Corollary (Proof of Lévy-Itô decomp).

Recall we just needed to show that given (a, σ, Π) we needed to show that we can construct a Lévy process whose characteristic exponent is

$$\Psi(\theta) = \int_{(-1,1)} (1 - e^{i\theta x} - i\theta x)\, \Pi(dx).$$

To do this, we can now follow our heuristic argument by choosing

$$\lambda_n = \Pi\left(x : |x| \in [2^{-(n+1)}, 2^{-n})\right)$$

$$F_n(dx) = \frac{1}{\lambda_n} \Pi(dx)\Big|_{|x| \in [2^{-(n+1)}, 2^{-n})}$$

we can apply previous theorem providing

$$\sum_{n \geq 0} \lambda_n \int x^2 F_n(dx) < \infty$$

and then the required result.

Note that

$$\sum_{n \geq 0} \lambda_n \int x^2 F_n(dx) = \sum_{n \geq 0} \int_{|x| \in [2^{-(n+1)}, 2^{-n})} x^2 \, \Pi(dx) = \int_{|x| \in (0,1)} x^2 \, \Pi(dx) < \infty$$

by def^n of Π!

Path variation

Lévy–Itô decomposition: Any (a, σ, Π), $a \in \mathbb{R}$, $\sigma^2 \geq 0$

$\int_{\mathbb{R}\setminus\{0\}} (1 \wedge x^2)\,\Pi(dx) < \infty$

$$X_t = X_t^{(1)} + X_t^{(2)} + X_t^{(3)}$$

$X_t^{(1)} = \sigma B_t - at$

$X_t^{(2)} = \sum_{i=1}^{N_t} \xi_i$ where $\{N_t : t \geq 0\}$ is a PP rate $\Pi(x : |x| \geq)$, ξ_i's iid d.f. $\Pi|_{|x|\geq 1}$

$X^{(3)}$ = as described on previous slide.

$\sigma^2 > 0 \implies$ UBV paths.

$\sigma^2 = 0$ then UBV paths $\iff X^{(3)}$ has UBV paths

$\varphi(f)$ is total variation of cadlag function f on $[0,1]$

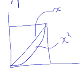
$\varphi(f) = \varphi(f^c) + \sum_{s \leq 1} |\Delta f_s|$

where $f^c = f - \sum_{s \leq \cdot} \Delta f_s$

$\Delta f_s = f_s - f_{s-}$

Theorem $X^{(3)}$ has UBV paths iff

$$\int_{|x|<1} |x|\,\Pi(dx) = \infty.$$



Proof Recall $X^{(3)} = \|\cdot\|\text{-}\lim \sum_{n=0}^{k} M^{(n)}$

where $M_t^{(n)} = \sum_{i=1}^{N_t^{(n)}} \xi_i^{(n)} - t\int_{|x|\in(2^{-(n+1)}, 2^{-n}]} x\,\Pi(dx)$

Define $M_t^{(n,+)} = \sum_{i=1}^{N_t^{(n)}} \xi_i^{(n)} \mathbb{1}(\xi_i^{(n)} > 0) - t\int_{x\in(2^{-(n+1)}, 2^{-n}]} x\,\Pi(dx)$

$M_t^{(n,-)} = \cdots\cdots \mathbb{1}(\xi_i^{(n)} < 0) - \int_{-x\in(2^{-(n+1)}, 2^{-n}]} \cdots$

Note that $M^{(n,+)}$ and $M^{(n,-1)}$ are compensated CPP which are also L^2-mgs.

Note that $\left(\boxed{\sum_{n=0}^{k} M^{(n,\pm)}} : k = 1, 2, 3, \ldots\right)$ gives us a Cauchy sequence in $\mathcal{M}_T^2$ (fix $T > 0$) $X_t^{(k,\pm)}$

and hence they converge under $\|\cdot\|$ to respective L^2-mgs & Lévy processes denoted by $X^{(\pm)}$

Now note

$$C_t^{(k,\pm)} := \sum_{n=0}^{k} \sum_{i=1}^{N_t^{(n)}} \xi_i^{(n)} \mathbb{1}(\xi_i^{(n)} \gtrless 0)$$

$$= \underbrace{\boxed{X_t^{(k,\pm)}}}_{\text{always converges } \|\cdot\|} + t\sum_{n=0}^{k} \int_{\pm x \in [2^{-(n+1)}, 2^{-n})} x\,\Pi(dx)$$

