Weiner-Hopf Factorisation

The random walk : $S_n = \sum_{i=1}^{n} \xi_i$ $\quad \xi_i \overset{iid}{\sim} \xi_1$

$S_0 = 0$.

F is the dist^n of ξ_1, assume $F(0,\infty), F(-\infty,0) > 0$ and $F\{0\} = 0$.

$(p>0)$ T_p ~ geometrically distributed (p)

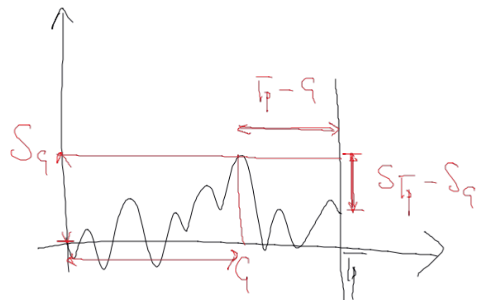
$$\Pr(T_p = k) = p(1-p)^k \quad : k = 0,1,2,\ldots$$

$$G = \min\{k = 0,1,\ldots,T_p : S_k = \max_{j=0,1,\ldots,T_p} S_j\}$$

$$D = \max\{k = 0,1,\ldots,T_p : S_k = \min_{j=0,1,\ldots,T_p} S_j\}.$$

$$N = \inf\{n>0 : S_n > 0\}.$$

Theorem (Wiener-Hopf factorisation). (G, S_G) is indep. of $(T_p - G, S_{T_p} - S_G)$.



Moreover both pairs are infinitely divisible* and the latter has the same law as (D, S_D)

* For each $n \geq 1$

$$\Theta = \Theta_{1,n} + \cdots\cdots + \Theta_{n,n}$$

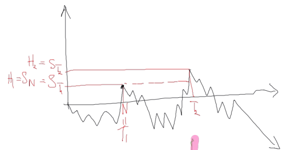
where $\Theta_{i,n}$ are iid

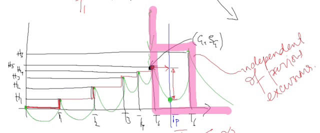
$\Theta \in \mathbb{R}^d$!

Ladder processes

$T_0 = 0, \quad T_n = \inf\{n > T_{n-1} : S_n > S_{T_{n-1}}\}$

$H_n = S_{T_n}$ on $T_n < \infty$. $(n \geq 1)$





$T_n - T_{n-1} \sim N : T_{n-1} < \infty$

$H_n - H_{n-1} \sim S_N : T_{n-1} < \infty$

An excursion "straddles" T_p if the left end pt is at time $k \leq T_p$ and the right end pt. is at time $l > T_p$

By the SMP & lack of memory property, can write

$$(G, S_G) = \sum_{i=1}^{\nu} (N^{(i)}, H^{(i)})$$

where $\nu \sim$ geometrically distributed with parameter $1 - \mathbb{P}(N \leq T_p) = \mathbb{P}(N > T_p)$

$(N^{(i)}, H^{(i)})$ are iid having the same dist^n as (N, S_N) conditioned on $\{N \leq T_p\}$.

Statement $(G, S_G) \perp\!\!\!\perp (T_p - G, S_{T_p} - S_G)$

is obvious by the independence of excursions in "Bernoulli trials of excursions whose lengths exceed T_p".

(G, S_G) inf div follows from:

Exercise Suppose ν is geometric $\perp\!\!\!\perp$ of iid $\{X_i\}$ ($\in \mathbb{R}^d$) then $\sum_{i=1}^{\nu} X_i$ is inf. div.

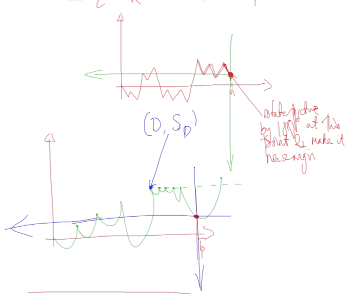
In particular ν is inf div!

