Meromorphic Lévy Processes and a new Wiener-Hopf Monte-Carlo simulation method

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- Other motivations from queuing theory, population models etc.
- One is fundamentally interested in the joint distribution

  $$P(X_t \in dx, \overline{X}_t \in dy)$$

  for any Lévy process (X, P).

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where $a \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and Π is a measure concentrated on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$...

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- ...in which case there are fast-Fourier methods for inverting $\exp\{-\Psi(\theta)t\}$ to give $P(X_t \in dx)$.
- For the case of $P(\overline{X}_t \in dx)$, recent methods have concentrated on Fourier inversion of the, so called, Wiener-Hopf factors.

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- Recall that it turns out that one may always uniquely decompose

$$\frac{q}{q+\Psi(\theta)} = E(\mathrm{e}^{\mathrm{i}\theta \overline{X}_{\mathbf{e}_q}}) \times E(\mathrm{e}^{\mathrm{i}\theta \underline{X}_{\mathbf{e}_q}})$$

where $\mathbf{e}_q$ is an independent and exponentially distributed random variable with rate $q > 0$ and $\underline{X}_t = \inf_{s \le t} X_s$.

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- *If one is in possession of close analytical expressions for these factors*, Fourier inversion, first in θ and then in q would be an option for accessing the law of $\overline{X}_t$ and $\underline{X}_t$. However one is rarely in possession of the factors (even after 60 years of research into this topic), and even then there is the issue of the double Fourier inversion.

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- *If one is in possession of close analytical expressions for these factors*, Fourier inversion, first in θ and then in q would be an option for accessing the law of $\overline{X}_t$ and $\underline{X}_t$. However one is rarely in possession of the factors (even after 60 years of research into this topic), and even then there is the issue of the double Fourier inversion.
- There are no convenient formulae which contain both X_t and $\overline{X}_t$ which could be Fourier inverted.

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- Note that

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is a Gamma (Erlang) distribution with parameters n and q and by the strong law of Large numbers, for $t > 0$,

$$\mathbf{g}(n, n/t) = \sum_{i=1}^{n} \frac{t}{n} \mathbf{e}^{(i)} \to t$$

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  almost surely.
- Hence for a suitably large n, we have in distribution

$$(X_{\mathbf{g}(n,n/t)}, \overline{X}_{\mathbf{g}(n,n/t)}) \simeq (X_t, \overline{X}_t).$$

  Indeed since t is not a jump time with probability 1, we have that $(X_{\mathbf{g}(n,n/t)}, \overline{X}_{\mathbf{g}(n,n/t)}) \to (X_t, \overline{X}_t)$ almost surely.

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- A reformulation of the Wiener-Hopf factorization states that

$$X_{\mathbf{e}_q} \stackrel{d}{=} S_q + I_q$$

where S_q is independent of I_q and they are respectively equal in distribution to $\overline{X}_{\mathbf{e}_q}$ and $\underline{X}_{\mathbf{e}_q}$.

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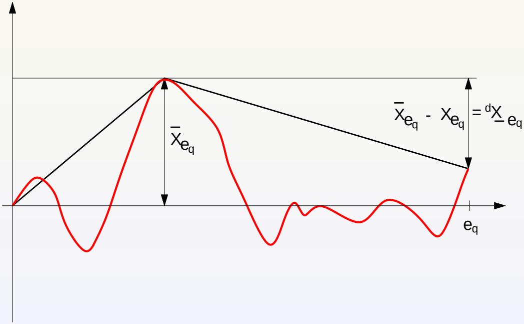

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  where S_q is independent of I_q and they are respectively equal in distribution to $\overline{X}_{\mathbf{e}_q}$ and $\underline{X}_{\mathbf{e}_q}$.
- Taking advantage of the above, the fact that X has stationary and independent increments and the fact that, as a time, $\mathbf{g}(n, n/t)$ can be seen as n subsequent independent exponential time periods we have the following:

