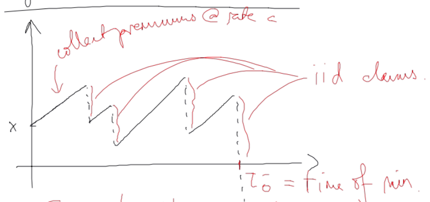
Scale functions & Ruin problems for insurance-risk model



Cramér-Lundberg model. (1905)

Common assumption: $\psi'(0+) > 0$.

[Henceforth we can replace C-L model by a general SNLP with Laplace exponent ψ & $\psi'(0+)$ of means that $\mathbb{E}(X_1) > 0$ which implies that $\lim_{t\to\infty} X_t = \infty$.]

Notation

$$\tau_0^- = \inf\{t > 0 : X_t < 0\} \quad : \quad X_{\tau_0^-} \leq 0$$

$$\tau_x^+ = \inf\{t > 0 : X_t > x\} \quad : \quad X_{\tau_x^+} = x.$$

Esscher transform:

$$\mathcal{E}_t(\beta) = e^{\beta X_t - \psi(\beta)t} \quad : t \geq 0 \text{ is a mg.}$$

$$= e^{\beta B_t - \frac{1}{2}\beta^2 t} \quad \text{in the BM case.}$$

Proof is the same as for BM case!

induce: $$\left.\frac{d\mathbb{P}^\beta}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{\beta X_t - \psi(\beta)t} \qquad t \geq 0.$$

$$\mathcal{F}_t = \sigma(X_s : s \leq t)$$

Then $(X, \mathbb{P}^\beta)$ is still a spec. neg. Lévy process! [Recall Girsanov!]

If $\psi_\beta(\lambda)$ is the Laplace exponent of $(X, \mathbb{P}^\beta)$

Then $\psi_\beta(\lambda) = \psi(\beta+\lambda) - \psi(\beta)$

$$\mathbb{E}^\beta\left(e^{\lambda X_1}\right) = \mathbb{E}\left(e^{\beta X_1 - \psi(\beta)} e^{\lambda X_1}\right)$$

$$= e^{\psi(\beta+\lambda) - \psi(\beta)}.$$

Check by hand (pull a general formula for ψ)

$$\sigma^\beta = \sigma$$

$$\Pi^\beta(dx) = e^{\beta x}\Pi(dx).$$

Theorem: Assume X is a general SNLP s.t. $\psi'(0^+) > 0$.

There exists a family of functions $W^{(q)} : \mathbb{R} \to [0, \infty)$ and

$$Z^{(q)} = 1 + q \int_0^x W^{(q)}(y)\, dy$$

$[W^{(0)} =: W]$

defined for each $q \geq 0$ such that the following hold

(i) for any $q \geq 0$, we have $W^{(q)}(x) = 0$ $x < 0$ and o/w on $[0, \infty)$ is defined as a right cts non-decreasing function whose Laplace transform satisfies

$$\int_0^\infty e^{-\beta x} W^{(q)}(y)\, dy = \frac{1}{\psi(\beta) - q} \quad \forall \beta > \Phi(q)$$

(ii) $\forall x \in \mathbb{R}$, $q \geq 0$

$$\mathbb{E}_x\left(e^{-q\tau_0^-} \mathbb{1}(\tau_0^- < \infty)\right) = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x)$$

(In the case that $q = 0$ we understand the RHS in the limiting sense; $\frac{q}{\Phi(q)} = \frac{\psi(\Phi(q))}{\Phi(q)} \xrightarrow[\Phi(q) \to 0 \text{ as } \psi'(0) > 0]{q \downarrow 0} \psi'(0^+)$

$$\mathbb{P}_x(\tau_0^- < \infty) = 1 - \psi'(0^+) W(x). \quad (*)$$

(iii) $\forall x \leq a$, $q \geq 0$,

$$\mathbb{E}_x\left(e^{-q\tau_a^+} \mathbb{1}(\tau_a^- < \tau_0^-)\right) = \frac{W^{(q)}(x)}{W^{(q)}(a)}$$

$$\mathbb{E}_x\left(e^{-q\tau_0^-} \mathbb{1}(\tau_0^- < \tau_a^+)\right) = Z^{(q)}(x) - Z^{(q)}(a) \frac{W^{(q)}(x)}{W^{(q)}(a)}$$

Results still true when condition $\psi'(0^+) > 0$ removed except $(*)$ which reads $\mathbb{P}_x(\tau_0^- < \infty) = 1$ when $\psi'(0) \leq 0$

Proof : Recall that when $\psi'(0+) > 0$,

$$\mathbb{E}\left(e^{\beta \underline{X}_\infty}\right) = \psi'(0+)\frac{\beta}{\psi(\beta)}$$

define the non-decreasing function $\longrightarrow$ right cts !

