Applications

1. M/G/1 Queue

customers arriving as a $PP(\lambda)$ → server

service times are iid with common distⁿ F.

<u>Workload model</u>: W_t = Amount of "work" that has still to be processed in the queue @ time t

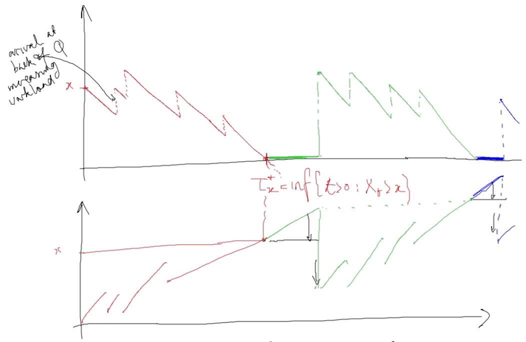
"work" means service time.

Describe W_t pathwise through the following L.P.

$$X_t = t - \sum_{i=1}^{N_t} \xi_i \qquad \text{where } \xi_i \overset{iid}{\sim} F \quad \perp\!\!\!\perp\ N_t\ PP(\lambda)$$

If $W_0 = x$ then

$$W_t = W_t^x = (x \vee \overline{X}_t) - X_t$$

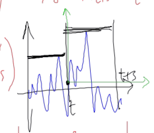


<u>Exercise</u>: Show that W_t^x is a Markov (and ultimately a strong Markov process).

$$W_{t+s}^x = x \vee \overline{X}_{t+s} - X_{t+s}$$

$$\overline{X}_{t+s} = \overline{X}_t \vee (X_t + \overline{X}'_s) \quad : X'_s = X_{t+s} - X_t$$

$$W_{t+s}^x = (x \vee \overline{X}_t) \vee (X_t + \overline{X}'_s) - (X_t + X'_s)$$



$$= \begin{cases} (X_t + \overline{X}'_s) - (X_t + X'_s) = \overline{X}'_s - X'_s & \text{if } \quad \text{bigger} \\ (x \vee \overline{X}_t) - (X_t + X'_s) = (x \vee \overline{X}_t - X_t) - X'_s & \text{o/w} \end{cases}$$

$$= (x \vee \overline{X}_t - X_t) \vee \overline{X}'_s - X'_s$$

$$= W_s'^{(W_t^x)}$$

where $W_s'^{(y)}$ is $\perp\!\!\!\perp (W_u^x : u \le t)$ has same law as $W_s^{(y)}$.

Note also that

$$\overline{X}_t = \int_0^t \mathbb{1}(\overline{X}_s = X_s)\, ds$$

since:

$$\overline{X}_t = \int_0^t \mathbb{1}(\overline{X}_s = X_s)\, dX_s$$

$X_t = t - S_t$ CPP

$$= \int_0^t \mathbb{1}(\overline{X}_s = X_s)\, ds - \boxed{\int_0^t \mathbb{1}(\overline{X}_s = X_s)\, dS_s}$$

$$\to 0 \le \int_0^t \mathbb{1}(\overline{X}_s = X_s)\, dS_s \le \int_0^t \mathbb{1}(\Delta S_s = 0)\, dS_s = 0$$

Introduce the parameter $\rho := \lambda \mathbb{E}\, \xi_1$ "traffic intensity".

$$\left.\begin{array}{l} 0 < \rho < 1 \\ \rho = 1 \\ \rho > 1 \end{array}\right\} \iff \left\{\begin{array}{l} \psi'(0^+) > 0 \\ \psi'(0^+) = 0 \\ \psi'(0^+) < 0 \end{array}\right.$$

where $\psi(\beta) = \beta - \lambda \int_0^\infty (1 - e^{-\beta x})\, F(dx)$

in particular $\mathbb{E}(X_1) = \psi'(0+)$.

Recall $\Phi(q) = \sup\{\beta \ge 0 : \psi(\beta) = q\}$
$q \ge 0$

$\Phi(0) > 0 \iff \psi'(0^+) < 0$.

<u>Theorem</u> Fix $y>0$, suppose $\rho>1$.