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- **Theorem.** For all $n \in \{1, 2, \cdots\}$ and $q > 0$,

$$(X_{\mathbf{g}(n,q)}, \overline{X}_{\mathbf{g}(n,q)}) \stackrel{d}{=} (V(n,q), J(n,q))$$

where

$$V(n,q) = \sum_{j=1}^{n} \{S_q^{(j)} + I_q^{(j)}\} \text{ and } J(n,q) := \bigvee_{i=0}^{n-1} \left(\sum_{j=1}^{i} \{S_q^{(j)} + I_q^{(j)}\} + S_q^{(i+1)} \right).$$

Here, $S_q^{(0)} = I_q^{(0)} = 0$, $\{S_q^{(j)} : j \geq 1\}$ are an i.i.d. sequence of random variables with common distribution equal to that of $\overline{X}_{\mathbf{e}_q}$ and $\{I_q^{(j)} : j \geq 1\}$ are another i.i.d. sequence of random variable with common distribution equal to that of $\underline{X}_{\mathbf{e}_q}$.

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- **Corollary.** We have as $n \uparrow \infty$

$$(V(n, n/t), J(n, n/t)) \to (X_t, \overline{X}_t)$$

where the convergence is understood in the distributional sense.

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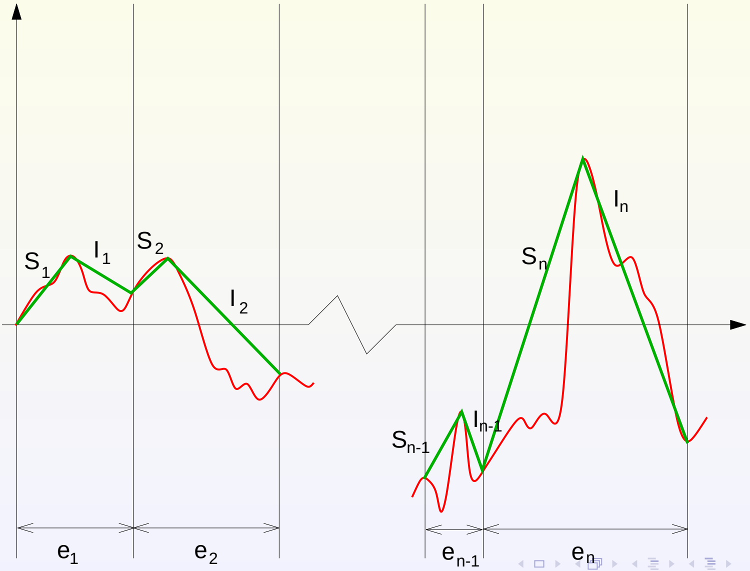

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- This numerical procedure has disposed of one (numerical) Fourier inverse computation.
- This still leaves the problem of simulating from the unknown distribution $\overline{X}_{\mathbf{e}_{n/t}}$ and $\underline{X}_{\mathbf{e}_{n/t}}$ i.e. we are still one (numerical) Fourier transform away from $(X_t, \overline{X}_t)$

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$$\pi(x) = \mathbb{I}_{\{x>0\}} \sum_{n\geq 1} a_n \rho_n \mathrm{e}^{-\rho_n x} + \mathbb{I}_{\{x<0\}} \sum_{n\geq 1} \hat{a}_n \hat{\rho}_n \mathrm{e}^{\hat{\rho}_n x}, \qquad (1)$$

where all the coefficients a_n, $\hat{a}_n$, ρ_n, $\hat{\rho}_n$ are positive, the sequences $\{\rho_n\}_{n\geq 1}$ and $\{\hat{\rho}_n\}_{n\geq 1}$ are stricly increasing, and $\rho_n \to +\infty$ and $\hat{\rho}_n \to +\infty$ as $n \to +\infty$.

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- We allow the case of a finite number summands (on either or both sides of the origin) with obvious modifications to the above.
- To ensure that $\int_{\mathbb{R}} x^2 \pi(x)\mathrm{d}x$ converges we need to impose the additional constraint that

$$\sum_{n\geq 1} a_n \rho_n^{-2} + \sum_{n\geq 1} \hat{a}_n \hat{\rho}_n^{-2} < \infty$$

Meromorphic Lévy processes (equivalent definition)

(i) The characteristic exponent $\Psi(z)$ is a meromorphic function which has poles at points $\{-\mathrm{i}\rho_n, \mathrm{i}\hat{\rho}_n\}_{n\ge1}$, where ρ_n and $\hat{\rho}_n$ are positive real numbers.