$$W(x) := \mathbb{P}_x\left(\underline{X}_\infty \geq 0\right)/\psi'(0^+)$$
$$= \mathbb{P}\left(-\underline{X}_\infty \leq x\right)/\psi'(0^+)$$

$(0 < x \leq a)$

$$\mathbb{P}_x\left(\underline{X}_\infty \geq 0\right) = \mathbb{E}_x\, \mathbb{P}_x\left(\underline{X}_\infty \geq 0 \,\middle|\, \mathcal{F}_{\tau_a^+}\right)$$

$$= \mathbb{E}_x\left[\mathbb{1}_{(\tau_a^+ < \tau_0^-)}\, \mathbb{P}_a\left(\underline{X}_\infty \geq 0\right)\right]$$
$$+ \mathbb{E}_x\left[\mathbb{1}_{(\tau_0^- < \tau_a^+)}\, \boxed{\mathbb{P}_{X_{\tau_0^-}}\left(\underline{X}_\infty \geq 0\right)}\right]$$

$$\left(\mathbb{1}_{(X_{\tau_0^-} < 0)} + \mathbb{1}_{(X_{\tau_0^-} = 0)}\right)$$

$\boxed{} = 0!$ $\qquad = \mathbb{P}_0\left(\underline{X}_\infty \geq 0\right)$

This situation occurs $\Leftarrow\!\!\Rightarrow \sigma > 0$. in which case $\mathbb{P}_0(\underline{X}_\infty \geq 0) = 0$

$$= \mathbb{E}_x\left(\mathbb{1}_{(\tau_a^+ < \tau_0^-)}\right) \mathbb{P}_a\left(\underline{X}_\infty \geq 0\right)$$

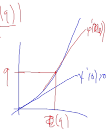
In conclusion :

$$\frac{W(x)}{W(a)} = \mathbb{P}_x\left(\tau_a^+ < \tau_0^-\right).$$

Now consider $(X, \mathbb{P}^{\Phi(q)})$, $q > 0$.

$$\psi_{\Phi(q)}(\lambda) = \psi(\lambda + \Phi(q)) - \underbrace{\psi(\Phi(q))}_{=q}$$
$$= \boxed{\psi(\lambda + \Phi(q)) - q}$$
$$\psi'_{\Phi(q)}(0^+) = \psi'(\Phi(q)) > 0$$

by convexity.



$(X, \mathbb{P}^{\Phi(q)})$ is a SNLP drifting to $+\infty$.

we have just proved that

$$\mathbb{P}_x^{\Phi(q)}\left(\tau_a^+ < \tau_0^-\right) = \frac{W_{\Phi(q)}(x)}{W_{\Phi(q)}(a)}$$

where assuming $\int_0^\infty e^{-\beta x} W_{\Phi(q)}(x)\,dx = \frac{1}{\psi(\beta + \Phi(q)) - q}$

pwf (1) for $q=0$

$$\mathbb{P}_x\left(e^{\Phi(q)\overbrace{(X_{\tau_a^+} - x)}^{(a-x)} - \overbrace{\psi(\Phi(q))}^{q}\tau_a^+}\, \mathbb{1}_{(\tau_a^+ < \tau_0^-)}\right)$$
$$= \mathbb{P}_x\left(e^{\Phi(q)(a-x) - q\tau_a^+}\, \mathbb{1}_{(\tau_a^+ < \tau_0^-)}\right)$$

In conclusion

$$\mathbb{E}_x\left(e^{-q\tau_a^+}\mathbb{1}_{(\tau_a^+ < \tau_0^-)}\right) = \frac{e^{\Phi(q)x} W_{\Phi(q)}(x)}{e^{\Phi(q)a} W_{\Phi(q)}(a)}.$$

Define $W^{(q)}(x) := e^{\Phi(q)x} W_{\Phi(q)}(x)$,

then
$$\int_0^\infty e^{-\beta x} W^{(q)}(x)\,dx = \int_0^\infty e^{-(\beta - \Phi(q))x} W_{\Phi(q)}(x)\,dx$$

providing $\beta > \Phi(q)$
$$= \frac{1}{\psi\left(\left(\beta - \cancel{\Phi(q)}\right) + \cancel{\Phi(q)}\right) - q}$$
$$= \frac{1}{\psi(\beta) - q}$$

First statement of part (iii) of Theorem is proved !!!

