

Gerber-Shiu Theory

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¹Based on several joint works with E. Biffis, R. Loeffen, C. Ott, Z. Palmowski, J.C. Pardo, J.L. Pérez, X. Zhou.

Classical Cramér-Lundberg process

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- The classical risk insurance ruin problem sees the wealth of an insurance problem modelled by the so-called Cramér-Lundberg process:

$$X_t := x + ct - \sum_{i=1}^{N_t} \xi_i,$$

with the understanding that x is the initial wealth, c is the rate at which premiums are collected and $\{N_t : t \geq 0\}$ is a Poisson process describing the arrival of the i.i.d. claims $\{\xi_i : i \geq 0\}$.

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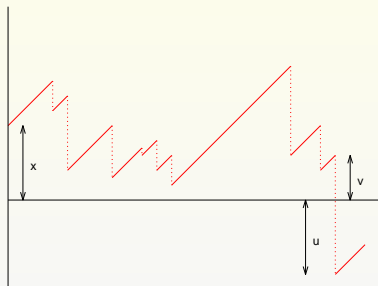
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- A classical field of study, so called Gerber-Shiu, theory, concerns the study of the joint law of

$$\tau_0^-, X_{\tau_0^-} \text{ and } X_{\tau_0^- -},$$

the time of ruin, the deficit at ruin and the wealth prior to ruin.

Ruin



- We are interested in (Gerber-Shiu penalty measure)

$$\mathbb{E}_x(e^{-q\tau_0^-}; -X_{\tau_0^-} \in du, X_{\tau_0^-} \in dv).$$

The general solution

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- Work with Laplace exponent instead of characteristic exponent, $\theta \geq 0$

$$\mathbb{E}(e^{\theta X_t}) = e^{\psi(\theta)t},$$

where

$$\psi(\lambda) = -a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{(x < 1)})\nu(dx),$$

$a \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a measure satisfying $\int_{(0,\infty)} (1 \wedge x^2)\nu(dx) < \infty$.

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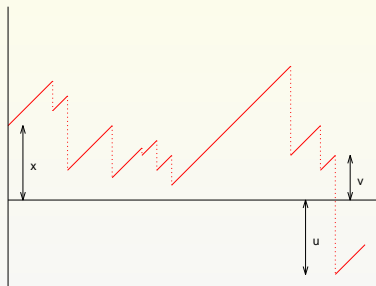
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- Theorem [scale functions]:** For each $q \geq 0$, there exists a continuous, non-decreasing function $W^{(q)} : [0, \infty) \rightarrow [0, \infty)$ satisfying

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x) dx = \frac{1}{\psi(\lambda) - q}$$

for $\lambda > \Phi(q) := \sup\{\theta : \psi(\theta) = q\}$.

The general solution



$$\mathbb{E}_x(e^{-q\tau_0^-}; -X_{\tau_0^-} \in du, X_{\tau_0^-} \in dv) = \left\{ e^{-\Phi(q)v} W^{(q)}(x) - W^{(q)}(x-y) \right\} \nu(v+du)$$

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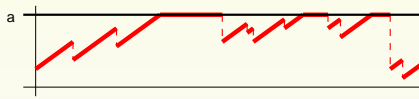


- Net present value of dividends paid until ruin

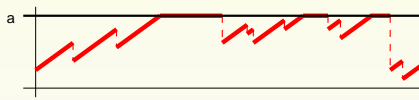
$$\mathbb{E}_x \left(\int_0^{\sigma^a} e^{-qt} dL_t \right)$$

for $x, q \geq 0$, where $\sigma^a = \inf\{t > 0 : X_t - L_t < 0\}$.

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where e_p is an independent and exponentially distributed random variable with some parameter $p > 0$.

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- Remarkably all integer moments² of I can be computed under \mathbb{P}_a . Specifically

$$\mathbb{E}_a \left[\left(\int_0^{\sigma^a} e^{-qt} dL_t \right)^n \right] = n! \prod_{k=1}^n \frac{W^{(kq)}(a)}{W^{(kq)'}(a)}$$

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- When $b = 0$, one minus this quantity gives a Parisian-type ruin probability.