(ii) For $q \ge 0$ function $q + \Psi(z)$ has roots at points $\{-\mathrm{i}\zeta_n, \mathrm{i}\hat{\zeta}_n\}_{n\ge1}$ where ζ_n and $\hat{\zeta}_n$ are nonnegative real numbers (strictly positive if $q > 0$). We will write $\zeta_n(q)$, $\hat{\zeta}_n(q)$ if we need to stress the dependence on q.

(iii) The roots and poles of $q + \Psi(\mathrm{i}z)$ satisfy the following interlacing condition

$$... - \rho_2 < -\zeta_2 < -\rho_1 < -\zeta_1 < 0 < \hat{\zeta}_1 < \hat{\rho}_1 < \hat{\zeta}_2 < \hat{\rho}_2 < ...$$

(iv) There exists $\alpha > 1/2$ such that $\rho_n \sim cn^\alpha$ and $\hat{\rho}_n \sim \hat{c}n^\alpha$ as $n \to +\infty$.

(v) The Wiener-Hopf factors are expressed as convergent infinite products,

$$\mathbb{E}\left[\mathrm{e}^{-z\overline{X}_{\mathbf{e}_q}}\right] = \prod_{n\ge1} \frac{1 + \frac{z}{\rho_n}}{1 + \frac{z}{\zeta_n}}$$

$$\mathbb{E}\left[\mathrm{e}^{z\underline{X}_{\mathbf{e}_q}}\right] = \prod_{n\ge1} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 + \frac{z}{\hat{\zeta}_n}}.$$

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- The density of the Lévy measure is

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- Including Gaussian and linear drift, one can verify that the characteristic exponent is a rational function and that hyper-exponential Lévy processes have finite activity jumps and paths of bounded variation unless $\sigma > 0$.
- Note that this class has been looked at by many other authors in the past and historically is starts life as the Kou process.

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- The characteristic exponent ($\Psi(\theta) = -\log \mathbb{E}(e^{i\theta X_1}), \theta \in \mathbb{R}$) is given by

$$\Psi(\theta) = iaz + \frac{1}{2}\sigma^2 z^2 + \frac{c_1}{\beta_1}\left\{\mathrm{B}(\alpha_1, 1-\lambda_1) - \mathrm{B}(\alpha_1 - \frac{i\theta}{\beta_1}, 1-\lambda_1)\right\} + \frac{c_2}{\beta_2}\left\{\mathrm{B}(\alpha_2, 1-\lambda_2) - \mathrm{B}(\alpha_2 + \frac{i\theta}{\beta_2}, 1-\lambda_2)\right\}$$

where $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta function, with parameter range $a \in \mathbb{R}, \sigma, c_i, \alpha_i, \beta_i > 0$ and $\lambda_1, \lambda_2 \in (0,3) \setminus \{1,2\}$.

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- The corresponding Lévy measure Π has density

$$\pi(x) = c_1 \frac{e^{-\alpha_1\beta_1 x}}{(1-e^{-\beta_1 x})^{\lambda_1}} \mathbf{1}_{\{x>0\}} + c_2 \frac{e^{\alpha_2\beta_2 x}}{(1-e^{\beta_2 x})^{\lambda_2}} \mathbf{1}_{\{x<0\}}.$$

  The β-class of Lévy processes includes another recently introduced family of Lévy processes known as Lamperti-stable processes.

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$$\Psi(\theta) = \frac{\Gamma(1-\beta+\gamma-\mathrm{i}\theta)}{\Gamma(1-\beta+\mathrm{i}\theta)} \frac{\Gamma(\hat{\beta}+\hat{\gamma}+\mathrm{i}\theta)}{\Gamma(\hat{\beta}+\mathrm{i}\theta)}$$

where $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$ belong to the admissible range

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- The Lévy density is given by

$$\pi(x) = \begin{cases} -\frac{\Gamma(\eta)}{\Gamma(\eta-\hat{\gamma})\Gamma(-\gamma)} e^{-(1-\beta+\gamma)x} {}_2F_1(1+\gamma, \eta; \eta-\hat{\gamma}; e^{-x}) & \text{if } x > 0 \\ -\frac{\Gamma(\eta)}{\Gamma(\eta-\gamma)\Gamma(-\hat{\gamma})} e^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1(1+\hat{\gamma}, \eta; \eta-\gamma; e^{-x}) & \text{if } x < 0 \end{cases}$$

where $\eta = 1-\beta+\gamma+\hat{\beta}+\hat{\gamma}$.

