

# LÉVY PROCESSES, THE WIENER-HOPF FACTORISATION AND APPLICATIONS: PART I

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## 1 Lévy Processes and Infinite Divisibility

Let us begin by recalling the definition of two familiar processes, a Brownian motion and a Poisson process.

A real-valued process  $B = \{B_t : t \geq 0\}$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is said to be a Brownian motion if the following hold:

- (i) The paths of  $B$  are  $\mathbb{P}$ -almost surely continuous.
- (ii)  $\mathbb{P}(B_0 = 0) = 1$ .
- (iii) For  $0 \leq s \leq t$ ,  $B_t - B_s$  is equal in distribution to  $B_{t-s}$ .
- (iv) For  $0 \leq s \leq t$ ,  $B_t - B_s$  is independent of  $\{B_u : u \leq s\}$ .
- (v) For each  $t > 0$ ,  $B_t$  is equal in distribution to a normal random variable with variance  $t$ .

A process valued on the non-negative integers  $N = \{N_t : t \geq 0\}$ , defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , is said to be a Poisson process with intensity  $\lambda > 0$  if the following hold:

- (i) The paths of  $N$  are  $\mathbb{P}$ -almost surely right continuous with left limits.
- (ii)  $\mathbb{P}(N_0 = 0) = 1$ .
- (iii) For  $0 \leq s \leq t$ ,  $N_t - N_s$  is equal in distribution to  $N_{t-s}$ .
- (iv) For  $0 \leq s \leq t$ ,  $N_t - N_s$  is independent of  $\{N_u : u \leq s\}$ .
- (v) For each  $t > 0$ ,  $N_t$  is equal in distribution to a Poisson random variable with parameter  $\lambda t$ .

On first encounter, these processes would seem to be considerably different from one another. Firstly, Brownian motion has continuous paths whereas a Poisson process does not. Secondly, a Poisson process is a non-decreasing process

and thus has paths of bounded variation<sup>1</sup> over finite time horizons, whereas a Brownian motion does not have monotone paths and in fact its paths are of unbounded variation over finite time horizons.

However, when we line up the definition of a Poisson process and a Brownian motion next to one another, we see that they also have a lot in common. Both processes have right continuous paths with left limits, are initiated from the origin and both have stationary and independent increments; that is properties (i), (ii), (iii) and (iv). We may use these common properties to define a general class of one-dimensional stochastic processes, which are called Lévy processes.

**Definition 1.1 (Lévy Process)** *A process  $X = \{X_t : t \geq 0\}$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is said to be a Lévy process if it possesses the following properties:*

- (i) *The paths of  $X$  are  $\mathbb{P}$ -almost surely right continuous with left limits.*
- (ii)  $\mathbb{P}(X_0 = 0) = 1$ .
- (iii) *For  $0 \leq s \leq t$ ,  $X_t - X_s$  is equal in distribution to  $X_{t-s}$ .*
- (iv) *For  $0 \leq s \leq t$ ,  $X_t - X_s$  is independent of  $\{X_u : u \leq s\}$ .*

*Unless otherwise stated, from now on, when talking of a Lévy process, we shall always use the measure  $\mathbb{P}$  (with associated expectation operator  $\mathbb{E}$ ) to be implicitly understood as its law.*

**Remark 1.1** Note that the properties of stationary and independent increments implies that a Lévy process is a Markov process. Thanks to almost sure right continuity of paths, one may show in addition that Lévy processes are also Strong Markov processes. The proof of this is somewhat technical and time consuming. We do not address it in this course.

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<sup>1</sup>For the reader who is not familiar with the notion of *path variation* for a stochastic process here is a brief reminder. First consider any function  $f : [0, \infty) \rightarrow \mathbb{R}$ . Given any partition  $\mathcal{P} = \{a = t_0 < t_1 < \dots < t_n = b\}$  of the bounded interval  $[a, b]$  we define the variation of  $f$  over  $[a, b]$  with partition  $\mathcal{P}$  by

$$V_{\mathcal{P}}(f, [a, b]) = \sum_{i=1}^n |f(t_i) - f(t_{i-1})|.$$

The function  $f$  is said to be of bounded variation over  $[a, b]$  if

$$V(f, [a, b]) := \sup_{\mathcal{P}} V_{\mathcal{P}}(f, [a, b]) < \infty$$

where the supremum is taken over all partitions of  $[a, b]$ . Moreover,  $f$  is said to be of *bounded variation* if the above inequality is valid for all bounded intervals  $[a, b]$ . If  $V(f, [a, b]) = \infty$  for all bounded intervals  $[a, b]$  then we say that  $f$  is of *unbounded variation*. Note moreover that if  $f$  is a right continuous function with left limits (and therefore has at most a countable number of discontinuities) then, defining  $f^c(x) = f(x) - \sum_{y \leq x} \Delta f(y)$  it can be shown that  $V(f, [a, b]) = V(f^c, [a, b]) + \sum_{x \in (a, b]} |\Delta f(x)|$  where  $\Delta f(x) = f(x) - f(x-)$ . For any given stochastic process  $X = \{X_t : t \geq 0\}$  we may adopt these notions in the almost sure sense. So for example, the statement  $X$  is a process of bounded variation (or has paths of bounded variation) simply means that as a random mapping,  $X : [0, \infty) \rightarrow \mathbb{R}$  is of bounded variation almost surely.

The term “Lévy process” honours the work of the French mathematician Paul Lévy who, although not alone in his contribution, played an instrumental role in bringing together an understanding and characterisation of processes with stationary independent increments. In earlier literature, Lévy processes can be found under a number of different names. In the 1940s, Lévy himself referred to them as a sub-class of *processus additifs* (additive processes), that is processes with independent increments. For the most part however, research literature through the 1960s and 1970s refers to Lévy processes simply as *processes with stationary independent increments*. One sees a change in language through the 1980s and by the 1990s the use of the term “Lévy process” had become standard.