Then the total time that the $M/G/1$ queue spends idle, when initial workload is y,

$$I = \int_0^\infty \mathbb{1}(W_t^y = 0)\, dt$$

has the <u>following</u> dist^n

$$\mathbb{P}(I \in dx) = \left(1 - e^{-\Phi(0)y}\right)\delta_0(dy) + \Phi(0)e^{-\Phi(0)(y+x)}\,dx.$$

Otherwise $0<\rho\leq 1$ then $\mathbb{P}(I=\infty)=1$.

Proof: Recall $\bar{X}_\infty \sim \exp(\Phi(0))$

when $\Phi(0) > 0$ (i.e. $\psi'(0^+) < 0$) $\;(\rho > 1)$.

$\mathbb{P}(\bar{X}_\infty = \infty) = 1$ if $\psi'(0^+) \geq 0$ $(0 < \rho \leq 1)$.

Let $I^y := \int_0^\infty \mathbb{1}(W_t^y = 0)\, dt$ $\quad (W_t^y = y \vee \bar{X}_t - X_t)$

$$I^0 = \int_0^\infty \mathbb{1}(W_t^0 = 0)\, dt = \int_0^\infty \mathbb{1}(\bar{X}_t = X_t)\, dt$$

$$\boxed{I^0 = \bar{X}_\infty}$$

$(\rho > 1,\ y > 0)$ $\quad I^y = \int_{\tau_y^+}^\infty \mathbb{1}(W_t^y = 0)\, dt \; \mathbb{1}(\tau_y^+ < \infty)$

$$\overset{W^y \text{ is SMP}}{=} \int_0^\infty \mathbb{1}\left(\tilde{W}_s^{(W^y_{\tau_y^+})} = 0\right) \; \mathbb{1}(\tau_y^+ < \infty)$$

where $\tilde{W}_s^{(x)} \perp\!\!\!\perp W^y_{\tau_y^+}$ $\quad$ note also $W^y_{\tau_y^+} = 0$

$$= \tilde{I}^{(0)} \, \mathbb{1}(\tau_y^+ < \infty)$$

where $\tilde{I}^{(0)} \overset{d}{=} I^{(0)} \perp\!\!\!\perp \{X_t : t \leq \tau_y^+\}$.

$$\boxed{I^{(y)} = \tilde{I}^{(0)} \, \mathbb{1}(\tau_y^+ < \infty)}$$

$I^{(y)} = 0 \iff \tau_y^+ = \infty$ $\quad$ w.p. $\mathbb{P}(\tau_y^+ = \infty)$

$= \mathbb{P}(\bar{X}_\infty \leq y)$

$= (1 - e^{-\Phi(0) y})$

o/w

$$\mathbb{P}(I^{(y)} \in dx) = \mathbb{P}(\tau_y^+ < \infty)\, \mathbb{P}(\tilde{I}^{(0)} \in dx)$$

$x > 0$

$$= e^{-\Phi(0) y}\, \mathbb{P}(\bar{X}_\infty \in dx)$$

$$= e^{-\Phi(0) y}\, \Phi(0) e^{-\Phi(0) x}\, dx$$

All together $\quad \mathbb{P}(I^{(y)} \in dx) = (1 - e^{-\Phi(0) x})\, \delta_0(dx)$

$$= \Phi(0) e^{-\Phi(0)(x+y)}\, dx\, \mathbb{1}_{(x>0)}$$

The case $y = 0$ is contained in this statement.

Stationary dist^n of Workload

$$\frac{q}{q+\Psi(\theta)} = \mathbb{E}(e^{i\theta \bar{X}_{e_q}})\,\mathbb{E}(e^{i\theta \underline{X}_{e_q}})$$

$$= \frac{\Phi(q)}{\Phi(q)-i\theta}\,\mathbb{E}(e^{i\theta \underline{X}_{e_q}})$$

$\boxed{\text{recall } \bar{X}_{e_q} \sim \exp(\Phi(q))}$

$$\Rightarrow \mathbb{E}(e^{i\theta \underline{X}_{e_q}}) = \frac{q}{\Phi(q)}\,\frac{\Phi(q)-i\theta}{q+\Psi(\theta)}$$

LHS enjoy an analytic extension s.t. $\theta = -i\beta$, $\beta \geq 0$

RHS

$\mathbb{E}(e^{i\theta X_1}) = e^{-\Psi(\theta)}$ ✓ $\quad \mathbb{E}(e^{\beta X_1}) = e^{\psi(\beta)}$ ✓