$\overline{x \leq a}$

Note $\mathbb{E}_x\left(e^{-q\tau_a^+}\mathbb{1}_{(\tau_a^+ < \tau_0^-)}\right) = \frac{W^{(q)}(x)}{W^{(q)}(a)}$

clearly $W^{(q)}(x)$ must be increasing in x

Also $W^{(q)}(x) = e^{\Phi(q)x} W_{\Phi(q)}(x)$ makes it right cts !

Pf of (1).

$$W(x) = \mathbb{P}_x(\underline{X}_\infty \geq 0) / \psi'(0^+)$$

$$\mathbb{E}\left(e^{\beta \underline{X}_\infty}\right) = \psi'(0^+) \frac{\beta}{\psi(\beta)}.$$

$$\|$$

$$\int_{[0,\infty)} e^{-\beta y} \mathbb{P}(-\underline{X}_\infty \in dy)$$

$\|$ integration by parts.

$$\int_0^\infty \beta e^{-\beta y} \mathbb{P}(-\underline{X}_\infty \leq y) \, dy$$

$$= \int_0^\infty \cancel{\beta} e^{-\beta y} \mathbb{P}_y(\underline{X}_\infty \geq 0) \, dy = \psi'(0^+) \frac{\cancel{\beta}}{\psi(\beta)}$$

Hence with my def^n $W(x) = \mathbb{P}_x(\underline{X}_\infty \geq 0) / \psi'(0^+)$

we have $\displaystyle\int_0^\infty e^{-\beta x} W(x) \, dx = \frac{1}{\psi(\beta)}$; $\beta > 0$.

$$\int_0^\infty e^{-\beta x} \boxed{W^{(q)}(x)\,dx} = \frac{1}{\psi(\beta)-q} \qquad \beta > \Phi(q)$$

$$\Longrightarrow \int_0^\infty e^{-\beta x} \boxed{W^{(q)}(dx)} = \frac{\beta}{\psi(\beta)-q}$$

Think of $W^{(q)}(x)$ as a d.f. of a measure that we shall call $W^{(q)}(dx)$

Recall from the WHF

$$\mathbb{E}\left(e^{\beta \underline{X}_{e_q}}\right) = \frac{q}{\Phi(q)} \frac{\beta - \Phi(q)}{\psi(\beta) - q}$$

inverting

$$\mathbb{P}\left(-\underline{X}_{e_q} \in dx\right) = \frac{q}{\Phi(q)} W^{(q)}(dx) - qW^{(q)}(x)\,dx.$$

$$\mathbb{E}_x\left(e^{-q\tau_0^-} \mathbb{1}_{(\tau_0^- < \infty)}\right)$$

$$= \mathbb{P}_x\left(e_q > \tau_0^-\right) = \mathbb{P}_x\left(\underline{X}_{e_q} < 0\right)$$

$$= \mathbb{P}\left(-\underline{X}_{e_q} > x\right) = 1 - \mathbb{P}\left(-\underline{X}_{e_q} \leq x\right)$$

$$= 1 - \frac{q}{\Phi(q)} W^{(q)}(x) + q\int_0^x W^{(q)}(y)\,dy$$

$$= Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x) \quad \text{as required!}$$

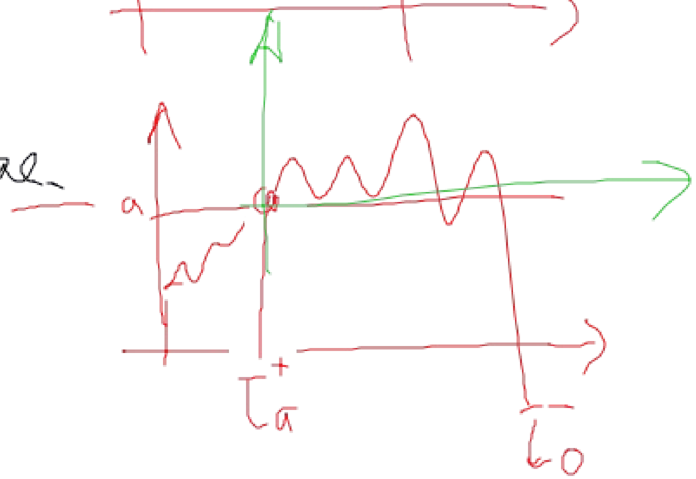
Hence (9.1) follows.

