Rigorous pf of Lévy–Itô decomp

1 Square integrable martingales

Fix time horizon $T < \infty$ assume that

$(\Omega, \mathcal{F}, \{\mathcal{F}_t^* : t \geq 0\}, \mathbb{P})$

where $\{\mathcal{F}_t^* : t \geq 0\}$ is a filtration which has been completed by the null sets of $\mathbb{P}$ and is right cts.

[Right cts means $\mathcal{F}_t^* = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}^*$]

<u>Define</u> $\mathcal{M}_T^2 = \mathcal{M}_T^2(\Omega, \mathcal{F}, \{\mathcal{F}_t^* : t \leq T\}, \mathbb{P})$ to consist of all real valued, zero-mean, a.s. right cts square integrable $\mathbb{P}$-mgs w.r.t. $\{\mathcal{F}_t^* : t \geq 0\}$.

i.e. $M \in \mathcal{M}_T^2$ then $M = \{M_t : t \in [0,T]\}$

s.t $\mathbb{E} M_t = 0 \ \forall t$, $\mathbb{E}(M_t^2) < \infty \ \forall t$

and $\mathbb{E}(M_t | \mathcal{F}_s^*) = M_s \quad \forall s \leq t \leq T$

Recall that $M' = \{M'_t : t \leq T\}$ is a <u>version</u> of M if $\{\exists t : M'_t \neq M_t\}$ is $\mathbb{P}$-measurable and has $\mathbb{P}$-prob. zero.

Can build up an equiv. class of processes in $\mathcal{M}_T^2$ by the identification of "versions"

We really want to understand $\mathcal{M}_T^2$ to be short for $\mathcal{M}_T^2/\sim$

Linear (vector) space over $\mathbb{R}$ [NB unique zero element cos of quotienting out versions]

<u>Claim</u> that $\mathcal{M}_T^2$ is a Hilbert space wrt the inner product $\langle M, N \rangle := \mathbb{E}(M_T N_T)$

Need to convince ourselves that

(i) $\langle \cdot , \cdot \rangle$ does indeed define an inner product

(ii) if $\|M\| := \langle M, M \rangle^{1/2}$ then need to check that all Cauchy sequences w.r.t $\|\cdot\|$ converge.

(i) <u>need</u> linearity: $a, b \in \mathbb{R}$, $M^{(1)}, M^{(2)}, M^{(3)} \in \mathcal{M}_T^2$

then $\langle aM^{(1)} + bM^{(2)}, M^{(3)} \rangle = a\langle M^{(1)}, M^{(3)} \rangle + b\langle M^{(2)}, M^{(3)} \rangle$

trivial to check

<u>need</u> $\langle M^{(1)}, M^{(2)} \rangle = \langle M^{(2)}, M^{(1)} \rangle$: trivial

<u>need</u> $\langle M^{(1)}, M^{(1)} \rangle \geq 0$: trivial

<u>need</u> $\langle M, M \rangle = 0 \quad (M \in \mathcal{M}_T^2)$

then $M \equiv 0$

This follows from Doob's maximal inequality (See Ash–Doleans-Dade: Probability & Measure) which says that $\forall M \in \mathcal{M}_T^2$

$$\mathbb{E}\left(\sup_{0 \leq s \leq T} M_s^2\right) \leq 4\,\mathbb{E}(M_T^2) = 4\langle M, M \rangle$$

Hence if $\langle M, M \rangle = 0 \Rightarrow \sup_{0 \leq s \leq T} M_s^2 = 0$ a.s.

$\Rightarrow M \equiv 0$

(ii) Need to check that if $\{M^{(n)} : n \geq 1\}$ is Cauchy sequence in $\mathcal{M}_T^2$ then $\exists M \in \mathcal{M}_T^2$ such that $\|M^{(n)} - M\| \xrightarrow[n \to \infty]{} 0$

$\{M^{(n)} : n \geq 1\}$ is a Cauchy sequence means

$$\|M^{(n)} - M^{(k)}\| \xrightarrow[n,k \to \infty]{} 0$$

$$\Longleftrightarrow \mathbb{E}\left(\left(M_T^{(n)} - M_T^{(k)}\right)^2\right)^{1/2} \xrightarrow[n,k \to \infty]{} 0$$

$\Longleftrightarrow \{M_T^{(n)} : n \geq 1\}$ as a sequence of r.v.s is a Cauchy sequence in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$

we know from our measure theoretic prob. courses or a hearty read of D. Williams (Prob. with Mgs) that $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ is a Hilbert space (hence its name as L^2 space!!!)