$(G, S_G) \stackrel{d}{=} (D, S_D)$

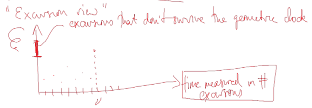
Intuition (rigorous?)

Feller's Duality Lemma:

$$\{S_{n-k} - S_n : k = 0, 1, 2, \dots, n\} \stackrel{law}{=} \{-S_k : k = 0, \dots, n\}$$



"Excursion view"



Weiner-Hopf Factorisation for LP

X – Lévy Process.

$e_p \sim \exp(p) \perp\!\!\!\perp X$.

$\overline{X}_t := \sup_{s\le t} X_s$, $\underline{X}_t := \inf_{s\le t} X_s$.

$\overline{G} = \inf\{ s<t : X_s = \overline{X}_t \}$

$\underline{G} = \sup\{ s<t : X_s = \underline{X}_t \}$

Theorem $(\overline{G}_{e_p}, \overline{X}_{e_p}) \perp\!\!\!\perp$

$(e_p - \overline{G}_{e_p}, X_{e_p} - \overline{X}_{e_p})$, both pairs are inf. div. & latter equals in distⁿ to

$$(\underline{G}_{e_p}, \underline{X}_{e_p}).$$

Why "factorisation"?

$$\mathbb{E}(e^{i\theta X_{e_p}}) = \mathbb{E}\left[e^{i\theta(X_{e_p} - \overline{X}_{e_p} + \overline{X}_{e_p})}\right]$$

$$= \mathbb{E}(e^{i\theta(X_{e_p} - \overline{X}_{e_p})})\,\mathbb{E}(e^{i\theta\overline{X}_{e_p}})$$

$$= \mathbb{E}(e^{i\theta \underline{X}_{e_p}})\,\mathbb{E}(e^{i\theta\overline{X}_{e_p}})$$

$$\int_0^\infty p e^{-pt}\,\mathbb{E}(e^{i\theta X_t})\,dt$$

Laplace transf. of Fourier transf.

The Ladder Process for LPs

X – LP s.t. $-X$ & X are not subordinators.

We want an analogue of

$$l_n = \max\{k : T_k \leq n\}$$

"The count of # of excursions from max in RW up to time n".

"Local time @ max"

Def^n (Local time @ max for LP)

A cts, non-decreasing, $[0,\infty)$-valued, $\mathbb{F}$-adapted process $L = \{L_t : t \geq 0\}$ is called local time @ the max (or just local time) if the following hold.

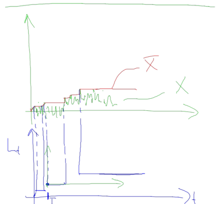
(i) The support of the Stieltjes measure dL_t is the closure of the times $\{t \geq 0 : \overline{X}_t = X_t\}$.

(ii) For every $\mathbb{F}^*$-stopping time T s.t. $\overline{X}_T = X_T$ on $\{T < \infty\}$, almost surely, the shifted process

$$\{L_{T+t} - L_T : t \geq 0\}$$

is independent of $\mathcal{F}^*_T$ on $\{T < \infty\}$ and has the same law as L.

$*$ $\mathcal{F}_t = \sigma\{X_s : s \leq t\}$
$\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$.



L is "flat" when $\overline{X}$ and X are not touching!

To prove that L exists, we continue under the assumption that X is spectrally negative i.e. $\Pi(0,\infty) = 0$ Permanent assumption!!

Note L can only be defined up to a multiplicative constant!

Theorem : L exists and may be taken as $L := \overline{X}$.

Proof : trivial to check

$L^{-1}_t = \inf\{s > 0 : L_s > t\}$

Theorem : If $\mathbb{E}(X_1) \geq 0$ then $L^{-1} = \{L^{-1}_t : t \geq 0\}$ is a subordinator. If $\mathbb{E}(X_1) < 0$ then L^{-1} has the law of a subordinator killed (sent to $+\infty$.) at an independent and exponentially distributed time

Don't be surprised!

$$l^{-1}_n = \inf\{k \geq 0 : l_k \geq n\} = T_n$$

on $\{T_{n-1} < \infty\}$; $T_n - T_{n-1} \overset{d}{=} N$ & indep of $\{T_k : k \leq n-1\}$.

