Theorem: If $\mathbb{E}(X_1) \geq 0$ then L^{-1} is a subordinator and o/w when $\mathbb{E}(X_1) < 0$ then L^{-1} is a subordinator killed (sent to $+\infty$) at an indep & exponentially distributed time.

Proof: Note: $\forall x \geq 0$ $\mathbb{E}\left(e^{-qL_x^{-1}} \mathbb{1}_{(L_x^{-1}<\infty)}\right) = e^{-\Phi(q)x}$

$$\mathbb{E}\left(e^{-qL_x^{-1}} \,\middle|\, L_x^{-1} < \infty\right) = e^{-(\Phi(q)-\Phi(0))x}$$

$\Phi(q) - \Phi(0)$ must be the Laplace exponent of a non-neg i.d. r.v. Indeed

$$\mathbb{E}\left(e^{-qL_1^{-1}} \,\middle|\, L_1^{-1} < \infty\right) = e^{-(\Phi(q)-\Phi(0))}$$

$$= \mathbb{E}\left(e^{-qL_{1/n}^{-1}} \,\middle|\, L_{1/n}^{-1}\right)^n$$

Law $\mu(dx) = \mathbb{P}\left(L_1^{-1} \in dx \,\middle|\, L_1^{-1} < \infty\right)$ is the i.d. distⁿ with Laplace exponent $\Phi(q) - \Phi(0)$.

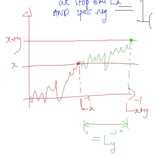
Note further: On the event $\{L_x^{-1} < \infty\}$

$$\mathbb{1}_{(L_x^{-1}<\infty)} \mathbb{E}\left(e^{-q(L_{x+y}^{-1} - L_x^{-1})} \mathbb{1}_{(L_{x+y}^{-1}<\infty)} \,\middle|\, \mathcal{F}_{L_x^{-1}}^{y}\right)$$

St. indep incr. at stop time L_x^{-1} AND spec neg

$$= \mathbb{1}_{(L_x^{-1}<\infty)} \mathbb{E}\left(e^{-qL_y^{-1}} \mathbb{1}_{(L_y^{-1}<\infty)}\right)$$

$$= \mathbb{1}_{(L_x^{-1}<\infty)} \underbrace{e^{-(\Phi(q)-\Phi(0))y} e^{-\Phi(0)y}}$$



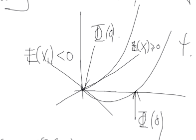
(i) independence of increments

(ii) stationarity of increments

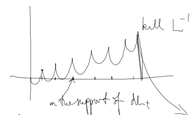
(iii) increments have the law of a $\boxed{\text{subordinator}}$ killed at an indep exp time with rate $\Phi(0)$

in particular the unique subordinator whose distⁿ at time 1 has i.d. distⁿ with Laplace exp $\Phi(q) - \Phi(0)$.

Note in particular if $\Phi(0) = 0$, then no killing! Also $\Phi(0) = 0 \iff \mathbb{E}(X_1) = \psi'(0^+) \geq 0$



Case $\mathbb{E}(X_1) < 0$



Corollary: $\overline{X}_{\mathbf{e}_q} \sim \exp(\Phi(q))$.

Pf: $\{\overline{X}_{\mathbf{e}_q} > x\} = \{\mathbf{e}_q > \tau_x^+\}$

$\tau_x^+ = \inf\{t > 0 : X_t > x\}$

$$\mathbb{P}(\overline{X}_{\mathbf{e}_q} > x) = \mathbb{P}(\mathbf{e}_q > \tau_x^+)$$