Hence $\exists M_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ such that

r.v.

$$\mathbb{E}\left(\left(M_T^{(n)} - M_T\right)^2\right)^{1/2} \xrightarrow[n \to \infty]{} 0.$$

Now define $M_t := \mathbb{E}(M_T \mid \mathcal{F}_t^*)$ $t \in [0,T]$

right cts modification of this conditional expectation
(uses the assumptions on $\mathcal{F}_t^*$)

Note that by def^n $\|M^{(n)} - M\| \xrightarrow[n\to\infty]{} 0$

Note also that $M = \{M_t : t \in [0,T]\}$ is clearly a mg, clearly zero mean 'cos $\mathbb{E}M_t = \mathbb{E}M_T = 0$

it is right cts a.s by construction, (this follows from M_T being a limit in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$)

and $\mathbb{E}(M_t^2) = \mathbb{E}\left(\mathbb{E}(M_T \mid \mathcal{F}_t^*)^2 \right)$

$$\underset{\text{Jensen}}{\leq} \mathbb{E}\,\mathbb{E}(M_T^2 \mid \mathcal{F}_t^*)$$

$$= \mathbb{E}(M_T^2) < \infty$$

In conclusion: we have shown that if $\{M^{(n)}, n \geq 1\}$ is Cauchy then $\exists\, M \in \mathcal{M}_T^2$ s.t.

$$M^{(n)} \xrightarrow[n\to\infty]{\|\cdot\|} M$$

Henceforth we shall assume that $\{\xi_i : i \geq 1\}$ is an iid sequence of r.v.s with common d.f. F, $N = \{N_t : t \geq 0\}$ is a PP with rate $\lambda > 0$.

<u>Lemma</u> Suppose $\int |x| F(dx) < \infty$

(i) $M := \{M_t : t \geq 0\}$ where

$$M_t = \sum_{i=1}^{N_t} \xi_i - \lambda t \int x F(dx)$$

is a martingale.

(ii) Assume further that $\int x^2 F(dx) < \infty$

Then M is a square integrable mg with

$$\mathbb{E}(M_t^2) = \lambda t \int x^2 F(dx)$$

Pf/(i) Note that M is a Lévy Process and hence

$$\mathbb{E}(M_t | \mathcal{F}_s) = \mathbb{E}(M_t - M_s + M_s | \mathcal{F}_s)$$

$[\mathcal{F}_s = \sigma(M_u : u \leq s)]$

$$= \mathbb{E}(M_t - M_s | \mathcal{F}_s) + M_s$$

$$\overset{\text{L.P.}}{=} \mathbb{E}(M_{t-s}) + M_s$$

Note however that

$$\mathbb{E}(M_u) = \mathbb{E}\left(\sum_{i=1}^{N_u} \xi_i\right) - \lambda u \int x F(dx)$$

$$= \lambda u \int x F(dx) - \lambda u \int x F(dx) = 0$$

Strictly speaking need to check that $\mathbb{E}|M_t| < \infty$ $\forall t$.

Similarly to above

$$\mathbb{E}\left|\sum_{i=1}^{N_u} \xi_i - \lambda u \int x F(dx)\right|$$

$$\leq \mathbb{E}\left|\sum_{i=1}^{N_u} \xi_i\right| + \lambda u \int |x| F(dx)$$

$$\leq \mathbb{E} \sum_{i=1}^{N_u} |\xi_i| + \lambda u \int |x| F(dx)$$

$$= 2\lambda u \int |x| F(dx) < \infty$$

(ii) suffices to check that $\mathbb{E}(M_t^2) = \lambda t \int x^2 F(dx)$

$$\mathbb{E} M_t^2 = \mathbb{E}\Big[\Big(\underbrace{\sum_{i=1}^{N_t} \xi_i}_{a}\Big)^2\Big] - \boxed{\lambda^2 t^2 \left(\int x F(dx)\right)^2}$$

$\mathbb{E}(a-b)^2 = \mathbb{E}a^2 - \mathbb{E}2ab + \mathbb{E}b^2$

$$= \mathbb{E}\left[\sum_{i=1}^{N_t} \xi_i^2 + \sum_{i,j} \mathbb{1}_{(i \neq j)} \xi_i \xi_j\right] - \lambda^2 t^2 \left(\int x^2 F(dx)\right)^2$$

$$= \lambda t \int x^2 F(dx) + \mathbb{E}(N_t^2 - N_t)\left(\int x F(dx)\right)^2 - \lambda^2 t^2 \left(\int x^2 F(dx)\right)^2$$

$$= \lambda t \int x^2 F(dx) + \cancel{(\lambda t)^2 \left(\int x F(dx)\right)^2} - \cancel{\lambda^2 t^2 \left(\int x F(dx)\right)^2}$$

Now suppose that for $n = 1, 2, 3, \ldots$

$N^{(n)} = \{N_t^{(n)} : t \geq 0\}$ is a seq of PP (indep!) with rates $\lambda_n \geq 0$ [we understand $N^{(n)} \equiv 0$ when $\lambda_n = 0$]. Moreover $\{\xi_i^{(n)} : i \geq 1\}$ are sequences of iid r.v. (also indep in n-index) with common d.f. $F^{(n)}$

Define $M_t^{(n)} = \sum_{i=1}^{N_t^{(n)}} \xi_i^{(n)} - \lambda_n t \int x \, F_n(dx)$.