From Definition 1.1 alone it is difficult to see just how rich a class of processes the class of Lévy processes forms. De Finetti [4] introduced the notion of an *infinitely divisible* distribution and showed that they have an intimate relationship with Lévy processes. This relationship gives a reasonably good impression of how varied the class of Lévy processes really is. To this end, let us now devote a little time to discussing infinitely divisible distributions.

**Definition 1.2** *We say that a real-valued random variable  $\Theta$  has an infinitely divisible distribution if for each  $n = 1, 2, \dots$  there exist a sequence of i.i.d. random variables  $\Theta_{1,n}, \dots, \Theta_{n,n}$  such that*

$$\Theta \stackrel{d}{=} \Theta_{1,n} + \dots + \Theta_{n,n}$$

where  $\stackrel{d}{=}$  is equality in distribution. Alternatively, we could have expressed this relation in terms of probability laws. That is to say, the law  $\mu$  of a real-valued random variable is infinitely divisible if for each  $n = 1, 2, \dots$  there exists another law  $\mu_n$  of a real valued random variable such that  $\mu = \mu_n^{*n}$ . (Here  $\mu_n^*$  denotes the  $n$ -fold convolution of  $\mu_n$ ).

In view of the above definition, one way to establish whether a given random variable has an infinitely divisible distribution is via its characteristic exponent. Suppose that  $\Theta$  has characteristic exponent  $\Psi(u) := -\log \mathbb{E}(e^{iu\Theta})$  for all  $u \in \mathbb{R}$ . Then  $\Theta$  has an infinitely divisible distribution if for all  $n \geq 1$  there exists a characteristic exponent of a probability distribution, say  $\Psi_n$ , such that  $\Psi(u) = n\Psi_n(u)$  for all  $u \in \mathbb{R}$ .

The full extent to which we may characterise infinitely divisible distributions is described by the characteristic exponent  $\Psi$  and an expression known as the Lévy–Khinchine formula.<sup>2</sup>

**Theorem 1.1 (Lévy–Khinchine formula)** *A probability law  $\mu$  of a real-valued random variable is infinitely divisible with characteristic exponent  $\Psi$ ,*

$$\int_{\mathbb{R}} e^{i\theta x} \mu(dx) = e^{-\Psi(\theta)} \text{ for } \theta \in \mathbb{R},$$

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<sup>2</sup>Note, although it is a trivial fact, it is always worth reminding oneself of that one works in general with the Fourier transform of a measure as opposed to the Laplace transform on account of the fact that the latter may not exist.

if and only if there exists a triple  $(a, \sigma, \Pi)$ , where  $a \in \mathbb{R}$ ,  $\sigma \geq 0$  and  $\Pi$  is a measure concentrated on  $\mathbb{R} \setminus \{0\}$  satisfying  $\int_{\mathbb{R}} (1 \wedge x^2) \Pi(dx) < \infty$ , such that

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbf{1}_{(|x|<1)}) \Pi(dx)$$

for every  $\theta \in \mathbb{R}$ .

The measure  $\Pi$  is called the Lévy (characteristic) measure and it is important to note that it is unique. The proof of the Lévy–Khintchine characterisation of infinitely divisible random variables is quite lengthy and we exclude it. The interested reader is referred to [20] or [23], to name but two of many possible references. Let us now discuss in further detail the relationship between infinitely divisible distributions and processes with stationary independent increments.

From the definition of a Lévy process we see that for any  $t > 0$ ,  $X_t$  is a random variable belonging to the class of infinitely divisible distributions. This follows from the fact that for any  $n = 1, 2, \dots$ ,

$$X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + \dots + (X_t - X_{(n-1)t/n}) \quad (1.1)$$

together with the fact that  $X$  has stationary independent increments. Suppose now that we define for all  $\theta \in \mathbb{R}$ ,  $t \geq 0$ ,

$$\Psi_t(\theta) = -\log \mathbb{E}(e^{i\theta X_t})$$

then using (1.1) twice we have for any two positive integers  $m, n$  that

$$m\Psi_1(\theta) = \Psi_m(\theta) = n\Psi_{m/n}(\theta)$$

and hence for any rational  $t > 0$ ,

$$\Psi_t(\theta) = t\Psi_1(\theta). \quad (1.2)$$

If  $t$  is an irrational number, then we can choose a decreasing sequence of rationals  $\{t_n : n \geq 1\}$  such that  $t_n \downarrow t$  as  $n$  tends to infinity. Almost sure right continuity of  $X$  implies right continuity of  $\exp\{-\Psi_t(\theta)\}$  (by dominated convergence) and hence (1.2) holds for all  $t \geq 0$ .

In conclusion, any Lévy process has the property that for all  $t \geq 0$

$$\mathbb{E}(e^{i\theta X_t}) = e^{-t\Psi(\theta)},$$

where  $\Psi(\theta) := \Psi_1(\theta)$  is the characteristic exponent of  $X_1$ , which has an infinitely divisible distribution.

**Definition 1.3** *In the sequel we shall also refer to  $\Psi(\theta)$  as the characteristic exponent of the Lévy process.*

It is now clear that each Lévy process can be associated with an infinitely divisible distribution. What is not clear is whether given an infinitely divisible distribution, one may construct a Lévy process  $X$ , such that  $X_1$  has that distribution. This latter issue is affirmed by the following theorem which gives the Lévy–Khintchine formula for Lévy processes.

**Theorem 1.2 (Lévy–Khintchine formula for Lévy processes)** *Suppose that  $a \in \mathbb{R}$ ,  $\sigma \geq 0$  and  $\Pi$  is a measure concentrated on  $\mathbb{R} \setminus \{0\}$  such that  $\int_{\mathbb{R}} (1 \wedge x^2) \Pi(dx) < \infty$ . From this triple define for each  $\theta \in \mathbb{R}$ ,*

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbf{1}_{(|x| < 1)}) \Pi(dx).$$

*Then there exists a probability space on which a Lévy process is defined having characteristic exponent  $\Psi$ .*

The proof of this theorem is rather complicated but very rewarding as it also reveals much more about the general structure of Lévy processes. In Section 3 we will prove a stronger version of this theorem, which also explains the path structure of the Lévy process in terms of the triple  $(a, \sigma, \Pi)$ .

