2.1. (Lévy Processes). A reduced stochastic process defined on some finite space \( \Omega \) is a stochastic process such that:

\( X_t = x \) for some \( x \) in \( \Omega \).

\( X_t \) is right continuous and adapted.

\( \lim_{t \to 0} X_t = x \) a.s.

Example 1: Brownian Motion

1. BM has no memory: Conditioning on \( X_t \) gives its future.

2. BM has independent increments.

\( X_t = B_t \) is a BM if and only if \( X_t = B_t - B_s \) is a BM for all \( 0 \leq s < t \).

3. Poisson Process: \( \{N_t : t \geq 0\} \)

\( N_t = \sum_{i=1}^{N_t} 1 \)

4. Compound Poisson Process: \( \{X_t : t \geq 0\} \)

\( X_t = \sum_{i=1}^{N_t} S_i \)

5. (Exercise) Markovian Diffusion Processes

\( X_t = \sum_{i=1}^{N_t} S_i \)

6. (Exercise) Independent Increments Lévy Process

Suppose \( \{X_t : t \geq 0\} \) is a Lévy process for \( i = 1, \ldots, n \) and \( X_t \) is independent.

\( X_t = \sum_{i=1}^{n} X_t^{(i)} \) a.s.
Infinitely divisible distributions

**Definition**: A $\mathbb{R}$-valued r.v. $\Theta$ is inf. div.

If for all $n \geq 1$, $\Theta$ is inf. div.

$$\Theta = \Theta_{1,n} + \Theta_{2,n} + \cdots + \Theta_{n,n}$$

where for each $n$, $\Theta_{i,n}$ are iid

Note: for each $n$, the common distr. of the $\Theta_{i,n}$ need not be the same.

**Examples**

1. $\Theta \sim N(0, \sigma^2)$

   For each $\Theta_n \sim N(0, \sigma^2/n)$

   $$E(e^{\delta \Theta}) = e^{-\frac{1}{2} \sigma^2} = (e^{-\frac{1}{2} \sigma^2})$$

   FT. of $N(0, \sigma^2/n)$

2. Poisson: If $\Theta \sim P_0(\lambda)$ : $\lambda > 0$

   $$E(s^{\Theta}) = \exp\{ \lambda (s-1) \} : s \in [0, 1]$$

   $$= \left[ \exp\{ \lambda (s-1) \} \right]$$

   iid across $\Theta_n$ in the Poisson

Hence for each $n \geq 1$, $\Theta_{i,n} \sim P_0(\lambda/n)$
Has big is the class of inf. div.
dusts?

Theorem (Lévy-Khintchine Formula)
A probability law (\text{dist}) of an \( \mathbb{R} \)-valued r.v. is inf. div. with characteristic exponent \( \Psi(\theta) \):

\[
\Psi(\theta) := \int e^{i\theta x} \mu(dx) = e^{-\Psi(\theta)}
\]

is well defined \( \forall \theta \in \mathbb{R} \) if and only if there exists a triple \((\alpha, \sigma, \Pi)\) such that \( \alpha \in \mathbb{R}, \sigma > 0 \) and \( \Pi \) is a s-finite measure concentrated on \( \mathbb{R} \setminus \{0\} \) satisfying:

\[
\int (1 + x^2) \Pi(dx) < \infty
\]

such that:

\[
\Psi(\theta) = \alpha \theta + \frac{1}{2} \sigma^2 \theta^2
\]

\[
+ \int (1 - e^{i\theta x} + i\theta x \mathbb{1}_{|x| < 1}) \Pi(dx)
\]

\( \forall \theta \in \mathbb{R} \).

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Respective remarks

What does \( \Pi \) mean for the Lévy process? Indeed, what does \( \alpha, \sigma \) mean for L.P.?

Recall linear BM, \(-\alpha t + \sigma B_t = X_t\)

\[
E(e^{i\Theta X_t}) = e^{-i\Theta \sigma t - \frac{1}{2} \sigma^2 \Theta^2 t} \quad \text{and} \quad N(0, \sigma^2_t)
\]

\[
= e^{-(i\Theta + \frac{1}{2} \sigma^2 \Theta^2) t}
\]

This suggests that a generic L.P. should be given by \(-\alpha t + \sigma B_t + \tilde{J}_t\)

where \( \tilde{J}_t \) is another independent L.P.

with char. exp. \[ \int \left(1 - e^{i\Theta x} + i\Theta x 1_{|x| \leq 1}\right) \Pi(dx) \]

For now, we mention that the process \( \tilde{J} \) has jumps discontinuously in such a way that \( \tilde{J} \)

The probability we experience a jump of size \( x \in \mathbb{R} \)

in an interval of time \((t, t+dt)\) will be approx \( \Pi(dx) dt + o(dt) \)

"baby-baby description"