It follows that $C_t^{(k,\pm)}$ converges a.s. $\forall t$

$$\iff \int_{\pm x \in [2^{-(n+1)}, 2^{-n})} x\,\Pi(dx) < \infty$$

$$\sum_{n=0}^{k} M_t^{(n,\pm)} = C_t^{(k,\pm)} - t\int_{\pm x \in (2^{-(k+1)}, 1)} x\,\Pi(dx)$$

$\to$ limits to a BV path Lévy process precisely when $\pm\int_{\pm x \in (0,1)} x\,\Pi(dx) < \infty$

$\to$ Combining a.s. statement for

$$\sum_{n=0}^{k} M_t^{(n)} = \left(C_t^{(k,+)} - C_t^{(k,-)}\right) - t\left(\int_{|x|\in(2^{-(k+1)}, 1)} x\,\Pi(dx)\right)$$

$\int_{|x|\in(0,1)} |x|\,\Pi(dx) < \infty$

$\Updownarrow$

$\int_{|x|\in(0,1)} x\,\Pi(dx) < \infty$

$\to$ ( ) converging a.s. $\iff \int_{|x|\in(0,1)} x\,\Pi(dx) < \infty$

Hence $X^{(3)}$ has BV paths $\iff \int_{|x|\in(0,1)} |x|\,\Pi(dx) < \infty$.

<u>Corollary</u> Any (a, σ, Π) – Lévy process, X has paths of UBV $\iff$

$$\sigma^2 > 0 \quad \text{OR} \quad \int_{(-1,1)} |x| \, \Pi(dx) = \infty.$$

Moreover, if X has paths of BV, then it can always be meaningfully written in the form

$$X_t = \boxed{-at} + \underbrace{\sum_{s \leq t} \Delta X_t}_{} - \boxed{t \cdot \int_{(-1,1)} x \, \Pi(dx)}$$

$$\Delta X_t = X_t - X_{t-}$$

jumps of all sizes

This is an absolutely convergent sum

$$= \lim_{k \to \infty} \left(C_t^{(k,+)} - C_t^{(k,-)} \right)$$

$$\sum_{s \leq t} |\Delta X_s| = \lim_{k \to \infty} C_t^{(k,+)} + C_t^{(k,-)}$$

compensation for small jumps

This is an absolutely cgt integral

<u>Subordinators</u> are Lévy processes with non-decr. paths. (necessarily have BV paths)

<u>Lemma</u> X is a subordinator iff $\Pi(-\infty, 0) = 0$ and

$$\infty > \delta := -a - \int_{(0,1)} x \, \Pi(dx) \geq 0,$$

In that case we write

$$X_t = \sum_{s \leq t} \Delta X_s + \delta t$$

and the Lévy–Khintchine exponent can be re-arranged to look like

$$\Psi(\theta) = -i\delta\theta + \int_{(0,\infty)} (1 - e^{i\theta x}) \, \Pi(dx)$$

<u>Spectrally negative Lévy process</u>

<u>Def</u>/ A SNLP is simply a Lévy process with no positive jumps (ie $\Pi(0, \infty) = 0$)

we are also exclude from our defn the case that

$$X_t = \delta t \quad : \delta > 0$$

and $X_t = -S_t$ where S_t is a subordinator

(i.e. we exclude from our defn the case of monotone paths)

Q: why does a LP of non-decr. path have BV?

<u>A1</u>: $X_t = \delta t + \sum_{s \leq t} \Delta X_s$

countable discontinuities

$$\varphi(X, [0,t]) = \delta t + \sum_{s \leq t} |\Delta X_s| = X_t < \infty$$

A2 A function f has finite variation

$\iff$ can be written in the form $f = f^{(1)} - f^{(2)}$

where $f^{(i)}$ is non-decreasing for $i = 1, 2$.

Poisson Random Measures

<u>Def</u>

Def^n Suppose that $(S, \mathcal{S}, \eta)$ is an arbitrary σ-finite measure space.