**Remark 1.2** What does the integral test on  $\Pi$  really say? Analytically speaking it says two things. Firstly that  $\Pi(|x| \geq 1) < \infty$  and secondly that

$$\int_{(-1,1)} x^2 \Pi(dx) < \infty. \tag{1.3}$$

Note that these two conditions are sufficient to ensure that the integral in the Lévy-Khintchine formula converges since the integrand is  $O(1)$  for  $|x| \geq 1$  and  $O(x^2)$  for  $|x| < 1$ .

In principle (1.3) means that one could have  $\Pi(-1, 1) < \infty$  or indeed  $\Pi(-1, 1) = \infty$ . If it is the case that  $\Pi(-1, 1) = \infty$  then (1.3) necessarily implies that  $\Pi(|x| \in (\varepsilon, 1)) < \infty$  but  $\Pi(-\varepsilon, \varepsilon) = \infty$  for all  $0 < \varepsilon < 1$ . As we shall eventually see, the measure  $\Pi$  describes the sizes and rate with which jumps of the Lévy process occurs. Naively we can say that in a small period of time  $dt$  a jump of size  $x$  will occur with probability  $\Pi(dx)dt + o(dt)$ . If it were the case that  $\Pi(-1, 1) = \infty$  then the latter interpretation would suggest that the smaller the jump size the greater the intensity and so the discontinuities in the path of the Lévy process is predominantly made up of arbitrarily small jumps. The Lévy-Itô decomposition discussed later will make this rather informal point of view more rigorous.

## 2 Some Examples of Lévy Processes

To conclude our introduction to Lévy processes and infinite divisible distributions, let us proceed to some concrete examples. Some of these will also be of use later to verify certain results from the forthcoming fluctuation theory we will present.

### 2.1 Poisson Processes

For each  $\lambda > 0$  consider a probability distribution  $\mu_\lambda$  which is concentrated on  $k = 0, 1, 2, \dots$  such that  $\mu_\lambda(\{k\}) = e^{-\lambda} \lambda^k / k!$ . That is to say the Poisson

distribution. An easy calculation reveals that

$$\begin{aligned} \sum_{k \geq 0} e^{i\theta k} \mu_\lambda(\{k\}) &= e^{-\lambda(1-e^{i\theta})} \\ &= \left[ e^{-\frac{\lambda}{n}(1-e^{i\theta})} \right]^n. \end{aligned}$$

The right-hand side is the characteristic function of the sum of  $n$  independent Poisson processes, each of which with parameter  $\lambda/n$ . In the Lévy–Khintchine decomposition we see that  $a = \sigma = 0$  and  $\Pi = \lambda\delta_1$ , the Dirac measure supported on  $\{1\}$ .

Recall that a Poisson process,  $\{N_t : t \geq 0\}$ , is a Lévy process with distribution at time  $t > 0$ , which is Poisson with parameter  $\lambda t$ . From the above calculations we have

$$\mathbb{E}(e^{i\theta N_t}) = e^{-\lambda t(1-e^{i\theta})}$$

and hence its characteristic exponent is given by  $\Psi(\theta) = \lambda(1 - e^{i\theta})$  for  $\theta \in \mathbb{R}$ .

## 2.2 Compound Poisson Processes

Suppose now that  $N$  is a Poisson random variable with parameter  $\lambda > 0$  and that  $\{\xi_i : i \geq 1\}$  is an sequence of i.i.d. random variables (independent of  $N$ ) with common law  $F$  having no atom at zero. By first conditioning on  $N$ , we have for  $\theta \in \mathbb{R}$ ,

$$\begin{aligned} \mathbb{E}(e^{i\theta \sum_{i=1}^N \xi_i}) &= \sum_{n \geq 0} \mathbb{E}(e^{i\theta \sum_{i=1}^n \xi_i}) e^{-\lambda} \frac{\lambda^n}{n!} \\ &= \sum_{n \geq 0} \left( \int_{\mathbb{R}} e^{i\theta x} F(dx) \right)^n e^{-\lambda} \frac{\lambda^n}{n!} \\ &= e^{-\lambda \int_{\mathbb{R}} (1-e^{i\theta x}) F(dx)}. \end{aligned} \tag{2.1}$$

Note we use the convention here and later that for any  $n = 0, 1, 2, \dots$

$$\sum_{n+1}^n = 0.$$

We see from (2.1) that distributions of the form  $\sum_{i=1}^N \xi_i$  are infinitely divisible with triple  $a = -\lambda \int_{0 < |x| < 1} x F(dx)$ ,  $\sigma = 0$  and  $\Pi(dx) = \lambda F(dx)$ . When  $F$  has an atom of unit mass at 1 then we have simply a Poisson distribution.

Suppose now that  $N = \{N_t : t \geq 0\}$  is a Poisson process with intensity  $\lambda$  and consider a compound Poisson process  $\{X_t : t \geq 0\}$  defined by

$$X_t = \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0.$$

Using the fact that  $N$  has stationary independent increments together with the mutual independence of the random variables  $\{\xi_i : i \geq 1\}$ , for  $0 \leq s < t < \infty$ , by writing

$$X_t = X_s + \sum_{i=N_s+1}^{N_t} \xi_i$$

it is clear that  $X_t$  is the sum of  $X_s$  and an independent copy of  $X_{t-s}$ . Right continuity and left limits of the process  $N$  also ensure right continuity and left limits of  $X$ . Thus compound Poisson processes are Lévy processes. From the calculations in the previous paragraph, for each  $t \geq 0$  we may substitute  $N_t$  for the variable  $N$  to discover that the Lévy–Khintchine formula for a compound Poisson process takes the form  $\Psi(\theta) = \lambda \int_{\mathbb{R}} (1 - e^{i\theta x}) F(dx)$ . Note in particular that the Lévy measure of a compound Poisson process is always finite with total mass equal to the rate  $\lambda$  of the underlying process  $N$ .