Put everything on the obvious product space and if $\{\mathcal{F}_t^{(n)} : t \geq 0\}$ is the natural filtration for $M^{(n)}$, then define $\mathcal{F}_t^* = \sigma\left(\bigcup_n \mathcal{F}_t^{(n)}\right)$ with completion and forced right continuity.

Theorem If $\sum_{n\geq 1} \lambda_n \int x^2 F_n(dx) < \infty$

then there exists a Lévy process defined on the aforementioned product space which is also a square integrable mg and whose char. exponent is given by

$$\Psi(\theta) = \int_{\mathbb{R}} \left(1 - e^{i\theta x} + i\theta x\right)\left(\sum_{n\geq 1} \lambda_n F_n(dx)\right) \quad \forall \theta \in \mathbb{R}.$$

Moreover, for all $T>0$,

$$\lim_{k\uparrow\infty} \mathbb{E}\left(\sup_{t\leq T}\left(X_t - \sum_{n\geq 1}^{k} M_t^{(n)}\right)^2\right) = 0$$

~~Pf~~ First note that

$$X_t^{(k)} := \sum_{n=1}^{k} M_t^{(n)}$$

then easy to check that since $M^{(n)}$ are L^2-mgs

~~then so~~ is $X^{(k)}$

$\boxed{\text{Fix } T>0}$

Now we claim that $X^{(k)}$ is a Cauchy sequence in

$\mathcal{M}_T^2$ (natural product space with filtration $\{\mathcal{F}_t^* : t \geq 0\}$)

$$\underset{\text{wlog } k \geq l}{\| X^{(k)} - X^{(l)} \|} = \sqrt{\mathbb{E}\left[\left(X_T^{(k)} - X_T^{(l)} \right)^2 \right]}$$

$$= \sqrt{\mathbb{E}\left[\left(\sum_{n=l+1}^{k} M_T^{(n)} \right)^2 \right]}$$

$$\overset{\text{Lemma}}{=} \sqrt{\sum_{n=l+1}^{k} T \lambda_n \int x^2 F_n(dx)}$$

$$= \sqrt{T \int x^2 \left(\sum_{n=l+1}^{k} \lambda_n F_n(dx) \right)}$$

$$\xrightarrow{l,k \to \infty} 0$$

because $\int x^2 \left(\sum_{n \geq 1} \lambda_n F_n(dx) \right) < \infty$

Hence $\exists\, X = \lim_{n \to \infty} {}_{\|\cdot\|} X^{(n)} \in \mathcal{M}_T^2$ as required

In particular thanks to Doob's maximal inequality

$$\mathbb{E}\Big(\sup_{t \leq T} (\underbrace{X_t - X_t^{(n)}}_{\in \mathcal{M}_T^2})^2 \Big) \leq 4 \| X - X^{(n)} \| \xrightarrow{n \to \infty} 0$$

as required!

$\Rightarrow X_t^{(n)} \Rightarrow X_t$ for each $t \in [0,T]$.

$$\mathbb{E}\left(e^{i\theta(X_t - X_s)}\right)$$

$$\overset{DCT}{=} \lim_{n\to\infty} \mathbb{E}\left(e^{i\theta\left(X_t^{(n)} - X_s^{(n)}\right)}\right) \qquad 0 \leq s \leq t \leq T$$

$$= \lim_{n\to\infty} \mathbb{E}\left(e^{i\theta X_{t-s}^{(n)}}\right) \qquad \left[\text{sums of indep. L.P.s are L.Ps}\right]$$

$$= \mathbb{E}\left(e^{i\theta X_{t-s}}\right)$$

This shows stationarity & independence of increments.

$\Rightarrow X \in \mathcal{M}_T^2$ is also a Lévy Process.

$$\mathbb{E}\left(e^{i\theta X_t}\right) = \lim_{n\to\infty} e^{-\sum_{k=1}^{n} \int \left(1 - e^{i\theta x} + i\theta x\right) \lambda_k F_k(dx)}$$

$$= \exp - \int \left(1 - e^{i\theta x} + i\theta x\right) \left[\sum_{k=1}^{\infty} \lambda_k F_k(dx)\right]$$

$O(x^2)$

Note that the limit exists because $\int x^2 \sum_{n=1}^{\infty} \lambda_n F_n(dx) < \infty$