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- More generally, define $\bar{X}_t = \sup_{s \leq t} X_s$ and

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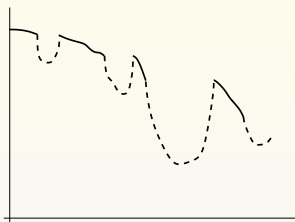
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- We consider two regimes

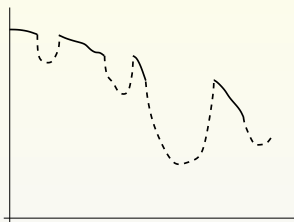
Heavy tax: $\gamma : \mathbb{R} \rightarrow (1, \infty)$

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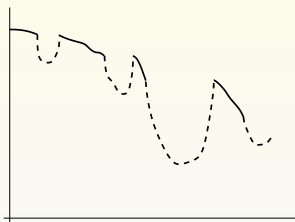


Perturbation through Heavy and Light tax



$$U_t = X_t - \int_{(0,t]} \gamma(\bar{X}_u) d\bar{X}_u = \int_{(0,t]} (1 - \gamma(\bar{X}_u)) d\bar{X}_u + (X_t - \bar{X}_t)$$

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- This offers the following path decomposition: The process U , with $U_0 = x > 0$, follows the deterministic **and monotone** curve

$$\bar{\gamma}(s) = x + \int_x^s (1 - \gamma(s)) ds, \quad s \geq x$$

interlaced with excursions of X from its maximum \bar{X} .

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- **Theorem:** Take the light tax regime. Fix $x > 0$.

$$\mathbb{P}_x(T_0^- < \infty) = 1 - \exp\left(-\int_x^\infty \frac{W'(\bar{\gamma}(s))}{W(\bar{\gamma}(s))} ds\right).$$

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- In the heavy tax regime it is (intuitively) trivial to deduce that

$$\mathbb{P}_x(T_0^- < \infty) = 1$$

Net present value of tax paid until ruin

Suppose that $U_0 = x$ and define

$$a^*(x) = \inf\{s \geq x : \bar{\gamma}(s) < 0\} \in (0, \infty].$$

Theorem: Take either light or heavy tax. For $q \geq 0$,

$$\mathbb{E}_x \left[\int_0^{T_0^-} e^{-qu} \gamma(\bar{X}_u) d\bar{X}_u \right] = \int_x^{a^*(x)} \exp \left(- \int_x^t \frac{W^{(q)'(\bar{\gamma}(s))}{W^{(q)}(\bar{\gamma}(s))} ds \right) \gamma(t) dt.$$

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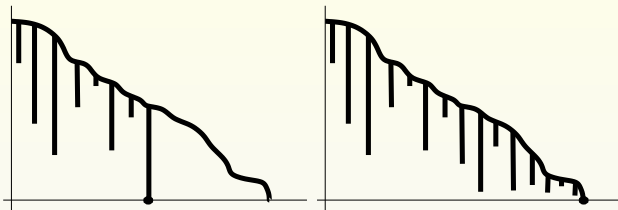
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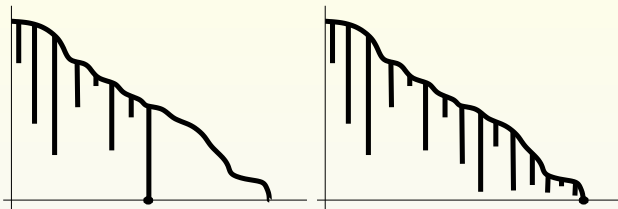
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- In the case of light tax, creeping occurs by the same mechanism as for a pure Lévy process during an excursion of U from the increasing curve $\bar{\gamma}$. Hence there is creeping if and only if a Gaussian component is present in X .
- In the case of heavy tax, creeping can occur during an excursion of U from $\bar{\gamma}$ (in which case a Gaussian component is needed), **OR**, if $\bar{\gamma}$ decreases sharply enough to the origin, then U can meet the origin continuously whilst moving along the curve $\bar{\gamma}$.



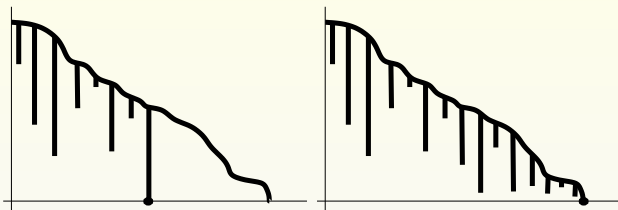


- **Theorem:** In the case of heavy tax, for $x > 0$, assume that

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Then

$$\mathbb{P}_x(\text{type II creeping at } 0) = \exp\left(-\int_x^{a^*(x)} \frac{W'(\bar{\gamma}(s))}{W(\bar{\gamma}(s))} ds\right).$$



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- **Corollary:** If we choose γ is continuous then $\mathbb{P}_x(\text{type II creeping at } 0) > 0$ if and only if X is a Lévy process with bounded variation paths.

Conclusion:

Gerber-Shiu theory is applied excursion theory