If a drift of rate  $c \in \mathbb{R}$  is added to a compound Poisson process so that now

$$X_t = \sum_{i=1}^{N_t} \xi_i + ct, \quad t \geq 0$$

then it is straightforward to see that the resulting process is again a Lévy process. The associated infinitely divisible distribution is nothing more than a shifted compound Poisson distribution with shift  $c$ . The Lévy–Khintchine exponent is given by

$$\Psi(\theta) = \lambda \int_{\mathbb{R}} (1 - e^{i\theta x}) F(dx) - ic\theta.$$

If further the shift is chosen to centre the compound Poisson distribution then  $c = \lambda \int_{\mathbb{R}} x F(dx)$  and then

$$\Psi(\theta) = \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x) \lambda F(dx).$$

### 2.3 Linear Brownian Motion

Take the probability law

$$\mu_{s,\gamma}(dx) := \frac{1}{\sqrt{2\pi s^2}} e^{-(x-\gamma)^2/2s^2} dx$$

supported on  $\mathbb{R}$  where  $\gamma \in \mathbb{R}$  and  $s > 0$ ; the well-known Gaussian distribution with mean  $\gamma$  and variance  $s^2$ . It is well known that

$$\begin{aligned} \int_{\mathbb{R}} e^{i\theta x} \mu_{s,\gamma}(dx) &= e^{-\frac{1}{2}s^2\theta^2 + i\theta\gamma} \\ &= \left[ e^{-\frac{1}{2}\left(\frac{s}{\sqrt{n}}\right)^2\theta^2 + i\theta\frac{\gamma}{n}} \right]^n \end{aligned}$$

showing again that it is an infinitely divisible distribution, this time with  $a = -\gamma$ ,  $\sigma = s$  and  $\Pi = 0$ .

We immediately recognise the characteristic exponent  $\Psi(\theta) = s^2\theta^2/2 - i\theta\gamma$  as also that of a linear Brownian motion,

$$X_t := sB_t + \gamma t, \quad t \geq 0,$$

where  $B = \{B_t : t \geq 0\}$  is a standard Brownian motion. It is a trivial exercise to verify that  $X$  has stationary independent increments with continuous paths as a consequence of the fact that  $B$  does.

## 2.4 Gamma Processes

For  $\alpha, \beta > 0$  define the probability measure

$$\mu_{\alpha, \beta}(\mathrm{d}x) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\alpha x} \mathrm{d}x$$

concentrated on  $(0, \infty)$ ; the gamma- $(\alpha, \beta)$  distribution. Note that when  $\beta = 1$  this is the exponential distribution. We have

$$\begin{aligned} \int_0^\infty e^{i\theta x} \mu_{\alpha, \beta}(\mathrm{d}x) &= \frac{1}{(1 - i\theta/\alpha)^\beta} \\ &= \left[ \frac{1}{(1 - i\theta/\alpha)^{\beta/n}} \right]^n \end{aligned}$$

and infinite divisibility follows. For the Lévy–Khintchine decomposition we have  $\sigma = 0$  and  $\Pi(\mathrm{d}x) = \beta x^{-1} e^{-\alpha x} \mathrm{d}x$ , concentrated on  $(0, \infty)$  and  $a = -\int_0^1 x \Pi(\mathrm{d}x)$ . However this is not immediately obvious. The following lemma proves to be useful in establishing the above triple  $(a, \sigma, \Pi)$ . Its proof is Exercise 3.

**Lemma 2.1 (Frullani integral)** *For all  $\alpha, \beta > 0$  and  $z \in \mathbb{C}$  such that  $\Re z \leq 0$  we have*

$$\frac{1}{(1 - z/\alpha)^\beta} = e^{-\int_0^\infty (1 - e^{-zx}) \beta x^{-1} e^{-\alpha x} \mathrm{d}x}.$$

To see how this lemma helps note that the Lévy–Khintchine formula for a gamma distribution takes the form

$$\Psi(\theta) = \beta \int_0^\infty (1 - e^{i\theta x}) \frac{1}{x} e^{-\alpha x} \mathrm{d}x = \beta \log(1 - i\theta/\alpha)$$

for  $\theta \in \mathbb{R}$ . The choice of  $a$  in the Lévy–Khintchine formula is the necessary quantity to cancel the term coming from  $i\theta \mathbf{1}_{(|x| < 1)}$  in the integral with respect to  $\Pi$  in the general Lévy–Khintchine formula.



According to Theorem 1.2 there exists a Lévy process whose Lévy–Khintchine formula is given by  $\Psi$ , the so-called *gamma process*.

Suppose now that  $X = \{X_t : t \geq 0\}$  is a gamma process. Stationary independent increments tell us that for all  $0 \leq s < t < \infty$ ,  $X_t = X_s + \tilde{X}_{t-s}$  where  $\tilde{X}_{t-s}$  is an independent copy of  $X_{t-s}$ . The fact that the latter is strictly positive with probability one (on account of it being gamma distributed) implies that  $X_t > X_s$  almost surely. Hence a gamma process is an example of a Lévy process with almost surely non-decreasing paths (in fact its paths are strictly increasing). Another example of a Lévy process with non-decreasing paths is a compound Poisson process where the jump distribution  $F$  is concentrated on  $(0, \infty)$ . Note however that a gamma process is not a compound Poisson process on two counts. Firstly, its Lévy measure has infinite total mass unlike the Lévy measure of a compound Poisson process, which is necessarily finite (and equal to the arrival rate of jumps). Secondly, whilst a compound Poisson process with positive jumps does have paths, which are almost surely non-decreasing, it does not have paths that are almost surely strictly increasing.

Lévy processes whose paths are almost surely non-decreasing (or simply non-decreasing for short) are called *subordinators*.