$$= \mathbb{E}\left(e^{-q\tau_x^+} \mathbb{1}_{(\tau_x^+<\infty)}\right) + 0$$

$$= \mathbb{E}\left(e^{-qL_x^{-1}} \mathbb{1}_{(L_x^{-1}<\infty)}\right)$$

$$= e^{-\Phi(q)x}$$

$$\Rightarrow \overline{X}_{\mathbf{e}_q} \sim \exp(\Phi(q)).$$

Corollary: If $\mathbb{E}(X_1) < 0$, $\overline{X}_\infty \sim \exp(\Phi(0))$

If $\mathbb{E}(X_1) \geq 0$ $\mathbb{P}(\overline{X}_\infty = \infty) = 1$

~~Pf~~

If $E(X_1) < 0 \Longrightarrow \Phi'(0) > 0$

$$\lim_{q \downarrow 0} \mathbb{P}\left(\overline{X}_{e_q} > x\right) = e^{-\Phi'(0)x}$$

$$\|$$

$$\lim_{q \downarrow 0} q \int_0^\infty e^{-qt}\, \mathbb{P}\left(\overline{X}_t > x\right) dt = \mathbb{P}\left(\overline{X}_\infty > x\right)$$

$$\|$$

$$\lim_{q \downarrow 0} \mathbb{P}\left(\overline{X}_{e_{1/q}} > x\right) = \mathbb{P}\left(\overline{X}_\infty > x\right)$$

Ladder height process:

$$H_t = \begin{cases} X_{L^{-1}_t} & L^{-1}_t < \infty \\ \infty & L^{-1}_t = \infty \end{cases}$$

In general H is a $\underbrace{\text{(killed)}}_{\text{at rate } \Phi(0)}$ subordinator

In the spec neg. case

$$H_t = t \quad \text{on } L^{-1}_t < \infty.$$

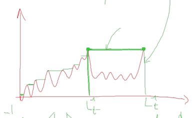
because L^{-1}_t is a first passage time also and X cannot pass t by jumping over it!

Excursions

Def^n For each moment of local time $t > 0$, we define

$$\epsilon_t = \begin{cases} \boxed{\{X_{L^{-1}_{t-}+s} - X_{L^{-1}_{t-}}} : 0 < s \leq \Delta L^{-1}_t\} & \text{IF } \Delta L^{-1}_t \\ \partial & \text{IF } \Delta L^{-1}_t = 0 \end{cases}$$

$(\Delta L^{-1}_t = L^{-1}_t - L^{-1}_{t-})$



$\Delta L^{-1}_t > 0 \iff$ an excursion from the maximum occurs.

Remark ϵ_t is a process in time, indexed by local time

$$\epsilon_t = \{\epsilon_t(s) : s \leq \Delta L^{-1}_t\}$$

Def^n We say $\mathcal{E}$ is the space of excursions meaning: The space of càdlàg mappings

$$\epsilon : [0, \zeta] \to (-\infty, 0] \text{ for some } \zeta \in (0, \infty]$$

where $\zeta = \zeta(\epsilon)$ is the excursion length.

Also define $\bar{\epsilon} = -\inf_{s \in (0,\zeta)} \epsilon(s)$.

Theorem There exists a σ-algebra, Σ, and a σ-finite measure, n, such that $(\mathcal{E}, \Sigma, n)$ is a measure space and Σ is rich enough to contain sets of the form

$$\{\epsilon \in \mathcal{E} : \zeta(\epsilon) \in A, \bar{\epsilon} \in B\}$$

with A, B Borel sets in $[0, \infty]$.

Moreover:

(i) If $\mathbb{E}(X_1) \geq 0$ then

$$(*) \quad \{(t, \epsilon_t) : t \geq 0, \epsilon_t \neq \partial\}$$

is a PPP on $[0, \infty) \times \mathcal{E}$ with intensity $dt \times dn$.

(ii) $\mathbb{E}(X_1) < 0$ then $(*)$ is a PPP with intensity $dt \times dn$ stopped at the first arrival of an excursion in

$$\mathcal{E}_\infty := \{\epsilon \in \mathcal{E} : \zeta(\epsilon) = \infty\}.$$

I.e. $(*)$ has the same law as a PPP on $\mathcal{E} \setminus \mathcal{E}_\infty$ with intensity $dt \times dn|_{\mathcal{E} \setminus \mathcal{E}_\infty}$ stopped at an indep. & exp. dist'd time with rate $n(\mathcal{E}_\infty)$.

