the *Lévy–Khintchine formula*.

The link between a Lévy processes and infinitely divisible distributions becomes clear when one notes that for each $t > 0$ and any $n = 1, 2, ...,$

$$X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + \cdots + (X_t - X_{(n-1)t/n}).$$

As a result of the fact that X has stationary independent increments it follows that X_t is infinitely divisible.

It can be deduced from the above observation that any Lévy process has the property that for all $t \geq 0$

$$\mathbb{E}\left(\mathrm{e}^{\mathrm{i}u \cdot X_t}\right) = \mathrm{e}^{-t\Psi(u)}, \tag{2}$$

where $\Psi\left(\theta\right) := \Psi_1\left(\theta\right)$ is the characteristic exponent of X_1, which has an infinitely divisible distribution. The converse of this statement is also true thus constituting the Lévy-Khintchine formula for Lévy processes.

Theorem 2 (Lévy–Khintchine formula for Lévy processes) *$a \in \mathbb{R}^d$, Σ is a $d \times d$ matrix whose eigen values are all non-negative and Π is a measure concentrated on $\mathbb{R}^d \backslash \{0\}$ satisfying $\int_{\mathbb{R}^d} \left(1 \wedge |x|^2\right) \Pi(\mathrm{d}x) < \infty$. Then there exists a Lévy process having characteristic exponent*

$$\Psi\left(u\right) = \mathrm{i}a \cdot u + \frac{1}{2} u \cdot \Sigma u + \int_{\mathbb{R}^d} (1 - \mathrm{e}^{\mathrm{i}u \cdot x} + \mathrm{i}u \cdot x \mathbf{1}_{(|x|<1)}) \Pi(\mathrm{d}x).$$

Two fundamental examples of Lévy processes, which shall be shown in the next section to form the 'building blocks' of all other Lévy processes, are Brownian motion and compound Poisson processes. A Brownian motion is the Lévy process associated with the characteristic exponent

$$\Psi(u) = \frac{1}{2} u \cdot \Sigma u$$

and therefore has increments over time periods of length t which are Gaussian distributed with covariance matrix Σt. It can be shown that, up to the addition of a linear drift, Brownian motions are the only Lévy processes which have continuous paths.

A compound Poisson process is the Lévy process associated with the characteristic exponent

$$\Psi(u) = \int_{\mathbb{R}^d} (1 - e^{\mathrm{i}u \cdot x}) \lambda F(\mathrm{d}x)$$

where $\lambda > 0$ and F is a probability distribution. Such processes may be described path-wise by the piecewise linear process

$$\sum_{i=1}^{N_t} \xi_i,\ t \geq 0$$

where $\{\xi_i : i \geq 1\}$ are a sequence of i.i.d. random variables with common distribution F and $\{N_t : t \geq 0\}$ is a Poisson process with rate λ; the latter being the process with initial value zero and with unit increments whose inter-arrival times are independent and exponentially distributed with parameter λ.

It is a straightforward exercise to show that the sum of any finite number of independent Lévy processes is also a Lévy process. Under some circumstances, one may show that a countably infinite sum of Lévy processes also converges in an appropriate sense to a Lévy process. This idea forms the basis of the Lévy-Itô decomposition discussed in the next section where, as alluded to above, the Lévy processes which are summed together are either a Brownian motion with drift or a compound Poisson process with drift.

3 The Lévy-Itô decomposition

Hidden in the Lévy–Khintchine formula is a representation of the path of a given Lévy process. Every Lévy process may always be written as the independent sum of up to a countably infinite number of other Lévy processes, at most one of which will be a linear Brownian motion and the remaining processes will be compound Poisson processes with drift.

Let Ψ be the characteristic exponent of some infinitely divisible distribution with associated triple (a, Σ, Π). The necessary assumption that $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\Pi(\mathrm{d}x) < \infty$ implies that $\Pi(A) < \infty$ for all Borel A such that 0 is in the interior of A^c and in particular that $\Pi(\{x : |x| \geq 1\}) \in [0, \infty)$. With this in mind, it is not difficult to see that, after some simple reorganization, for $u \in \mathbb{R}^d$, the Lévy–Khintchine formula can be written in the form

