Corollary

Suppose N is a PRM on $(S, \mathcal{S}, \eta)$. Then for each $A \in \mathcal{S}$, $N(\cdot \cap A)$ is a PRM on $(A, A \cap \mathcal{S}, \eta(\cdot\,; A))$.

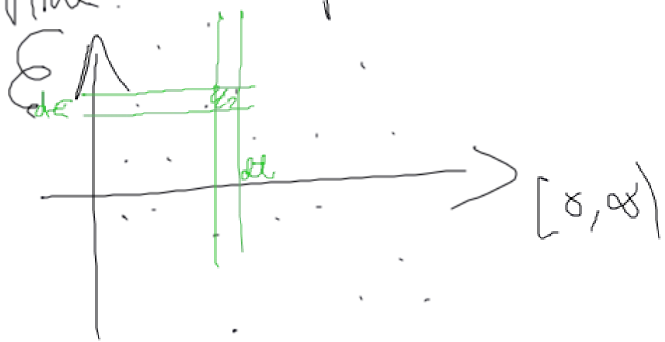
Further, if $A, B \in \mathcal{S}$ and $A \cap B = \phi$, then $N(\cdot \cap A) \perp\!\!\!\perp N(\cdot \cap B)$.

Corollary. Suppose N is a PRM on $(S, \mathcal{S}, \eta)$. Then the support of N is P a.s. countable. If in addition η is a finite measure, then the support of N is a.s. finite (in fact it consists of a Poisson distributed # of pts.)

Remark: If η has atoms i.e. $\exists\, s \in S$ s.t. $\{s\} \in \mathcal{S}$ and $\eta\{s\} > 0$, then we can get a "pile-up" of points at $\{s\}$. If η is diffuse (no atoms) then the probability that $N(\{s\}) > 1$ is zero. $\forall\, s \in S$ such that $\{s\} \in \mathcal{S}$.

Poisson Point Processes

In the case that $S = [0,\infty) \times \mathcal{E}$ for some space $\mathcal{E}$, we can think of points in the support of an associated PRM as arriving in time.



The support of an associated PRM on $S = [0,\infty) \times \mathcal{E}$ is called a Poisson Point Process if the intensity measure $\eta(dt \times de) = dt \times n(de)$ for some σ-finite measure n.

diffuse!

Intuitively: A "point" arrives in PPP at time t and pos^n $e \in \mathcal{E}$ with prob $dt\, n(de) + o(dt)$

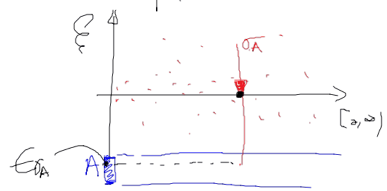
Thinning Theorem for PPP

Notation Given measure (σ-finite) n on $\mathcal{E}$
(t, ϵ_t) are the points supporting the PRM on $[0,\infty) \times \mathcal{E}$ with intensity $dt \times dn$.
If we define $\epsilon_t = $ "∂" for t such that t is not a "time" in the support of our PRM

Then we can think of $\{\epsilon_t : t \geq 0\}$ as our PPP on $\mathcal{E}$.

Theorem: Suppose $A \in \mathcal{E}$ is an n-measurable set such that $n(A) < \infty$. Let

$$\sigma_A = \inf\{t > 0 : \epsilon_t \in A\}.$$



(i) σ_A is a stopping time w.r.t. the natural filtration of the underlying PPP

$$\mathcal{G}_t = \sigma\left(N([0,s] \times B) = k \;:\; \begin{array}{l} s \leq t \\ B \in \Sigma, k = 0,1,2,\ldots \end{array} \right)$$

Σ: σ-algebra for $\mathcal{E}$

Moreover $\sigma_A \sim \exp(n(A))$.

(ii) $\epsilon_{\sigma_A} \perp\!\!\!\perp \{\epsilon_t : t < \sigma_A\}$ and has law given by

$$\frac{n(\cdot\,; A)}{n(A)}$$

(iii) $\{\epsilon_t : t < \sigma_A\}$ has the law of a PPP with intensity $dt \times n(d\epsilon;\, \mathcal{E} \setminus A)$ which is stopped at an independent and exponentially distributed random time with parameter $n(A)$.

Pf (Sketch Pf).

Suppose $f : [0,\infty) \times \mathcal{E} \to [0,\infty)$ is bounded and measurable w.r.t. $dt \times dn$.

$$\mathbb{E}\left(\boxed{e^{-\int_0^t \int_{\mathcal{E}} f(s,\epsilon)\, N(ds \times d\epsilon)}} ; \sigma_A > t ; \epsilon_{\sigma_A} \in B\right)$$

$B \subseteq A$ and B is n-measurable.

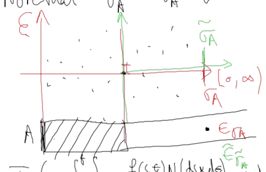
$$= \mathbb{E}\left(e^{-\int_0^t \int_{\mathcal{E}} f(s,\epsilon)\, N(ds \times d\epsilon)} ; N([0,t] \times A) = 0 ; \tilde{\epsilon}_{\tilde{\sigma}_A} \in B\right)$$

$\Rightarrow \sigma_A$ is a stopping time.

where $\{\tilde{\epsilon}_s : s \geq 0\} = \{\epsilon_{t+s} : s \geq 0\}$

$\tilde{\sigma}_A = \inf\{s > 0 : \tilde{\epsilon}_s \in A\}$.