Remember that L^{-1} is subordinator (possibly killed). Obviously $\Delta L^{-1}_t = \zeta(\epsilon_t)$ when $\Delta L^{-1}_t > 0$. Hence if $\Lambda(dx)$ is the Lévy measure of L^{-1} then $\Lambda(dx) = n(\zeta \in dx)$.

Also, killing rate $\Phi(0) = n(\mathcal{E}_\infty)$

Also, if N is the Poisson random measure corresponding to the PPP of excursions, then

$$L^{-1}_t = \int_{[0,t]} \int_{\mathcal{E}} \zeta(\epsilon) \, N(ds \times d\epsilon) + \int_0^{L^{-1}_t} \mathbb{1}_{(\epsilon_s = \partial)} \, ds$$

On the other hand, L^{-1} is a subordinator. Hence

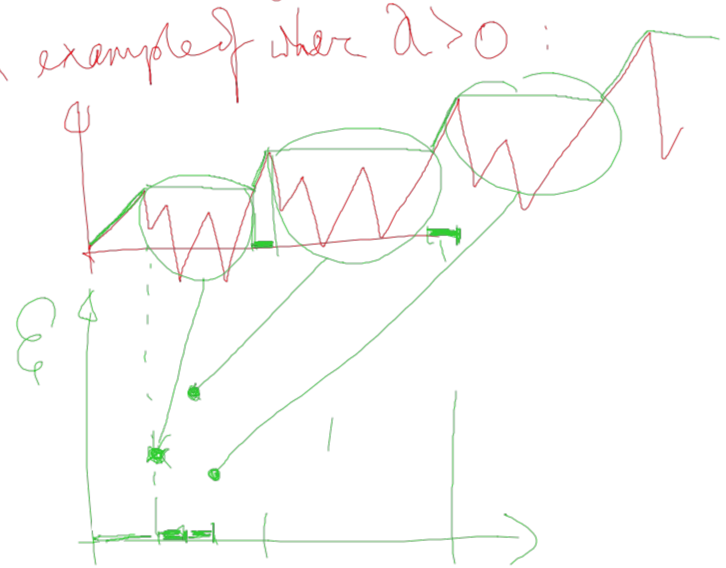
$$L_t^{-1} = \lambda t + \sum_{s \le t} \Delta L_s^{-1}$$

for some $\lambda \ge 0$.

Clearly $\displaystyle\sum_{s \le t} \Delta L_s^{-1} = \int_{[0,t]} \int_{\mathcal{E}} \zeta(\epsilon)\, N(ds \times d\epsilon)$.

Hence $\displaystyle\lambda t = \int_0^{L_t^{-1}} \mathbb{1}_{(\epsilon_s = \partial)}\, ds$

An example of where $\lambda > 0$:



BM case: $\Phi(q) = \sqrt{2q}$ $\quad \psi(\theta) = \frac{1}{2}\theta^2$

Hence L^{-1} is stable subordinator!

no drift!

Pf of WHF

Now consider the PPP of MARKED excursions. I.e. a PPP on

$$\Big([0,\infty) \times \mathcal{E} \times [0,\infty),\ \mathcal{B}[0,\infty) \times \Sigma \times \mathcal{B}[0,\infty),\ dt \times dn \times d\eta \Big)$$

$x \geq 0 \quad \eta(dx) = p e^{-px}\, dx.$

Claim:

$$\{ (t, \epsilon_t) : t \leq L_{e_q},\ \epsilon_t \neq \partial \}$$

has the same law as

$$\{ (t, \epsilon_t, e_p^{(t)}) : t \leq \sigma_1 \wedge \sigma_2,\ \epsilon_t \neq \partial \}$$

mark

projected onto $[0,\infty) \times \mathcal{E}$, where

$$\sigma_1 = \inf\{ t > 0 : \int_0^{L_t^{-1}} \mathbb{1}_{(\epsilon_s = \partial)}\, ds > e_p \}$$

and $\sigma_2 = \inf\{ t > 0 : \zeta(\epsilon_t) > e_p^{(t)} \}.$

Note

(i) $\sigma_1 \perp\!\!\!\perp \sigma_2$

(ii) $\mathbb{P}(\sigma_1 > t) = \mathbb{P}\Big(\int_0^{L_t^{-1}} \mathbb{1}_{(\epsilon_s = \partial)}\, ds < e_p \Big)$

