Lévy Processes, The Wiener-Hopf factorisation and (hopefully) applications.

<u>Defn</u> (Lévy Processes). $\mathbb{R}$-valued stochastic process defined on some prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ with the following properties: If $\{X_t : t \geq 0\}$ is its trajectory then

(i) $\mathbb{P}(X_0 = 0) = 1$

(ii) Given $0 \leq s \leq t < \infty$. $X_t - X_s \perp\!\!\!\perp \{X_u : u \leq s\}$

(iii) ——— " ——— $X_t - X_s \overset{d}{=} X_{t-s}$

(iv) X has paths that are right-cts with left limits a.s.

Examples

1. <u>Brownian Motion</u>

<u>Note</u>: BM has more than càdlàg paths – it has cts paths
BM has increments which are Gaussian distributed
$X_t \sim N(0,t)$

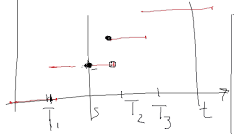
2. <u>Linear BM</u>: $a \in \mathbb{R}$, $b > 0$

$X_t = at + bB_t$ where $\{B_t : t \geq 0\}$ is a BM

3. Poisson Process : $\{N_t : t \geq 0\}$

$\boxed{T_n - T_{n-1} \overset{iid}{\sim} e_q}$ $q > 0$

$N_t \in \{0, 1, 2, \ldots\}$ $\forall t \geq 0$



$N_t - N_s \overset{d}{=} N_{t-s}$

$N_t - N_s \perp\!\!\!\perp \{N_u : u \leq s\}$

by lack of memory property

4. <u>(Exercise) Compound Poisson process</u>

$$X_t := \sum_{i=1}^{N_t} \xi_i$$ where ξ_i's are iid $\mathbb{R}$-valued r.v.s.

5 (Exercise) Mixture of linear BM and <u>independent</u> CPP

$$X_t = at + bB_t + \sum_{i=1}^{N_t} \xi_i$$

6. (Exercise). Independent sum of Lévy processes

Suppose $X^{(i)}$ is a Lévy process for $i = 1, \ldots, n$ and they are <u>mutually</u> independent, then

$$X_t := \sum_{i=1}^{n} X_t^{(i)} \quad : t \geq 0$$ is a Lévy process

Infinitely divisible distributions

Def^n A $\mathbb{R}$-valued r.v. Θ is inf. div. if for all $n = 1, 2, 3, \dots$

$$\Theta \overset{d}{=} \Theta_{1,n} + \Theta_{2,n} + \dots + \Theta_{n,n}$$

where for each n, $\Theta_{i,n}$ are iid

Note for each n, the common dist^n of the $\Theta_{i,n}$ need not be the same

Examples (1) $\Theta \sim N(0, \sigma^2)$

for each $n \geq 1$ $\quad \Theta_{i,n} \sim N\left(0, \frac{\sigma^2}{\sqrt{n}}\right)$

adding iid r.v.s n together

$$\mathbb{E}\left(e^{i\lambda\Theta}\right) = e^{-\frac{1}{2}\sigma^2} = \left(e^{-\frac{1}{2}\frac{\sigma^2}{n}}\right)^{n}$$

F.T. of $N\left(0, \frac{\sigma^2}{n}\right)$

(2) Poisson: If $\Theta \sim P_o(q)$: $q > 0$

$$\mathbb{E}\left(s^{\Theta}\right) = \exp\{q(s-1)\} \quad : s \in [0,1]$$

$$= \left[\exp\left\{\frac{q}{n}(s-1)\right\}\right]^{n}$$

$P_o(q/n)$

iid sum in the PGF

Hence for each $n \geq 1$, $\Theta_{i,n} \sim P_o(q/n)$

How big is the class of inf. div. dist^{n}s ?

Theorem (Lévy-Khintchine formula)

A probability law (dist^n) of an $\mathbb{R}$-valued r.v. is inf. div. with characteristic exponent $\Psi(\theta)$:

$$\int_{\mathbb{R}} e^{i\theta x}\mu(dx) =: e^{-\Psi(\theta)}$$

(well defined $\forall \theta \in \mathbb{R}$)

if and only if there exists a triple (a, σ, Π) where $a \in \mathbb{R}$, $\sigma \geq 0$ and Π is a σ-finite measure concentrated on $\mathbb{R}\setminus\{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge x^2)\,\Pi(dx) < \infty$ such that

$$\Psi(\theta) = ai\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left(1 - e^{i\theta x} + i\theta x \mathbb{1}_{(|x|<1)}\right)\Pi(dx)$$

$\forall \theta \in \mathbb{R}$.

Margaret.Sloper@maths.ox.ac.uk

The connection between L-K formula and Lévy Processes.