Hence since $-\Psi(-i\beta) = \psi(\beta)$

$$\boxed{\mathbb{E}(e^{\beta \underline{X}_{e_q}}) = \frac{q}{\Phi(q)}\,\frac{\Phi(q)-\beta}{q-\psi(\beta)}}$$

$$e_q \overset{d}{=} \frac{1}{q} e_1$$

$$\mathbb{E}(e^{\beta \underline{X}_{e_q}}) = \mathbb{E}(e^{\beta \underline{X}_{e_1/q}})$$

take $q \downarrow 0$

$$\lim_{q\downarrow 0} \mathbb{E}(e^{\beta \underline{X}_{e_q}}) \overset{\text{DCT + monotonicity}}{=} \mathbb{E}(e^{\beta \underline{X}_\infty})$$

$\|$

$$\lim_{q\downarrow 0} \frac{q}{\Phi(q)}\,\frac{\Phi(q)-\beta}{q-\psi(\beta)} \overset{\psi'(0^+)\geq 0}{=} \lim_{q\downarrow 0} \frac{\psi(\Phi(q))}{\Phi(q)} \cdot \frac{\Phi(0)-\beta}{-\psi(\beta)}$$

$$\overset{\psi'(0^+)<0}{=} \frac{q}{\Phi(q)}\,\frac{\Phi(0)-\beta}{-\psi(\beta)}$$

$$\mathbb{E}(e^{-\beta(-\underline{X}_\infty)}) = \begin{cases} \psi'(0^+)\dfrac{\beta}{\psi(\beta)} & \psi'(0^+) \geq 0 \\ 0 & \psi'(0) < 0. \end{cases}$$

Theorem Suppose $\rho < 1$, $y \geq 0$, then

(i) W^y has a st. distⁿ equal to that of the r.v. W^y_∞ whose L.T. is given by

$$\mathbb{E}\left(e^{-\beta W^y_\infty}\right) = \psi'(0+)\frac{\beta}{\psi(\beta)}$$

(ii) Suppose $\rho \geq 1$, $y \geq 0$, then W^y does not converge in distⁿ.

Proof $\rho < 1 \Longleftrightarrow \psi'(0) > 0 \Longrightarrow \Phi(0) = 0$

$\Longrightarrow \mathbb{P}(\bar{X}_\infty = \infty) = 1$

$\Longrightarrow \mathbb{P}(\tau^+_y < \infty) = 1$

$\Longrightarrow$ for all t suff. large ($t > \tau^+_y$) $W^y_t = \bar{X}_t - X_t$

$$\lim_{t\to\infty}{}^{*}\, \mathbb{E}\left(e^{-\beta W^y_t}\right) = \lim_{t\to\infty}{}^{*}\, \mathbb{E}\left(e^{-\beta(\bar{X}_t - X_t)}\right)$$

$$\overset{\text{Duality}}{=} \lim_{t\to\infty}{}^{*}\, \mathbb{E}\left(e^{\beta \underline{X}_t}\right)$$

$$= \psi'(0+)\,\beta / \psi(\beta)$$

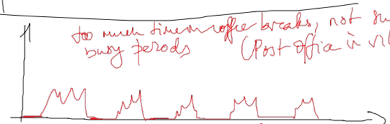
* these limits exist as soon as the last equality is true!

Pf for $\rho \geq 1$ is now obvious

$\rho < 1$

$I = \infty$ a.s.

$\psi'(0+) > 0$



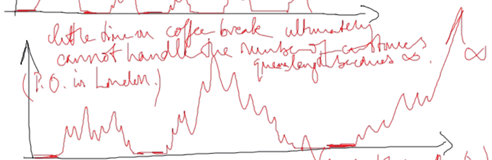
too much time on coffee breaks, not such large busy periods (Post office in village).

little time on coffee break ultimately cannot handle the number of customers queue length becomes ∞. (P.O. in London.)

$\rho > 1$

$I < \infty$ a.s

$\psi'(0+) < 0$



repeated coffee breaks (infinite total coffee time) experience larger & larger surges of customers (P.O. Manchester).

$\rho = 1$

$I = \infty$ a.s.

$\psi'(0+) = 0$.