$= \mathbb{P}(\partial t < e_p)$

$= e^{-\partial p t}$

$\sigma_1 \sim \exp(\partial p)$

(iii) σ_2 "=" σ_A with

$A = \{ \epsilon \in \mathcal{E} : \zeta(\epsilon) > e_p \}.$

hence $\sigma_2 \sim \exp\{ n \times \eta(\zeta > e_p) \}$

$= \exp\Big(\boxed{\int_0^\infty p e^{-px}\, n(\zeta > x)} \Big)$ k

(iv) $\sigma_1 \wedge \sigma_2 = \exp\{ \partial + k \}$

(v) Marked PPP up to time $\sigma_1 \wedge \sigma_2$ has same law as PPP with intensity

$$dt \times n(d\epsilon\ ;\ \zeta < x) \times \eta(dx)$$

and stopped at an indep exp. dist'd time rate $\partial + k$.

(vi) On the event $\{\sigma_2 < \sigma_1\}$

(†) $\{ (t, \epsilon_t, e_p^{(t)}) : t < \sigma_1 \wedge \sigma_2,\ \epsilon_t \neq \partial \}$

is indep. of $\epsilon_{\sigma_2} = \epsilon_{\sigma_1 \wedge \sigma_2}$

On the other hand, when $\sigma_1 < \sigma_2$

then $\epsilon_{\sigma_1} = \partial = \epsilon_{\sigma_1 \wedge \sigma_2}.$

Either way $\epsilon_{\sigma_1 \wedge \sigma_2} \perp\!\!\!\perp$ (†)

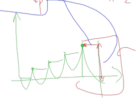
$$\bar{G}_{e_p} \overset{d}{=} L^{-1}_{(\sigma_1 \wedge \sigma_2)-} = \partial(\sigma_1 \wedge \sigma_2) + \int_{[0, \sigma_1 \wedge \sigma_2)} \int_{\mathcal{E}} \zeta(\epsilon)\, N(ds \times d\epsilon)$$

$$L_{e_p} = \bar{X}_{e_p} \overset{d}{=} \sigma_1 \wedge \sigma_2$$

Moreover

$$\boxed{e_p - G_{e_p}} \overset{d}{=} e_p^{(\sigma_1 \wedge \sigma_2)}$$

$$\boxed{X_{e_p} - \bar{X}_{e_p}} = \epsilon_{\sigma_1 \wedge \sigma_2}\big(e_p^{(\sigma_1 \wedge \sigma_2)} \wedge \zeta \big)$$



$\epsilon_{\sigma_1 \wedge \sigma_2}.$

$(*) \Rightarrow$

$$\underbrace{(\bar{X}_{e_p}, \bar{\varsigma}_{e_p})}_{\text{inf. div.}} \perp\!\!\!\perp \underbrace{(X - \bar{X}_{e_p}, \, e_p - \bar{\varsigma}_{e_p})}_{\overset{d}{=} (\underline{X}_{e_p}, \, \underline{\varsigma}_{e_p}) \ \text{inf. div!}} \quad (\dagger\dagger)$$

Exercise: $\bar{X}_{e_p} \overset{d}{=} \sigma_1 \wedge \sigma_2 \sim \exp(\lambda + k)$

$\bar{\varsigma}_{e_p} \overset{d}{=} L^{-1}_{(\sigma_1 \wedge \sigma_2)-}$

$\overset{d}{=} (\text{Subordinator})_{e_p}$

i.d.

$(\dagger\dagger)$ follows from duality

$$\{X_{(t-s)-} - X_t : 0 \le s \le t\}$$

$$\overset{d}{=} \{-X_s : 0 \le s \le t\}$$