Note that $\tilde{\sigma}_A = \sigma_A - t$ on $\{\sigma_A > t\}$.



$$\Rightarrow \mathbb{E}\left(e^{-\int_0^t \int_{\mathcal{E}\setminus A} f(s,\epsilon)\, N(ds \times d\epsilon)} ; N([0,t] \times A) = 0 ; \tilde{\epsilon}_{\tilde{\sigma}_A} \in B\right)$$

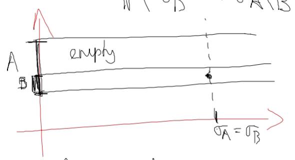
$$= \mathbb{E}\left(e^{-\int_0^t \int_{\mathcal{E}\setminus A} f(s,\epsilon)\, N(ds \times d\epsilon)}\right) \times \mathbb{P}\left(N([0,t] \times A) = 0\right) \times \mathbb{P}\left(\tilde{\epsilon}_{\tilde{\sigma}_A} \in B\right)$$

$$= \underbrace{\mathbb{E}\left(e^{-\int_0^t \int_{\mathcal{E}\setminus A} f(s,\epsilon)\, N(ds \times d\epsilon)}\right)}_{(i)} \times \underbrace{e^{-t\, n(A)}}_{(ii)} \times \underbrace{\mathbb{P}\left(\epsilon_{\sigma_A} \in B\right)}_{(iii)}$$

This factorisation tells us:
- (ii) $\sigma_A \sim \exp(n(A))$
- (i) & (ii) vs (iii): $\epsilon_{\sigma_A} \perp\!\!\!\perp \{\epsilon_t : t < \sigma_A\}$
- (i) & (ii): $\{\epsilon_t : t < \sigma_A\}$ $=$ the PPP$(dt \times n(d\epsilon ; A))$ stopped at indep $\exp(n(A))$-time.

Finally note that $\mathbb{P}(\epsilon_{\sigma_A} \in B)$

$$= \mathbb{P}(\sigma_B < \sigma_{A \setminus B})$$



By earlier conclusions:

$\sigma_{A\setminus B} \sim \exp(n(A\setminus B))$ — path functional of PRM on $[0,\infty) \times (\mathcal{E}\setminus B)$

$\sigma_B \sim \exp(n(B))$ & $\sigma_A \perp\!\!\!\perp \sigma_B$

Hence $\mathbb{P}(\sigma_B < \sigma_{A\setminus B}) = \dfrac{n(B)}{n(B) + n(A\setminus B)} = \dfrac{n(B)}{n(A)}$

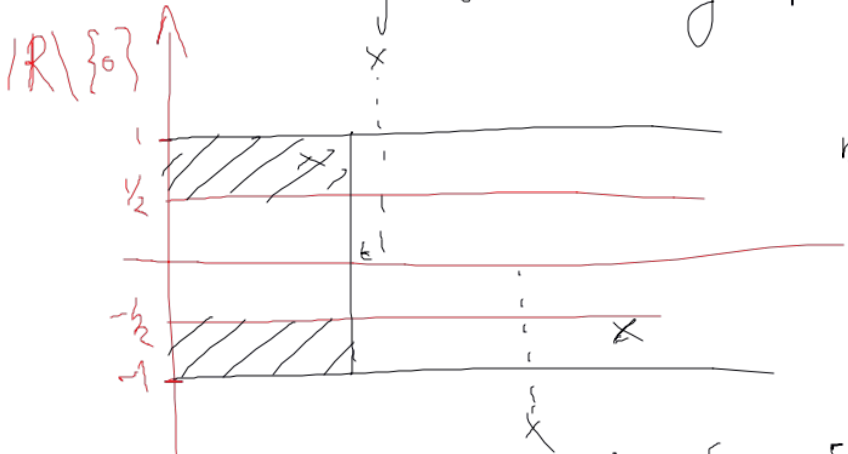
$$\Rightarrow \epsilon_{\sigma_A} \sim \frac{n(\cdot\, ; A)}{n(A)}$$

Example of a PPP

If X is any Lévy process with Lévy measure Π

Then define $\Delta X_t = X_t - X_{t-} \in \mathbb{R}\setminus\{0\}$ if t is a jump time

if t is not a jump time then $\Delta X_t = 0$ "∂"



big jump $|x| \geq 1$ have inter arrival times $\exp(\Pi(|x|\geq 1))$

jumps arrived in annulus $\{|x| \in [\tfrac{1}{2}, 1)\}$ with inter arrival time which is $\exp\left(\Pi(\tfrac{1}{2} \leq |x| < 1)\right)$,

A little thought shows that

$\{\Delta X_t : t \geq 0\}$ is a PPP with intensity $dt \times \Pi(dx)$ on $\mathbb{R}\setminus\{0\}$.