Distribution of extrema

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- For $x \geq 0$

$$\begin{aligned}\mathbb{P}(\overline{X}_{\mathsf{e}_q} \in \mathsf{d}x) &= \bar{\mathsf{a}}(\rho, \zeta)^T \times \bar{\mathsf{v}}(\zeta, x)\mathsf{d}x \\ \mathbb{P}(-\underline{X}_{\mathsf{e}_q} \in \mathsf{d}x) &= \bar{\mathsf{a}}(\hat{\rho}, \hat{\zeta})^T \times \bar{\mathsf{v}}(\hat{\zeta}, x)\mathsf{d}x.\end{aligned}$$

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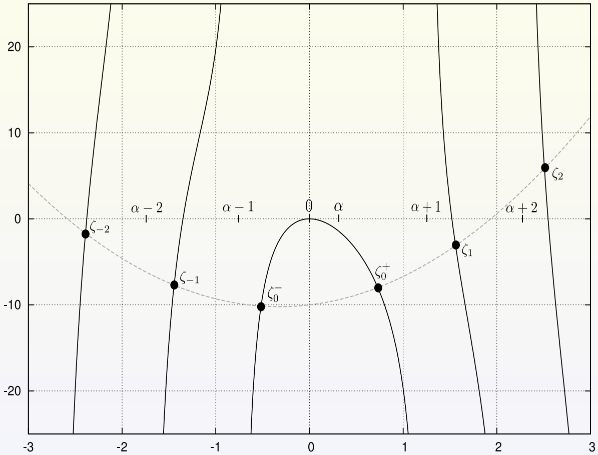
- Here $\bar{\mathsf{a}}(\rho, \zeta) = [\mathsf{a}_0(\rho, \zeta), \mathsf{a}_1(\rho, \zeta), \mathsf{a}_2(\rho, \zeta), ...]^T$ such that

$$\mathsf{a}_0(\rho, \zeta) = \lim_{n \to +\infty} \prod_{k=1}^{n} \frac{\zeta_k}{\rho_k}, \quad \mathsf{a}_n(\rho, \zeta) = \left(1 - \frac{\zeta_n}{\rho_n}\right) \prod_{\substack{k \geq 1 \\ k \neq n}} \frac{1 - \frac{\zeta_n}{\rho_k}}{1 - \frac{\zeta_n}{\zeta_k}}$$

$$\bar{\mathsf{v}}(\zeta, x) = \left[\delta_0(x), \zeta_1 \mathrm{e}^{-\zeta_1 x}, \zeta_2 \mathrm{e}^{-\zeta_2 x}, \ldots\right]^T,$$

where $\delta_0(x)$ is the Dirac delta function at $x = 0$. A similar expression holds for $\bar{\mathsf{a}}(\rho, \zeta)$.