For second part of (1a).

$$\mathbb{E}_x\left(e^{-q\tau_0^-}\mathbf{1}_{(\tau_0^-<\tau_a^+)}\right)$$

$$= \underbrace{\mathbb{E}_x\left(e^{-q\tau_0^-}\mathbf{1}_{(\tau_0^-<\infty)}\right)}_{\text{known}} - \underbrace{\mathbb{E}\left(e^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+<\tau_0^-)}\right)}_{\text{known}}\underbrace{\mathbb{E}_a\left(e^{-q\tau_0^-}\mathbf{1}_{(\tau_0^-<\infty)}\right)}_{\text{known}}$$

To complete pf:
fill in the known formulae



Recent ideas on how to generate concrete examples of W

$$\mathbb{P}_x(\tau_0^- < \infty) = 1 - \psi'(0+) W(x).$$

Recall $\psi(\lambda) = -a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} (e^{\lambda x} - 1 - \lambda x \mathbf{1}_{(|x|<1)}) \Pi(dx)$

Assumption that $\psi'(0) > 0$ ($\Rightarrow \mathbb{E}(X_1) < \infty$).
means we can re-write (exercise)

$$\psi(\lambda) = \psi'(0+)\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} (e^{\lambda x} - 1 - \lambda x) \Pi(dx)$$

$$\underset{\text{integrate by parts}}{=} \psi'(0)\lambda + \frac{1}{2}\sigma^2\lambda^2 + \lambda \int_0^\infty (1 - e^{-\lambda y}) \Pi(-\infty, -y) dy$$

[Note that $\mathbb{E}(X_1) < \infty \iff \int_{-\infty}^{-1} |x| \Pi(dx) < \infty$

$\int_{(-1,0)} x^2 \Pi(dx) < \infty \iff \int_0^1 x \Pi(-\infty, -x) dx < \infty$]

Lemma If X is a SNLP s.t. $\psi'(0+) > 0$.

Then $\boxed{\psi(\lambda) = \lambda \phi(\lambda)}$ $\qquad \int_0^1 x \Upsilon(dx) < \infty$

where $\phi(\lambda) = k + \delta\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x}) \Upsilon(dx)$

with $k = \psi'(0+)$, $\delta = \frac{1}{2}\sigma^2$, $\qquad \Upsilon(dx) = \Pi(-\infty, -x) dx$

Recall that $\int_0^\infty e^{-\beta x} W(dx) = \frac{\beta}{\psi(\beta)} = \frac{\cancel{\beta}}{\cancel{\beta}\phi(\beta)}$

Hence if we know ϕ and can invert! then we know W

Idea: Can we choose ϕ ourselves, define a SNLP via $\psi(\lambda) = \lambda\phi(\lambda)$, and make sure that $1/\phi$ is invertible!?

By inspection we have the following theorem.

Theorem Suppose $\phi(\lambda) = k + \delta\lambda + \int_0^\infty (1 - e^{-\lambda x}) \Upsilon(dx)$

Then $\lambda\phi(\lambda)$ is the Laplace exponent of a SNLP (the parent process)

$\iff k \geq 0$, $\delta \geq 0$, $\Upsilon(dx) = \upsilon(x) dx$

when $\upsilon(x)$ is non-increasing.

In which case Π of the parent process satisfies $\Pi(-\infty, -x) = \upsilon(x)$

Moreover $\sigma = \sqrt{2\delta}$ and $\mathbb{E}(X_1) = k$.

Example of how to use:

$$\phi(\lambda) = \kappa + (\lambda+\gamma)^{\alpha} - \gamma^{\alpha} \quad : \alpha \in (0,1).$$

This is the Laplace exponent of Tempered Stable subordinator (no drift, killing at rate κ, and $\upsilon(x)\,dx = \frac{-1}{\Gamma(-\alpha)} \frac{e^{-\gamma x}}{x^{1+\alpha}}\,dx$)

Parent process: $\sigma = 0$, $\mathbb{E}(X_1) = \kappa$

$$\Pi(dx) = \frac{-1}{\Gamma(-\alpha)} \left(\frac{\gamma}{(-x)^{\alpha+1}} e^{\gamma x} + \frac{(\alpha+1)}{(-x)^{\alpha+2}} e^{\gamma x} \right) dx$$

$x < 0$

Mittag-Leffler functions: $E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n+\beta)}$ $(\alpha, \beta > 0)$.

Laplace-Mellin transform of M-L:

$$\int_0^{\infty} e^{-\theta x} x^{\alpha-1} E_{\alpha,\alpha}(\lambda x^{\alpha})\,dx = \frac{1}{\theta^{\alpha} - \lambda}$$

$\theta = \mu + \gamma$ $\qquad$ $\lambda = \gamma^{\alpha} - \kappa$

$$\int_0^{\infty} e^{-\mu x} \underbrace{\boxed{e^{-\gamma x} x^{\alpha-1} E_{\alpha,\alpha}\left((\gamma^{\alpha}-\kappa) x^{\alpha}\right)}}_{W(x)!}\,dx = \frac{1}{(\mu+\gamma)^{\alpha} - \gamma^{\alpha} + \kappa} = \frac{1}{\phi(\mu)}$$