## 2.5 Inverse Gaussian Processes

Suppose as usual that  $B = \{B_t : t \geq 0\}$  is a standard Brownian motion. Define the first passage time

$$\tau_s = \inf\{t > 0 : B_t + bt > s\}, \quad (2.2)$$

that is, the first time a Brownian motion with linear drift  $b > 0$  crosses above level  $s$ . Recall that  $\tau_s$  is a stopping time<sup>3</sup>. Moreover, since Brownian motion has continuous paths we know that  $B_{\tau_s} + b\tau_s = s$  almost surely. From the Strong Markov Property it is known that  $\{B_{\tau_s+t} + b(\tau_s + t) - s : t \geq 0\}$  is equal in law to  $B$  and hence for all  $0 \leq s < t$ ,

$$\tau_t = \tau_s + \tilde{\tau}_{t-s},$$

where  $\tilde{\tau}_{t-s}$  is an independent copy of  $\tau_{t-s}$ . This shows that the process  $\tau := \{\tau_t : t \geq 0\}$  has stationary independent increments. Continuity of the paths of  $\{B_t + bt : t \geq 0\}$  ensures that  $\tau$  has right continuous paths. Further, it is clear that  $\tau$  has almost surely non-decreasing paths, which guarantees its paths have left limits as well as being yet another example of a subordinator. According to its definition as a sequence of first passage times,  $\tau$  is also the almost sure right inverse of the path of the graph of  $\{B_t + bt : t \geq 0\}$  in the sense of (2.2). From this  $\tau$  earns its name as the inverse Gaussian process.

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<sup>3</sup>We assume that the reader is familiar with the notion of a stopping time for a Markov process. By definition, the random time  $\tau$  is a stopping time with respect to the filtration  $\{\mathcal{G}_t : t \geq 0\}$  if for all  $t \geq 0$ ,

$$\{\tau \leq t\} \in \mathcal{G}_t.$$

According to the discussion following Theorem 1.1 it is now immediate that for each fixed  $s > 0$ , the random variable  $\tau_s$  is infinitely divisible. Its characteristic exponent takes the form

$$\Psi(\theta) = s(\sqrt{-2i\theta + b^2} - b)$$

for all  $\theta \in \mathbb{R}$  and corresponds to a triple  $a = -2sb^{-1} \int_0^b (2\pi)^{-1/2} e^{-y^2/2} dy$ ,  $\sigma = 0$  and

$$\Pi(dx) = s \frac{1}{\sqrt{2\pi x^3}} e^{-\frac{b^2 x}{2}} dx$$

concentrated on  $(0, \infty)$ . The law of  $\tau_s$  can also be computed explicitly as

$$\mu_s(dx) = \frac{s}{\sqrt{2\pi x^3}} e^{sb} e^{-\frac{1}{2}(s^2 x^{-1} + b^2 x)}$$

for  $x > 0$ . The proof of these facts forms Exercise 6.

## 2.6 Stable Processes

Stable processes are the class of Lévy processes whose characteristic exponents correspond to those of stable distributions. Stable distributions were introduced by [17, 18] as a third example of infinitely divisible distributions after Gaussian and Poisson distributions. A random variable,  $Y$ , is said to have a stable distribution if for all  $n \geq 1$  it observes the distributional equality

$$Y_1 + \cdots + Y_n \stackrel{d}{=} a_n Y + b_n, \quad (2.3)$$

where  $Y_1, \dots, Y_n$  are independent copies of  $Y$ ,  $a_n > 0$  and  $b_n \in \mathbb{R}$ . By subtracting  $b_n/n$  from each of the terms on the left-hand side of (2.3) one sees in particular that this definition implies that any stable random variable is infinitely divisible. It turns out that necessarily  $a_n = n^{1/\alpha}$  for  $\alpha \in (0, 2]$ ; see [3], Sect. VI.1. In that case we refer to the parameter  $\alpha$  as the *index*. A smaller class of distributions are the *strictly stable* distributions. A random variable  $Y$  is said to have a strictly stable distribution if it observes (2.3) but with  $b_n = 0$ . In that case, we necessarily have

$$Y_1 + \cdots + Y_n \stackrel{d}{=} n^{1/\alpha} Y. \quad (2.4)$$

The case  $\alpha = 2$  corresponds to zero mean Gaussian random variables and is excluded in the remainder of the discussion as it has essentially been dealt with in Sect. 2.3.

Stable random variables observing the relation (2.3) for  $\alpha \in (0, 1) \cup (1, 2)$  have characteristic exponents of the form

$$\Psi(\theta) = c|\theta|^\alpha \left(1 - i\beta \tan \frac{\pi\alpha}{2} \operatorname{sgn} \theta\right) + i\theta\eta, \quad (2.5)$$

where  $\beta \in [-1, 1]$ ,  $\eta \in \mathbb{R}$  and  $c > 0$ . Stable random variables observing the relation (2.3) for  $\alpha = 1$ , have characteristic exponents of the form

$$\Psi(\theta) = c|\theta| \left(1 + i\beta \frac{2}{\pi} \operatorname{sgn} \theta \log |\theta|\right) + i\theta\eta, \quad (2.6)$$

where  $\beta \in [-1, 1]$ ,  $\eta \in \mathbb{R}$  and  $c > 0$ . Here we work with the definition of the sign function  $\text{sgn } \theta = \mathbf{1}_{(\theta > 0)} - \mathbf{1}_{(\theta < 0)}$ . To make the connection with the Lévy–Khintchine formula, one needs  $\sigma = 0$  and

$$\Pi(dx) = \begin{cases} c_1 x^{-1-\alpha} dx & \text{for } x \in (0, \infty) \\ c_2 |x|^{-1-\alpha} dx & \text{for } x \in (-\infty, 0), \end{cases} \quad (2.7)$$

where  $c = c_1 + c_2$ ,  $c_1, c_2 \geq 0$  and  $\beta = (c_1 - c_2)/(c_1 + c_2)$  if  $\alpha \in (0, 1) \cup (1, 2)$  and  $c_1 = c_2$  if  $\alpha = 1$ . The choice of  $a \in \mathbb{R}$  in the Lévy–Khintchine formula is then implicit. Exercise 4 shows how to make the connection between  $\Pi$  and  $\Psi$  with the right choice of  $a$  (which depends on  $\alpha$ ). Unlike the previous examples, the distributions that lie behind these characteristic exponents are heavy tailed in the sense that the tails of their distributions decay slowly enough to zero so that they only have moments strictly less than  $\alpha$ . The value of the parameter  $\beta$  gives a measure of asymmetry in the Lévy measure and likewise for the distributional asymmetry (although this latter fact is not immediately obvious). The densities of stable processes are known explicitly in the form of convergent power series. See [31], [23] and [22] for further details of all the facts given in this paragraph. With the exception of the defining property (2.4) we shall generally not need detailed information on distributional properties of stable processes in order to proceed with their fluctuation theory. This explains the reluctance to give further details here.