Computing roots



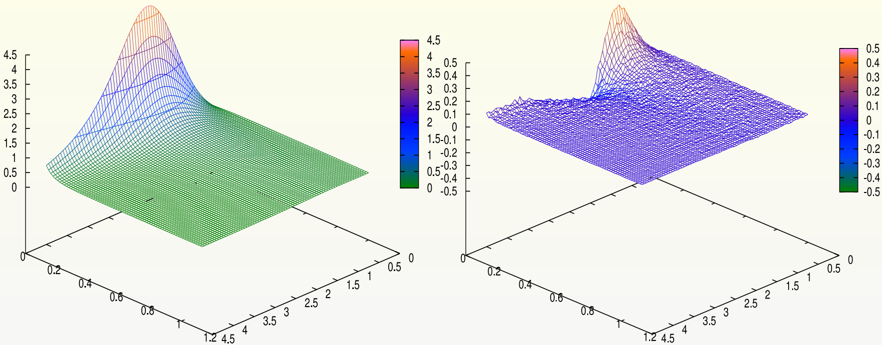
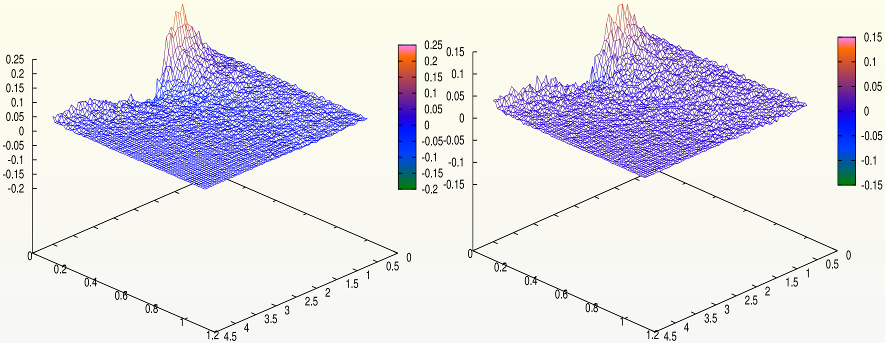


Figure: Computing the joint density of $(\overline{X}_1, \overline{X}_1 - X_1)$ for parameter Set 1. Here $\overline{X}_1 \in [0, 1]$ and $\overline{X}_1 - X_1 \in [0, 4]$. Fourier method benchmark (left), $N = 20$, WH-MC error (right).



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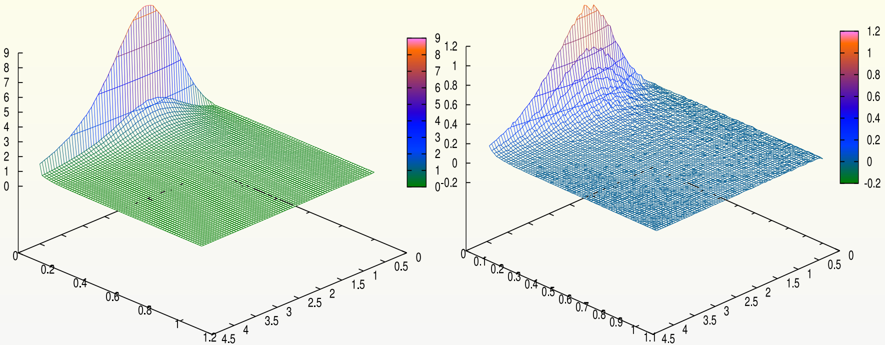

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- Does not artificially build in an atom at zero in the numerical distribution of $\overline{X}_t$.

More identities: One sided exit problem

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- Define a matrix $\mathbf{A} = \{a_{i,j}\}_{i,j\geq 0}$ as

$$a_{i,j} = \begin{cases} 0 & \text{if } i=0,\ j\geq 0 \\ \mathsf{a}_i(\rho,\zeta)\mathsf{b}_0(\zeta,\rho) & \text{if } i\geq 1,\ j=0 \\ \dfrac{\mathsf{a}_i(\rho,\zeta)\mathsf{b}_j(\zeta,\rho)}{\rho_j-\zeta_i} & \text{if } i\geq 1,\ j\geq 1 \end{cases} \qquad (2)$$

Then for $c>0$ and $y\geq 0$ we have

$$\mathbb{E}\left[e^{-q\tau_c^+}\mathbb{I}\left(X_{\tau_c^+}-c\in \mathsf{d}y\right)\right] = \bar{\mathsf{v}}(\zeta,c)^T\times\mathbf{A}\times\bar{\mathsf{v}}(\rho,y)\mathsf{d}y. \qquad (3)$$