Fix $t>0$. Use a telescopic sum to write

$$X_t = \underbrace{\boxed{\left(X_{\frac{t}{n}} - X_0\right)}}_{\Theta_{1,n}} + \underbrace{\boxed{\left(X_{\frac{2t}{n}} - X_{\frac{t}{n}}\right)}}_{\Theta_{2,n}} + \cdots + \underbrace{\boxed{\left(X_{\frac{nt}{n}} - X_{\frac{(n-1)t}{n}}\right)}}_{\Theta_{n,n}} \quad \circledast$$

for any $n = 1, 2, 3, \dots$

$$\boxed{} \overset{\text{iid}}{\sim} X_{t/n}$$

Hence for each $t>0$, X_t is an inf div. r.v.

Define $\Psi_t(\theta) = -\log \mathbb{E}\left(e^{i\theta X_t}\right)$

Using $\circledast$

take $t = m \in \mathbb{N}$, $n = m$ in $\circledast$, and take FT

$$\mathbb{E}\left(e^{i\theta X_m}\right) = \mathbb{E}\left(e^{i\theta X_1}\right)^m = e^{-m\Psi_1(\theta)}$$

$$\| \quad e^{-\Psi_m(\theta)}$$

$$\Rightarrow \boxed{\Psi_m(\theta) = m\Psi_1(\theta)} \text{ — (1)}$$

take $t = m$ where $m, m' \in \mathbb{N}$, $n = m'$ & take FT

$$\mathbb{E}\left(e^{i\theta X_m}\right) = \mathbb{E}\left(e^{i\theta X_{m/m'}}\right)^{m'}$$

$$e^{-\Psi_m(\theta)} \qquad e^{-\Psi_{m/m'}(\theta)m'}$$

$$\boxed{\Psi_m(\theta) = m'\Psi_{m/m'}(\theta)} \text{ — (2)}$$

$$(1) \& (2) \Rightarrow m'\Psi_{m/m'}(\theta) = m\Psi_1(\theta)$$

$$\Rightarrow \Psi_t(\theta) = t\Psi_1(\theta) \quad \forall t \in \mathbb{Q} \cap [0,\infty)$$

Recall $\mathbb{E}\left(e^{i\theta X_t}\right) = e^{-\Psi_t(\theta)} = e^{-t\Psi_1(\theta)}$ $(t \in \mathbb{Q} \cap [0,\infty))$

suppose $t \geq 0$ is not rational, $\exists\, t_n \in \mathbb{Q} \cap [0,\infty)$ s.t. $t_n \downarrow t$, in which case by Bounded cgce theorem and right cts paths of X,

$$\lim_{t_n \downarrow t} \mathbb{E}\left(e^{i\theta X_{t_n}}\right) = \mathbb{E}\left(e^{i\theta X_t}\right) = e^{-\Psi_t(\theta)}$$

$$\lim_{t_n \downarrow t} e^{-t_n \Psi_1(\theta)} = e^{-t\Psi_1(\theta)}$$

In conclusion $\forall t \geq 0$ $\Psi_t(\theta) = t\,\Psi_1(\theta)$

and hence $\mathbb{E}\left(e^{i\theta X_t}\right) = e^{-t\Psi_1(\theta)}$

ie the dist^n of any Lévy process at time $t \geq 0$ is inf. div. & entirely determined by its dist^n at time $\underline{1}$.

The above analysis shows that each L.P. is associated to an inf div. dist^n and hence a triple (a, σ, Π), through X_1.

Is it true that given any (a, σ, Π) forming an inf. div. dist^n (via the L-K formula), we can construct a Lévy Process whose dist^n at time $\underline{1}$ is that of the chosen inf. div. dist^n?

<u>Theorem</u> Yes.

Pre-emptive remarks

What does Π mean for the Lévy process?
Indeed What does a, σ mean for L.P.?

Recall linear BM, $-at + \sigma B_t =: X_t$ $\sim^d N(0, \sigma^2 t)$

$$\mathbb{E}(e^{i\theta X_t}) = e^{-i\theta a t - \frac{1}{2}\sigma^2\theta^2 t}$$
$$= e^{-(i\theta + \frac{1}{2}\theta^2\sigma^2)t}$$

This suggests that a generic L.P. should be equal to $-at + \sigma B_t + J_t$
where J_t is another independent L.P.
with char. exponent $\int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbb{1}_{(|x|\le 1)})\Pi(dx)$

For now we mention that the process J_t has jump discontinuities in such a way that the probability we experience a jump of size $x \in \mathbb{R}$

in an interval of time $(t, t+dt)$
will be approx $\Pi(dx)\,dt + o(dt)$

"baby-baby description"