More examples of Lévy processes

CPP

$$X_t = \sum_{i=1}^{N_t} \xi_i$$

ξ_i iid

N_t Poisson process rate $\lambda > 0$.

$$\mathbb{E}(e^{i\theta X_t}) = \mathbb{E}\left(e^{i\theta \sum_{i=1}^{N_t} \xi_i}\right)$$

$$= \mathbb{E}\,\mathbb{E}\left(e^{i\theta \sum_{i=1}^{N_t} \xi_i} \,\Big|\, N_t\right)$$

$$= \mathbb{E} \prod_{i=1}^{N_t} \mathbb{E}(e^{i\theta \xi_i}) = \mathbb{E}\left[\mathbb{E}(e^{i\theta\xi_1})^{N_t}\right]$$

$$= e^{\lambda t(\mathbb{E}(e^{i\theta \xi_1}) - 1)}$$

Suppose further that ξ_i's have common distⁿ F

$$= e^{-\lambda t\left(1 - \int_{\mathbb{R}} e^{i\theta x} F(dx)\right)}$$

$$= \boxed{e^{-\left(\int_{\mathbb{R}} (1 - e^{i\theta x}) \boxed{\lambda F}(dx)\right) t}}$$

LK formula
$$\Psi(\theta) = \boxed{ia\theta + \tfrac{1}{2}\sigma^2\theta^2} + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbb{1}_{(|x|<1)})\Pi(dx)$$

An even closer looking formula:

Take $X_t = \sum_{i=1}^{N_t} \xi_i - ct$ where $c \in \mathbb{R}$.

Then $\mathbb{E}(e^{i\theta X_t}) = \exp\left\{-\int_{\mathbb{R}} (1 - e^{i\theta x})\lambda F(dx) - ic\theta\right\}$

Lets suppose further that $\mathbb{E}|\xi_1| < \infty$

hence $\int x \lambda F(dx) < \infty$. Set $c = \int x \lambda F(dx)$

Then $\mathbb{E}(e^{i\theta X_t}) = \exp\left\{-\int_{\mathbb{R}} (1 - e^{i\theta x} + ix\theta)\lambda F(dx)\right\}$

Looks similar to L-K formula !!

Subtle differences between L-K formula and exponent for CPP with drift $c = \int x \lambda F(dx)$

1. Indicator is missing!
2. In L-K formula the measure Π satisfies
$$\underbrace{\int_{\mathbb{R}} (1 \wedge x^2)\, \Pi(dx)}_{\Uparrow} < \infty$$

In the CPP case
$$\int_{\mathbb{R}} \underbrace{\boxed{\lambda F}}_{\Pi}(dx) < \infty$$

CPP would not allow for eg
$$\Pi(dx) = \frac{1}{|x|^{1+\alpha}}\, dx \qquad \alpha \in (1,2) \quad (*)$$

$(*)'$
$$\Pi(dx) = \begin{cases} \dfrac{c_+}{x^{1+\alpha}}\, dx & (x>0) \\[2ex] \dfrac{c_-}{|x|^{1+\alpha}}\, dx & (x<0) \end{cases}$$
$c_+, c_- \geq 0$
$c_+ + c_- > 0$

note
$$\int_{\mathbb{R}} \Pi(dx) = \infty$$

however
$$\int_{\mathbb{R}\setminus\{0\}} (1 \wedge x^2)\, \Pi(dx) < \infty$$
$$\Updownarrow$$
$$\int_{[-1,1]\setminus\{0\}} x^2\, \Pi(dx) < \infty \text{ AND } \Pi(x : |x| \geq 1) < \infty$$

and these two conditions checkout for $(*)$

Hence $(*)$ corresponds to a Lévy process that cannot be expressed as a CPP with drift. (modulo linear BM)

Note the Lévy process with Lévy measure given by $(*)$ with $\rho = 0$ is a Stable process.

In latex lecture notes, there is an exercise to help you show that for $(*)'$
$$\Psi(\theta) = |\theta|^{\alpha}\left(1 - i\beta \tan\frac{\pi\alpha}{2}\, \operatorname{sgn}\theta\right) + i\theta\eta$$
$\eta \in \mathbb{R}$ and $\beta \in [-1,1]$

Another example of a Lévy Process

Recall the Gamma (α, β) dist^n

$$\mu_{\alpha,\beta}(dx) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\alpha x}\, dx \quad \text{on } (0,\infty)$$

$$\int_{(0,\infty)} e^{i\theta x} \mu_{\alpha,\beta}(dx) = \frac{1}{(1-i\theta/\alpha)^\beta}$$

$$= \left[\frac{1}{(1-i\theta/\alpha)^{\beta/n}}\right]^n$$

$\sim$ Gamma $(\alpha, \beta/n)$

Lévy-Itô theorem ("Yes" theorem)

tells us that $\exists$ a Lévy process $\{X_t : t \geq 0\}$ such that

$$X_1 \overset{d}{\sim} \mu_{\alpha,\beta}$$

L-K exponent: $\mathbb{E}(e^{i\theta X_t}) = e^{-[\beta \log(1-i\theta/\alpha)]t}$

Exercise

Frullani Integral: $\forall \alpha, \beta > 0$, $z \in \mathbb{C}$ s.t. $\mathrm{Re}\, z \leq 0$

$$\frac{1}{(1-z/\alpha)^\beta} = \exp\left\{-\int_0^\infty (1-e^{zx}) \underbrace{\boxed{\frac{\beta}{x} e^{-\alpha x}\, dx}}_{\Pi(dx)}\right\}$$

$\int_{(0,\infty)} \frac{\beta}{x} e^{-\alpha x}\, dx = \infty$ (not cpp)

check easily $\int_{(0,\infty)} (1 \wedge x^2)\, \Pi(dx) < \infty$ ok.