More identities: Two-sided exit problem

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- Let $a > 0$ and define a matrix $\mathbf{B} = \mathbf{B}(\hat{\rho}, \zeta, a) = \{b_{i,j}\}_{i,j\geq 0}$ with

$$b_{i,j} = \begin{cases} \zeta_j \mathrm{e}^{-a\zeta_j} & \text{if } i = 0,\ j \geq 1 \\ 0 & \text{if } i \geq 0,\ j = 0 \\ \dfrac{\hat{\rho}_i \zeta_j}{\hat{\rho}_i + \zeta_j} \mathrm{e}^{-a\zeta_j} & \text{if } i \geq 1,\ j \geq 1 \end{cases}$$

  and similarly $\hat{\mathbf{B}} = \mathbf{B}(\rho, \hat{\zeta}, a)$.There exist matrices $\mathbf{C}_1$, $\mathbf{C}_2$ such that for $x \in (0, a)$ we have

$$\mathbb{E}_x \left[\mathrm{e}^{-q\tau_a^+} \mathbb{I}\left(X_{\tau_a^+} \in \mathrm{d}y \ ;\ \tau_a^+ < \tau_0^- \right) \right] = \left[\bar{\mathrm{v}}(\zeta, a-x)^T \times \mathbf{C}_1 + \bar{\mathrm{v}}(\hat{\zeta}, x)^T \times \mathbf{C}_2 \right] \times \bar{\mathrm{v}}(\rho, y-a)\mathrm{d}y$$

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- These matrices satisfy the following system of linear equations

$$\begin{cases} \mathbf{C}_1 & = \mathbf{A} - \hat{\mathbf{C}}_2 \mathbf{B} \mathbf{A} \\ \hat{\mathbf{C}}_2 & = -\mathbf{C}_1 \hat{\mathbf{B}} \hat{\mathbf{A}} \end{cases} \qquad \begin{cases} \hat{\mathbf{C}}_1 & = \hat{\mathbf{A}} - \mathbf{C}_2 \hat{\mathbf{B}} \hat{\mathbf{A}} \\ \mathbf{C}_2 & = -\hat{\mathbf{C}}_1 \mathbf{B} \mathbf{A} \end{cases}$$

  This system of linear equations can be solved iteratively with exponential convergence.

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- For $a > 0$, $y \leq a$ we define $R^{(q)}(a, \mathrm{d}y) := \int_0^\infty \mathrm{e}^{-qt} \mathbb{P}(X_t \in \mathrm{d}y; t < \tau_a^+) \mathrm{d}t$

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- Define a matrix $\mathbf{D} = \{d_{i,j}\}_{i,j\geq 0}$ as follows

$$d_{i,j} = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \mathsf{a}_i(\rho, \zeta)\dfrac{\zeta_i \hat{\zeta}_j}{\zeta_i + \hat{\zeta}_j}\mathsf{a}_j(\hat{\rho}, \hat{\zeta}) & \text{if } i \geq 1,\ j \geq 1 \end{cases}$$

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- Then if $y \leq a$ we have

$$\begin{aligned} qR^{(q)}(a, \mathrm{d}y) &= \Big[\bar{\mathsf{v}}(\zeta, 0 \vee y) \times \mathbf{D} \times \bar{\mathsf{v}}(\hat{\zeta}, 0 \vee (-y))) \\ &\quad - \bar{\mathsf{v}}(\zeta, a) \times \mathbf{D} \times \bar{\mathsf{v}}(\hat{\zeta}, a - y) \Big] \mathrm{d}y. \end{aligned}$$

Example of numerics

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- Choose an example from Kuznetsov's β-class that has bounded variation jump component and concentrate on four cases: With/without Gaussian component, drift to $\pm\infty$.

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- Choose an example from Kuznetsov's β-class that has bounded variation jump component and concentrate on four cases: With/without Gaussian component, drift to $\pm\infty$.
- For the above four cases, consider the following densities.
  - **(i)** density of the overshoot if the exit happens at the upper boundary

$$f_1(x,y) = \frac{\mathrm{d}}{\mathrm{d}y}\,\mathbb{E}_x\left[\mathrm{e}^{-q\tau_1^+}\mathbb{I}\left(X_{\tau_1^+} \le y\,;\,\tau_1^+ < \tau_0^-\right)\right]$$

  - **(ii)** probability of exiting from the interval $[0,1]$ at the upper boundary

$$f_2(x) = \mathbb{E}_x\left[\mathrm{e}^{-q\tau_1^+}\mathbb{I}\left(\tau_1^+ < \tau_0^-\right)\right]$$