$$\begin{aligned}
\Psi(\theta) &= \left\{ \mathrm{i}u \cdot a + \frac{1}{2} u \cdot \Sigma u \right\} \\
&\quad + \left\{ \lambda_0 \int_{|x|\geq 1} (1 - \mathrm{e}^{\mathrm{i}u\cdot x}) F_0(\mathrm{d}x) \right\} \\
&\quad + \sum_{n\geq 1} \left\{ \lambda_n \int_{2^{-n}\leq|x|<2^{-(n-1)}} (1 - \mathrm{e}^{\mathrm{i}u\cdot x}) F_n(\mathrm{d}x) \right. \\
&\qquad\qquad \left. + \mathrm{i}\lambda_n u \cdot \left(\int_{2^{-n}\leq|x|<2^{-(n-1)}} x F_n(\mathrm{d}x) \right) \right\}, \qquad (3)
\end{aligned}$$

where $\lambda_0 = \Pi(\{x : |x| \geq 1\})$, $F_0(\mathrm{d}x) = \Pi(\mathrm{d}x)/\lambda_0$ and for $n = 1, 2, 3, ...,$ $\lambda_n = \Pi(\{x : 2^{-n} \leq |x| < 2^{-(n-1)}\})$ and $F_n(\mathrm{d}x) = \Pi(\mathrm{d}x)/\lambda_n$ (with the understanding that the nth integral is absent if $\lambda_n = 0$). This decomposition suggests that the Lévy process $X = \{X_t : t \geq 0\}$ associated with Ψ may be written as the independent sum

$$X_t = Y_t + X_t^{(0)} + \lim_{k\uparrow\infty} \sum_{n=1}^{k} X_t^{(n)},\ t \geq 0 \tag{4}$$

where

$$Y_t = B_t^{\Sigma} - at,\ \ t \geq 0,$$

with $\{B_t^{\Sigma} : t \geq 0\}$ a d-dimensional Brownian motion with covariance matrix Σ,

$$X_t^{(0)} = \sum_{i=1}^{N_t^{(0)}} \xi_i^{(0)},\ \ t \geq 0,$$

with $\{N_t^{(0)} : t \geq 0\}$ as a Poisson process with rate λ_0 and $\{\xi_i^{(0)} : i \geq 1\}$ are independent and identically distributed with common distribution $F_0(\mathrm{d}x)$ concentrated on $\{x : |x| \geq 1\}$ and for $n = 1, 2, 3, ...$

$$X_t^{(n)} = \sum_{i=1}^{N_t^{(n)}} \xi_i^{(n)} - \lambda_n t \int_{2^{-n}\leq|x|<2^{-(n-1)}} xF_n(\mathrm{d}x),\ \ t \geq 0,$$

with $\{N_t^{(n)} : t \geq 0\}$ as a Poisson process with rate λ_n and $\{\xi_i^{(n)} : i \geq 1\}$ are independent and identically distributed with common distribution $F_n(\mathrm{d}x)$ concentrated on $\{x : 2^{-n} \leq |x| < 2^{-(n-1)}\}$. The limit in (4) needs to be understood in the appropriate context however.

It is a straightforward exercise to deduce that $X_{\cdot}^{(n)}$ is a square integrable martingale on account of the fact that it is a centered compound Poisson process together with the fact that x^2 is integrable in the neighbourhood of the origin against the measure Π. It is not difficult to see that $\sum_{n=1}^{k} X_{\cdot}^{(n)}$ is also a square integrable martingale. The convergence of $\sum_{n=1}^{k} X_{\cdot}^{(n)}$ as $k \uparrow \infty$ can happen in one of two ways. The two quantities

$$\lim_{k\uparrow\infty} \sum_{n=1}^{k} \sum_{i=1}^{N_t^{(n)}} |\xi_i^{(n)}| \text{ and } \lim_{k\uparrow\infty} \sum_{n=1}^{k} \int_{2^{-n}\leq|x|<2^{-(n-1)}} |x|\lambda_n F_n(\mathrm{d}x) \tag{5}$$

are either simultaneously finite or infinite (for all $t > 0$) where the random limit is understood in the almost sure sense. When both are finite, that is to say, when $\int_{|x|<1} |x|\Pi(\mathrm{d}x) < \infty$, then $\sum_{n=1}^{\infty} X_{\cdot}^{(n)}$ is well defined as the difference of a well defined stochastic processes with jumps and a linear drift. Conversely when $\int_{|x|<1} |x|\Pi(\mathrm{d}x) = \infty$, it can be shown that, thanks to the assumption $\int_{|x|<1} |x|^2\Pi(\mathrm{d}x) < \infty$, $\sum_{n=1}^{k} X_{\cdot}^{(n)}$ converges uniformly over finite time horizons in the L^2 norm as $k \uparrow \infty$. In that case, the two exploding limits in (5) compensate one another in the right way for their difference to converge in the prescribed sense.