Two examples of the aforementioned power series that tidy up to more compact expressions are centred Cauchy distributions, corresponding to  $\alpha = 1$ ,  $\beta = 0$  and  $\eta = 0$ , and stable- $\frac{1}{2}$  distributions, corresponding to  $\alpha = 1/2$ ,  $\beta = 1$  and  $\eta = 0$ . In the former case,  $\Psi(\theta) = c|\theta|$  for  $\theta \in \mathbb{R}$  and its law is given by

$$\frac{c}{\pi} \frac{1}{(x^2 + c^2)} dx \quad (2.8)$$

for  $x \in \mathbb{R}$ . In the latter case,  $\Psi(\theta) = c|\theta|^{1/2}(1 - \text{isgn } \theta)$  for  $\theta \in \mathbb{R}$  and its law is given by

$$\frac{c}{\sqrt{2\pi}x^3} e^{-c^2/2x} dx.$$

Note then that an inverse Gaussian distribution coincides with a stable- $\frac{1}{2}$  distribution for  $a = c$  and  $b = 0$ .

Suppose that  $\mathcal{S}(c, \alpha, \beta, \eta)$  is the distribution of a stable random variable with parameters  $c$ ,  $\alpha$ ,  $\beta$  and  $\eta$ . For each choice of  $c > 0$ ,  $\alpha \in (0, 2)$ ,  $\beta \in [-1, 1]$  and  $\eta \in \mathbb{R}$  Theorem 1.2 tells us that there exists a Lévy process, with characteristic exponent given by (2.5) or (2.6) according to this choice of parameters. Further, from the definition of its characteristic exponent it is clear that at each fixed time the  $\alpha$ -stable process will have distribution  $\mathcal{S}(ct, \alpha, \beta, \eta)$ .

## 2.7 Other Examples

There are many more known examples of infinitely divisible distributions (and hence Lévy processes). Of the many known proofs of infinite divisibility for

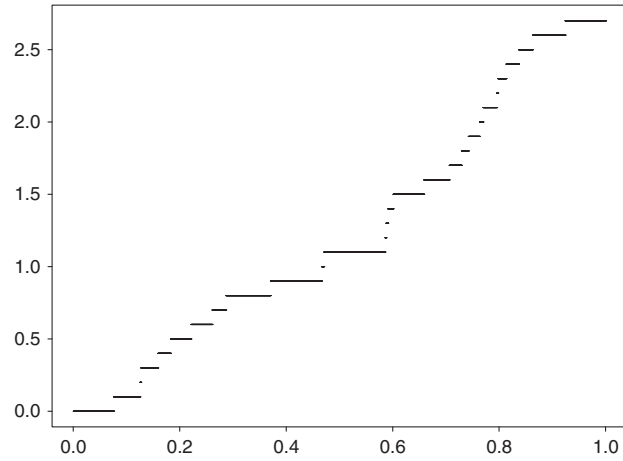


Figure 1: A sample path of a Poisson process;  $\Psi(\theta) = \lambda(1 - e^{i\theta})$  where  $\lambda$  is the jump rate.

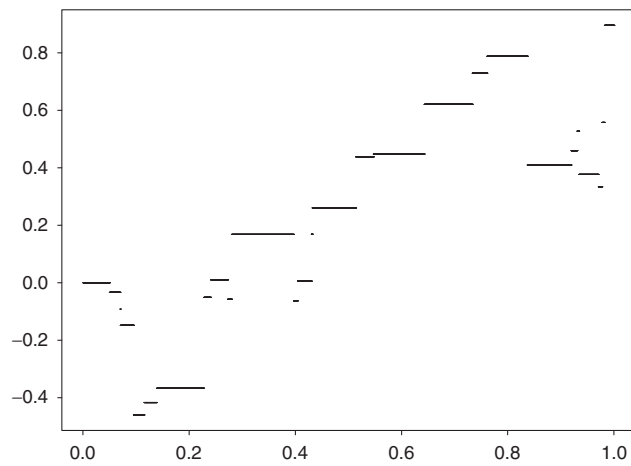


Figure 2: A sample path of a compound Poisson process;  $\Psi(\theta) = \lambda \int_{\mathbb{R}} (1 - e^{i\theta x}) F(dx)$  where  $\lambda$  is the jump rate and  $F$  is the common distribution of the jumps.

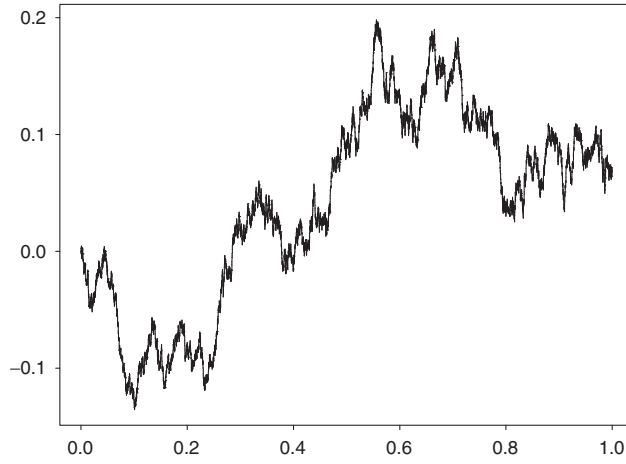


Figure 3: A sample path of a Brownian motion;  $\Psi(\theta) = \theta^2/2$ .