  - **(iii)** probability of exiting the interval $[0,1]$ by creeping across the upper boundary

$$f_3(x) = \mathbb{E}_x\left[\mathrm{e}^{-q\tau_1^+}\mathbb{I}\left(X_{\tau_1^+} = 1\,;\,\tau_1^+ < \tau_0^-\right)\right]$$

Double sided exit: $\sigma > 0$ and positive drift

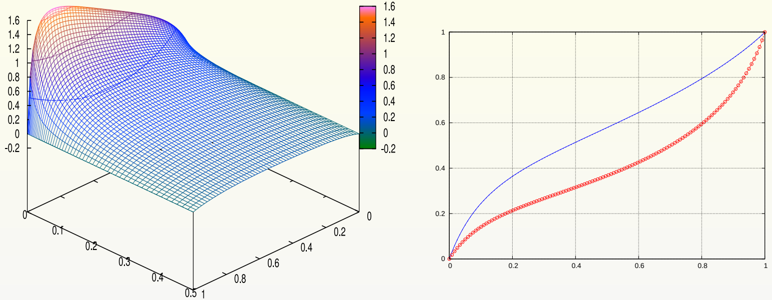

Figure: Unbounded variation case ($\sigma = 0.5$): computing the density of the overshoot $f_1(x, y)$ ($x \in (0, 1)$, $y \in (0, 0.5)$), probability of first exit $f_2(x)$ and probability of creeping $f_3(x)$ for parameter Set 1, positive drift $\mu = 1$

Double sided exit: $\sigma > 0$ and negative drift

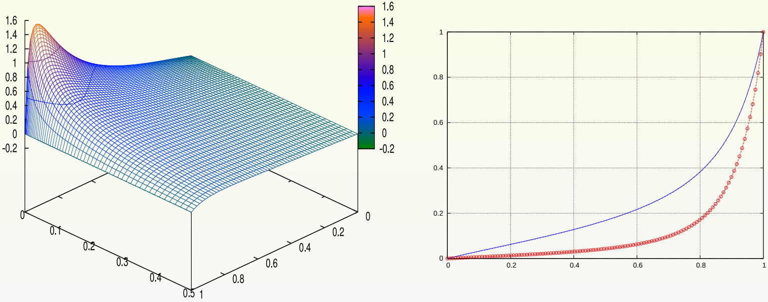

Figure: Unbounded variation case ($\sigma = 0.5$): computing the density of the overshoot $f_1(x, y)$ ($x \in (0, 1)$, $y \in (0, 0.5)$), probability of first exit $f_2(x)$ and probability of creeping $f_3(x)$ for parameter Set 2, negative drift $\mu = -1$.

Double sided exit: bounded variation and positive drift

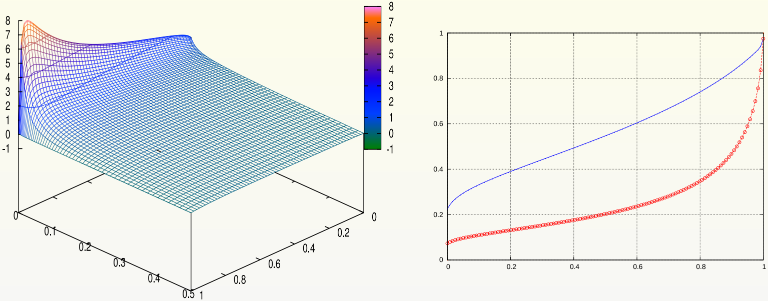

Figure: $\sigma = 0$, positive drift: computing the density of the overshoot $f_1(x, y)$ $(x \in (0, 1),\ y \in (0, 0.5))$, probability of first exit $f_2(x)$ and probability of creeping $f_3(x)$.