Let $N : \mathcal{S} \longrightarrow \mathbb{N} \cup \{\infty\}$ in such a way that $\{N(A) : A \in \mathcal{S}\}$ are random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then N is called a Poisson random measure on $(S, \mathcal{S}, \eta)$ [sometimes called a Poisson random measure on S with <u>intensity</u> η]

if (i) for mutually disjoint sets $A_1, \ldots, A_n \in \mathcal{S}$ the variables $N(A_1), \ldots, N(A_n)$ are independent.

(ii) for all $A \in \mathcal{S}$, $N(A) \sim \mathrm{Po}(\eta(A))$

[we allow $\eta(A) = \infty$, in which case we understand $N(A) = \infty$ a.s.]

(iii) $\mathbb{P}$-a.s., N is a measure!

<u>Theorem</u> There exists a PRM as given in the def^n before.

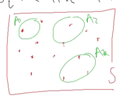
Pf/ Assume for simplicity that $\eta(S) < \infty$.
There is a standard construction (using product spaces) that allows us to say that there exists a pr. space $(\Omega, \mathcal{F}, \mathbb{P})$ on which the independent r.v.s

$$\nu \text{ and } \{v_1, v_2, \dots\}$$

are defined where

$$\nu \sim Po(\eta(S)) \text{ and } v_i \text{ are iid with}$$

common dist^n $\dfrac{\eta(\cdot)}{\eta(S)}$



Now define $N : \mathcal{S} \longrightarrow \mathbb{N} \cup \{\infty\}$
such that $\forall A \in \mathcal{S}$,

$$N(A) = \sum_{i=1}^{\nu} \mathbb{1}(v_i \in A)$$

Trivially N is (a.s.) a measure! (iii) ✓
Suppose now $A_1, \dots, A_k$ disjoint sets in $\mathcal{S}$

$$\mathbb{P}\big(N(A_1) = n_1, N(A_2) = n_2, \dots, N(A_k) = n_k\big)$$

$$= \mathbb{E}\,\mathbb{P}(\text{——} \mid \nu)$$

$$= \sum_{j \geq \sum_{i=1}^{k} n_i}^{\infty} e^{-\eta(S)} \frac{\boxed{\eta(S)^j}}{\boxed{j!}} \binom{\boxed{j}}{n_0 n_1 \dots n_k} \prod_{i=0}^{k} \left(\frac{\eta(A_i)}{\boxed{\eta(S)}}\right)^{n_i}$$

$n_0 = j - \sum_{i=1}^{k} n_i$

$A_0 = S \setminus (A_1 \cup \dots \cup A_k)$

$$= \underbrace{\boxed{\sum_{j \geq \sum_i n_i} e^{-\eta(A_0)} \frac{\eta(A_0)^{j - \sum_i n_i}}{(j - \sum_i n_i)!}}}_{= 1} \prod_{i=1}^{k} e^{-\eta(A_i)} \frac{\eta(A_i)^{n_i}}{n_i!}$$

$\eta(S) = \eta(A_0) + \eta(A_1) + \dots + \eta(A_k)$

In conclusion:

$$\mathbb{P}\big(N(A_1) = n_1, \dots, N(A_k) = n_k\big) = \prod_{i=1}^{k} e^{-\eta(A_i)} \frac{\eta(A_i)^{n_i}}{n_i!}$$

$\Longrightarrow$ $N(A_i)$'s are indep and $Po(\eta(A_i))$ distributed as required.

To get rid of the assumption that $\eta(S) < \infty$.
Suppose $(S, \mathcal{S}, \eta)$ is a σ-finite measure space and in particular decompose $S = \bigcup_i B_i$
where $\eta(B_i) < \infty$ and $B_i \cap B_j = \emptyset$ $i \neq j$
we know that $\exists$ a prob. space $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$ on which a Poisson Random measure, N_i, can constructed for the space $(B_i, B_i \cap \mathcal{S}, \eta(\cdot\,; B_i))$

Now define

$$N(\cdot) = \sum_i N_i(\cdot \cap B_i)$$

on product space $(\Omega, \mathcal{F}, \mathbb{P}) = \bigotimes (\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$

<u>Exercise</u>: check that N has the desired properties of PRM with intensity η.