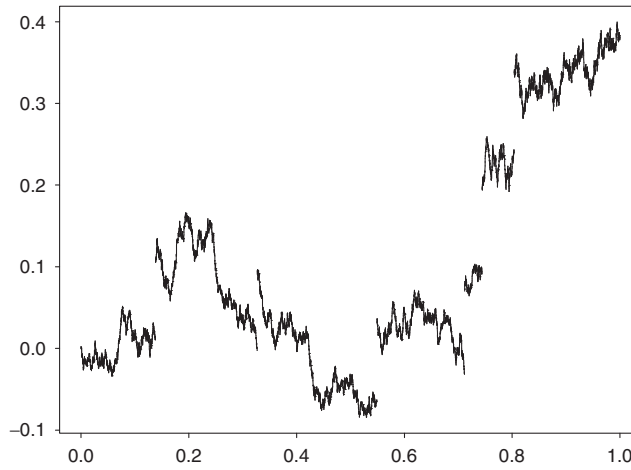


Figure 4: A sample path of the independent sum of a Brownian motion and a compound Poisson process;  $\Psi(\theta) = \theta^2/2 + \int_{\mathbb{R}} (1 - e^{i\theta x}) F(dx)$ .

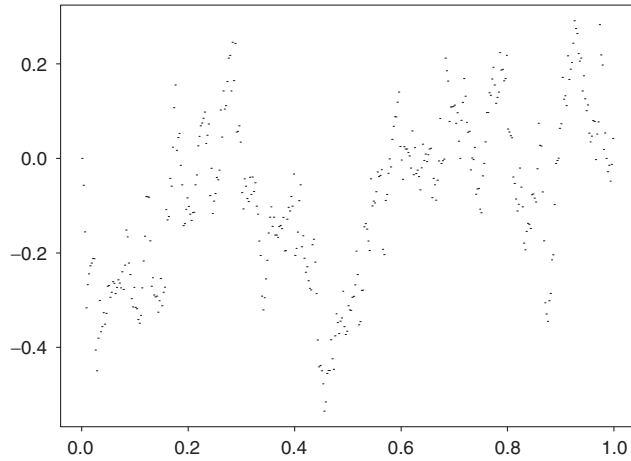


Figure 5: A sample path of a variance gamma processes. The latter has characteristic exponent given by  $\Psi(\theta) = \beta \log(1 - i\theta c/\alpha + \beta^2 \theta^2/2\alpha)$  where  $c \in \mathbb{R}$  and  $\beta > 0$ .

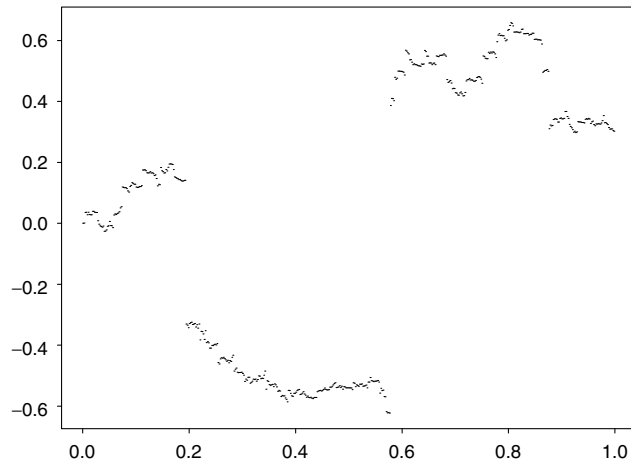


Figure 6: A sample path of a normal inverse Gaussian process;  $\Psi(\theta) = \delta(\sqrt{\alpha^2 - (\beta + i\theta)^2} - \sqrt{\alpha^2 - \beta^2})$  where  $\alpha, \delta > 0$ ,  $|\beta| < \alpha$ .

specific distributions, most of them are non-trivial, often requiring intimate knowledge of special functions. A brief list of such distributions might include generalised inverse Gaussian (see [5] and [14]), truncated stable (see [30], [8], [15], [1] and [2]), generalised hyperbolic (see [7]), Meixner (see [24]), Pareto (see [25] and [28]),  $F$ -distributions (see [10]), Gumbel (see [13] and [26]), Weibull (see [13] and [25]), lognormal (see [29]) and Student  $t$ -distribution (see [6] and [9]). See also the book of Steutel [27]

Despite being able to identify a large number of infinitely divisible distributions and hence associated Lévy processes, it is not clear at this point what the paths of Lévy processes look like. The task of giving a mathematically precise account of this lies ahead in the next section. In the meantime let us make the following informal remarks concerning paths of Lévy processes.

Exercise 1 shows that a linear combination of a finite number of independent Lévy processes is again a Lévy process. It turns out that one may consider any Lévy process as an independent sum of a Brownian motion with drift and a countable number of independent compound Poisson processes with different jump rates, jump distributions and drifts. The superposition occurs in such a way that the resulting path remains almost surely finite at all times and, for each  $\varepsilon > 0$ , the process experiences at most a countably infinite number of jumps of magnitude  $\varepsilon$  or less with probability one and an almost surely finite number of jumps of magnitude greater than  $\varepsilon$ , over all fixed finite time intervals. If in the latter description there is always an almost surely finite number of jumps over each fixed time interval then it is necessary and sufficient that one has the linear independent combination of a Brownian motion with drift and a compound Poisson process. Depending on the underlying structure of the jumps and the presence of a Brownian motion in the described linear combination, a Lévy process will either have paths of bounded variation on all finite time intervals or paths of unbounded variation on all finite time intervals.

We include six computer simulations to give a rough sense of how the paths of Lévy processes look. Figs. 1 and 2 depict the paths of Poisson process and a compound Poisson process, respectively. Figs. 3 and 4 show the paths of a Brownian motion and the independent sum of a Brownian motion and a compound Poisson process, respectively. Finally Figs. 5 and 6 show the paths of a variance gamma process and a normal inverse Gaussian processes. Both are pure jump processes (no Brownian component as described above). Variance gamma processes are discussed in more detail later normal inverse Gaussian processes are Lévy processes whose jump measure is given by  $\Pi(dx) = (\delta\alpha/\pi|x|) \exp\{\beta x\}K_1(\alpha|x|)dx$  for  $x \in \mathbb{R}$  where  $\alpha, \delta > 0$ ,  $\beta \leq |\alpha|$  and  $K_1(x)$  is the modified Bessel function of the third kind with index 1 (the precise definition of the latter is not worth the detail at this moment in the text). Both experience an infinite number of jumps over a finite time horizon. However, variance gamma processes have paths of bounded variation whereas normal inverse Gaussian processes have paths of unbounded variation. The reader should be warned however that computer simulations ultimately can only depict a finite number of jumps in any given path. All figures were very kindly produced by Antonis Papapantoleon.