Double sided exit: bounded variation and negative drift

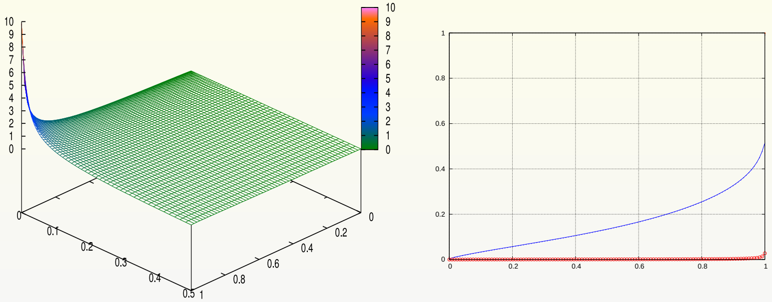

Figure: $\sigma = 0$, negative drift: computing the density of the overshoot $f_1(x, y)$ ($x \in (0, 1)$, $y \in (0, 0.5)$), probability of first exit $f_2(x)$ and probability of creeping $f_3(x)$.

Simulating processes with heavy tails

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- A little thought shows that a huge class of Lévy processes can be written as the independent sum of a β-process plus and independent compound Poisson process. Say,

$$Y_t = X_t + \sum_{i=1}^{N_t} \xi_i$$

where $\{N_t : t \geq 0\}$ is a Poisson process of rate γ and $\{\xi_i : i \geq 1\}$ and i.i.d. sequence.

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- Define iteratively for $n \geq 1$

$$\begin{aligned} V(n,\lambda) &= V(n-1,\lambda) + S^{(n)}_{\lambda+\gamma} + I^{(n)}_{\lambda+\gamma} + \xi_n(1-\beta_n) \\ J(n,\lambda) &= \max\left(V(n,\lambda)\,,J(n-1,\lambda)\,,V(n-1,\lambda) + S^{(n)}_{\lambda+\gamma}\right) \end{aligned}$$

where $V(0,\lambda) = J(0,\lambda) = 0$ and $\{\beta_n : n \geq 1\}$ are an i.i.d. sequence of Bernoulli random variables such that $\mathbb{P}(\beta_n = 1) = \lambda/(\gamma+\lambda)$. Then

$$(Y_{\mathbf{g}(n,\lambda)}, \overline{Y}_{\mathbf{g}(n,\lambda)}) \stackrel{d}{=} (V(T_n,\lambda), J(T_n,\lambda))$$

where $T_n = \min\{j \geq 1 \;:\; \sum_{i=1}^{j} \beta_i = n\}$.

Approximate simulation of the law of $(X_t, \overline{X}_t, \underline{X}_t)$

Define iteratively for $n \geq 1$

$$\begin{aligned} V(n,\lambda) &= V(n-1,\lambda) + S_\lambda^{(n)} + I_\lambda^{(n)} \\ J(n,\lambda) &= \max\left(J(n-1,\lambda), V(n-1,\lambda) + S_\lambda^{(n)}\right) \\ K(n,\lambda) &= \min\left(K(n-1,\lambda), V(n,\lambda)\right) \\ \tilde{J}(n,\lambda) &= \max\left(\tilde{J}(n-1,\lambda), V(n,\lambda)\right) \\ \tilde{K}(n,\lambda) &= \min\left(\tilde{K}(n-1,\lambda), V(n-1,\lambda) + I_\lambda^{(n)}\right), \end{aligned}$$

where $V(0,\lambda) = J(0,\lambda) = K(0,\lambda) = \tilde{J}(0,\lambda) = \tilde{K}(0,\lambda) = 0$. Then for any bounded function $f(x,y,z) : \mathbb{R}^3 \mapsto \mathbb{R}$ which is increasing in z-variable we have

$$\begin{aligned} \mathbb{E}[f(V(n,\lambda), J(n,\lambda), K(n,\lambda))] &\geq \mathbb{E}[f(X_{\mathbf{g}(n,\lambda)}, \overline{X}_{\mathbf{g}(n,\lambda)}, \underline{X}_{\mathbf{g}(n,\lambda)})] \\ \mathbb{E}[f(V(n,\lambda), \tilde{K}(n,\lambda), \tilde{J}(n,\lambda))] &\leq \mathbb{E}[f(X_{\mathbf{g}(n,\lambda)}, \underline{X}_{\mathbf{g}(n,\lambda)}, \overline{X}_{\mathbf{g}(n,\lambda)})]. \end{aligned}$$