Note : The Theorem is also a statement about first passage times:

$$\begin{aligned} L^{-1}_t &= \inf\{s > 0 : L_s > t\} \\ &= \inf\{s > 0 : \overline{X}_s > t\} \\ &= \inf\{s > 0 : X_s > t\} \\ &= \inf\{s > 0 : X_s \in (t,\infty)\} \end{aligned}$$

unique to spec. neg. case!!

Side results:

Introduce the Laplace exponent (ONLY for spec. neg.)

Go back to the Lévy–Itô decomp.

Recall the characteristic exponent was established by "adding" the char. exponents of lin. BM, CPP of "large" jumps & many CPP with compensation for "small" jumps.

Note that in spec. neg. case, Laplace exponent

$$\psi(\lambda) := \frac{1}{t} \log \mathbb{E}(e^{\lambda X_t})$$

will exist for all $\lambda \geq 0$ because in the above Lévy–Itô decomp, we only need to convince ourselves that the Laplace exponent of a CPP with negative jumps exists — which it trivially does!

Moreover If (a, σ, Π) are triple for X, then

$$\psi(\lambda) = -a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} (e^{\lambda x} - 1 - \lambda x \mathbb{1}_{(|x|<1)}) \Pi(dx)$$

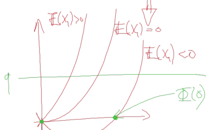
Exercise Using the fact that $\int_{(-\infty,0)} (1 \wedge x^2) \Pi(dx) < \infty$ show that on $[0,\infty)$, $\psi(\lambda)$ is:

(i) strictly convex

(ii) $\psi(0) = 0$

(iii) $\psi(+\infty) = \infty$

(iv) $\psi'(0+) = \mathbb{E}(X_1)$



$(q \geq 0)$ $\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\} \geq 0$

Note when $q = 0$ $\Phi(0) = \begin{cases} \Phi(0) > 0 & \mathbb{E}(X_1) < 0 \\ 0 & \mathbb{E}(X_1) \geq 0. \end{cases}$

Lemma For spec. neg. Lévy process, with $q \geq 0$, $(x \geq 0)$

$$\mathbb{E}(e^{-q L_x^{-1}} \mathbb{1}_{(L_x^{-1} < \infty)}) = e^{-\Phi(q)x}$$

Pf/ Exercise: $e^{\lambda X_t - \psi(\lambda)t}$ is a mg.

We know that L_x^{-1} is a stopping time

hence $t \wedge L_x^{-1}$ is —— " ——

Also know that $X_{L_x^{-1}} = x$ by spectral negativity

Doob's optimal sampling theorem says (amongst other things) that the mean of a +ve mg is preserved at bounded stopping times.

$$\mathbb{E}(e^{\lambda X_{t \wedge L_x^{-1}} - \psi(\lambda)(t \wedge L_x^{-1})}) = 1$$

note $\leq e^{\lambda x}$ (ie $X_{t \wedge L_x^{-1}} \leq x$)

DCT can be applied

$$1 = \lim_{t \to \infty} \mathbb{E}(\cdots) = \mathbb{E}\Big(\lim_{t \to \infty} e^{\lambda X_t - \psi(\lambda)t} \mathbb{1}_{(t < L_x^{-1})} + \lim_{t \to \infty} e^{\lambda X_{L_x^{-1}} - \psi(\lambda) L_x^{-1}} \mathbb{1}_{(L_x^{-1} < t)}\Big)$$

($\lambda X_t \leq \lambda x$)

$q > 0$

choose $\lambda = \Phi(q) > 0 \Rightarrow \psi(\lambda) = \psi(\Phi(q)) = q > 0$

$$1 = \mathbb{E}\left[e^{\Phi(q)x - q L_x^{-1}} \mathbb{1}_{(L_x^{-1} < \infty)}\right]$$