Either way, the properties of stationary and independent increments and almost surely right continuous paths with left limits that belong to $\sum_{n=1}^{k} X^{(n)}$ as a finite sum of Lévy processes, are also inherited by the limiting process as $k \uparrow \infty$. It is also the case that the limiting Lévy process is also a square integrable martingale just as the elements of the approximating sequence are.

4 Path variation

Consider any function $f : [0, \infty) \to \mathbb{R}^d$. Given any partition $\mathcal{P} = \{a = t_0 < t_2 < \cdots < t_n = b\}$ of the bounded interval $[a, b]$ define the variation of f over $[a, b]$ with partition $\mathcal{P}$ by

$$V_{\mathcal{P}}(f, [a, b]) = \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|.$$

The function f is said to be of bounded variation over $[a, b]$ if

$$V(f, [a, b]) := \sup_{\mathcal{P}} V_{\mathcal{P}}(f, [a, b]) < \infty$$

where the supremum is taken over all partitions of $[a, b]$. Moreover, f is said to be of *bounded variation* if the above inequality is valid for all bounded intervals $[a, b]$. If $V(f, [a, b]) = \infty$ for all bounded intervals $[a, b]$ then f is said to be of *unbounded variation*.

For any given stochastic process $X = \{X_t : t \geq 0\}$ we may adopt these notions in the almost sure sense. So for example, the statement X is a process of bounded variation (or has paths of bounded variation) simply means that as a random mapping, $X : [0, \infty) \to \mathbb{R}^d$ is of bounded variation almost surely.

In the case that X is a Lévy process the Lévy-Itô decomposition also gives the opportunity to establish a precise characterization of the path variation of a Lévy process. Since any Lévy process may written as the independent sum in (4) and any d-dimension Brownian motion is known to have paths of unbounded variation, it follows that any Lévy process for which $\Sigma \neq 0$ has unbounded variation. In the case that $\Sigma = 0$, since the paths of the component $X^{(0)}$ in (4) are independent and clearly of bounded variation (they are piecewise linear), the path variation of X is characterized by the way in which the component $\sum_{n=1}^{k} X_t^{(n)}$ converges. In the case that

$$\int_{|x|<1} |x| \Pi(\mathrm{d}x) < \infty$$

the Lévy process X will thus be of bounded variation and otherwise, when the above integral is infinite, the paths are of unbounded variation.

In the case that $d = 1$, as an extreme case of a Lévy process with bounded variation, it is possible that the process X has non-decreasing paths, in which case it is called a *subordinator*. As is apparent from the Lévy-Itô decomposition (4) this will necessarily occur when $\Pi(-\infty, 0) = 0$,

$$\int_{(0,1)} x\Pi(\mathrm{d}x) < \infty$$

and $\Sigma = 0$. In that case, reconsidering the decomposition (4), one may identify

$$X_t = \left(-a - \int_{(0,1)} x\Pi(\mathrm{d}x)\right) t + \lim_{k\uparrow\infty} \sum_{n=1}^{k} \sum_{i=1}^{N_t^{(n)}} \xi_i^{(n)}.$$

On account of the assumption $\Pi(-\infty, 0) = 0$, all the jumps $\xi_i^{(n)}$ are non-negative. Hence it is also a necessary condition that

$$-a - \int_{(0,1)} x\Pi(\mathrm{d}x) \geq 0$$

for X to have non-decreasing paths. These necessary conditions are also sufficient.

5 Lévy processes as semi-martingales

Recall that a semi-martingale with respect to a given filtration $\mathbb{F} := \{\mathcal{F}_t : t \geq 0\}$ is defined as the sum of an $\mathbb{F}$-local martingale and an $\mathbb{F}$-adapted process of bounded variation. The importance of semi-martingales is that they form a natural class of stochastic processes with respect to which one may construct a stochastic integral and thereafter perform calculus. Moreover, the theory of stochastic calculus plays a significant role in mathematical finance as it can be used as a key ingredient in justifying the pricing and hedging of derivatives in markets where risky assets are modelled as positive semi-martingales.

A popular choice of model for risky assets in recent years has been the exponential of a Lévy process. See for example the monograph of Cont and Tankov [4] for an extensive exposition on these types of models. Thanks to Itô's formula for semi-martingales, the exponential of a Lévy process is a semi-martingale as soon as it can be shown that a Lévy process is a semi-martingale. However reconsidering (4) and recalling that B^{Σ} and $\lim_{k\uparrow\infty}\sum_{n=1}^{k} X_{\cdot}^{(n)}$ are martingales and that $X_{\cdot}^{(0)} - a\cdot$ is an adapted process with bounded variation paths it follows immediately that any Lévy process is a semi-martingale.

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