Note also that since $X_1 \sim \mu_{\alpha,\beta}$ then $X_1 > 0$. In fact $X_t \sim \mu_{\alpha,\beta t}$ so $X_t > 0$!

St + indep increments

$t > s > 0$: $X_t - X_s \overset{d}{\sim} X_{t-s} > 0$

$\{X_t : t \geq 0\}$ has monotone increasing paths.

Another quick example

$X^{(i)}$ to be a Gamma process for $i = 1, 2$. Independent of one another

$$X_t = X_t^{(1)} - X_t^{(2)}$$

later we will see the independence of $X^{(1)}$ & $X^{(2)}$ means their jumps cannot cancel one another out hence X is a process which moves both upwards and down-wards and has paths of Bounded Variation

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Lévy-Itô decomposition
(Proving the "yes" theorem)

$$\Psi(\theta) = i\theta a + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{i\theta x} + i\theta x \mathbb{1}_{(|x|<1)}\right)\Pi(dx)$$

$a \in \mathbb{R}$, $\sigma \geq 0$ $\quad \Pi$ a (σ-finite) measure concentrated on $\mathbb{R}\setminus\{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge x^2)\,\Pi(dx) < \infty$

Re-write the general L-K formula as follows:

$$\Psi(\theta) = \left\{ i\theta a + \frac{1}{2}\sigma^2\theta^2 \right\} \sim -at + \sigma B_t$$

Brownian Motion

$$+ \left\{ \int_{|x|\geq 1} \left(1 - e^{i\theta x}\right) \frac{\Pi(dx)}{\Pi(x : |x| \geq 1)}\, \Pi(x : |x| \geq 1) \right\}$$

CPP arrival rate λ and $\sim$ jump dist^n $\Pi(dx)/\Pi(x:|x|\geq 1)$

$\sim F$ $\quad \sim \lambda$

$$+ \left\{ \int_{x \in (-1,1)} \left(1 - e^{i\theta x} + i\theta x\right) \Pi(dx) \right\}$$

Need to get an idea about a 3rd Lévy process corresponds to 3rd curly bracket.

$$\int_{(-1,1)} \left(1 - e^{i\theta x} + i\theta x\right) \Pi(dx)$$

$$= \sum_{n=0}^{\infty} \boxed{\int_{|x| \in [2^{-(n+1)}, 2^{-n})} \left(1 - e^{i\theta x} + i\theta x\right) \lambda_n F_n(dx)}$$

where $\lambda_n := \Pi\left(x : |x| \in [2^{-(n+1)}, 2^{-n})\right)$

$F_n(dx) := \left.\dfrac{\Pi(dx)}{\lambda_n}\right|_{[2^{-(n+1)}, 2^{-n})}$ as a measure on $[2^{-(n+1)}, 2^{-n})$

Note ① if $\lambda_n = 0$ we understand $\int_{|x| \in [2^{-(n+1)}, 2^{-n})} \cdots = 0$

② $\lambda_n < \infty$ (exercise) it follows from the fact that

$$\int_{(-1,1)} x^2 \, \Pi(dx) < \infty.$$

The formal statement of "Yes" Theorem

<u>Theorem</u> (Lévy-Itô decomposition).

Given any $a \in \mathbb{R}$, $\sigma \geq 0$ and σ-finite measure Π concentrated on $\mathbb{R} \setminus \{0\}$ & satisfying

$$\int (1 \wedge x^2) \, \Pi(dx) < \infty,$$

there exists a probability space on which 3 independent Lévy processes exist; $X^{(1)}$, $X^{(2)}$, $X^{(3)}$ where

$X_t^{(1)} = -at + \sigma B_t$

$X_t^{(2)}$ is a CPP with arrival rate $\Pi(|x| \geq 1)$ and jumps concentrated on $\{x : |x| \geq 1\}$ with dist^n $F(dx) = \dfrac{\Pi(dx)}{\Pi(|x| \geq 1)}$

and $X^{(3)}$ is a square integrable martingale with characteristic exponent

$$\int_{(-1,1)} (1 - e^{i\theta x} + i\theta x) \, \Pi(dx).$$

In particular $X_t := X_t^{(1)} + X_t^{(2)} + X_t^{(3)}$ is a Lévy process with the given characteristic exponent

$$\Psi(\theta) = i\theta x + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \, \mathbb{1}_{|x|<1}) \, \Pi(dx)$$

$\forall \theta \in \mathbb{R}$.