### 3 The Lévy-Itô decomposition intuitively

One of our main objectives for the remainder of this text is to establish a rigorous understanding of the structure of the paths of Lévy processes. The way we shall do this is to prove the assertion in Theorem 1.2 that given any characteristic exponent,  $\Psi$ , belonging to an infinitely divisible distribution, there exists a Lévy process with the same characteristic exponent. This will be done by establishing the so-called Lévy-Itô decomposition which describes the structure of a general Lévy process in terms of three independent auxiliary Lévy processes, each with different types of path behaviour.

According to Theorem 1.1, any characteristic exponent  $\Psi$  belonging to an infinitely divisible distribution can be written, after some simple reorganisation, in the form

$$\begin{aligned} \Psi(\theta) = & \left\{ ia\theta + \frac{1}{2}\sigma^2\theta^2 \right\} \\ & + \left\{ \Pi(\mathbb{R} \setminus (-1, 1)) \int_{|x| \geq 1} (1 - e^{i\theta x}) \frac{\Pi(dx)}{\Pi(\mathbb{R} \setminus (-1, 1))} \right\} \\ & + \left\{ \int_{0 < |x| < 1} (1 - e^{i\theta x} + i\theta x) \Pi(dx) \right\} \end{aligned} \quad (3.1)$$

for all  $\theta \in \mathbb{R}$  where  $a \in \mathbb{R}$ ,  $\sigma \geq 0$  and  $\Pi$  is a measure on  $\mathbb{R} \setminus \{0\}$  satisfying  $\int_{\mathbb{R}} (1 \wedge x^2) \Pi(dx) < \infty$ . Note that the latter condition on  $\Pi$  implies that  $\Pi(A) < \infty$  for all Borel  $A$  such that 0 is in the interior of  $A^c$  and particular that  $\Pi(\mathbb{R} \setminus (-1, 1)) \in [0, \infty)$ . In the case that  $\Pi(\mathbb{R} \setminus (-1, 1)) = 0$  one should think of the second bracket in (3.1) as absent. Call the three brackets in (3.1)  $\Psi^{(1)}$ ,  $\Psi^{(2)}$  and  $\Psi^{(3)}$ . The essence of the Lévy-Itô decomposition revolves around showing that  $\Psi^{(1)}$ ,  $\Psi^{(2)}$  and  $\Psi^{(3)}$  all correspond to the characteristic exponents of three different types of Lévy processes. Therefore  $\Psi$  may be considered as the characteristic exponent of the independent sum of these three Lévy processes which is again a Lévy process (cf. Exercise 1). Indeed, as we have already seen in the previous section,  $\Psi^{(1)}$  and  $\Psi^{(2)}$  correspond, respectively, to a linear Brownian motion with drift,  $X^{(1)} = \{X_t^{(1)} : t \geq 0\}$  where

$$X_t^{(1)} = \sigma B_t - at, \quad t \geq 0 \quad (3.2)$$

and a compound Poisson process, say  $X^{(2)} = \{X_t^{(2)} : t \geq 0\}$ , where,

$$X_t^{(2)} = \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0, \quad (3.3)$$

$\{N_t : t \geq 0\}$  is a Poisson process with rate  $\Pi(\mathbb{R} \setminus (-1, 1))$  and  $\{\xi_i : i \geq 1\}$  are independent and identically distributed with distribution  $\Pi(dx)/\Pi(\mathbb{R} \setminus (-1, 1))$  concentrated on  $\{x : |x| \geq 1\}$  (unless  $\Pi(\mathbb{R} \setminus (-1, 1)) = 0$  in which case  $X^{(2)}$  is the process which is identically zero).



The proof of existence of a Lévy process with characteristic exponent given by (3.1) thus boils down to showing the existence of a Lévy process,  $X^{(3)}$ , whose characteristic exponent is given by  $\Psi^{(3)}$ . Noting that

$$\begin{aligned} & \int_{0 < |x| < 1} (1 - e^{i\theta x} + i\theta x) \Pi(dx) \\ &= \sum_{n \geq 0} \left\{ \lambda_n \int_{2^{-(n+1)} \leq |x| < 2^{-n}} (1 - e^{i\theta x}) F_n(dx) \right. \\ & \quad \left. + i\theta \lambda_n \left( \int_{2^{-(n+1)} \leq |x| < 2^{-n}} x F_n(dx) \right) \right\}, \end{aligned} \quad (3.4)$$

where  $\lambda_n = \Pi(\{x : 2^{-(n+1)} \leq |x| < 2^{-n}\})$  and

$$F_n(dx) = \lambda_n^{-1} \Pi(dx)|_{\{x: 2^{-(n+1)} \leq |x| < 2^{-n}\}}$$

(again with the understanding that the  $n$ th integral is absent if  $\lambda_n = 0$ ). It would appear from (3.4) that the process  $X^{(3)}$  consists of the superposition of (at most) a countable number of independent compound Poisson processes with different arrival rates and additional linear drift. To understand the mathematical sense of this superposition we shall look at related square integrable martingales. This is done in subsequent sections, but let us now conclude this discussion with a statement of our objectives in the form of a theorem.

The identification of a Lévy process,  $X$  as the independent sum of processes  $X^{(1)}$ ,  $X^{(2)}$  and  $X^{(3)}$  is attributed to Lévy [19] and Itô [11] (see also [12]) and is thus known as the *Lévy–Itô decomposition*. Formally speaking and in a little more detail we quote the Lévy–Itô decomposition in the form of a theorem.

**Theorem 3.1** (*Lévy–Itô decomposition*) *Given any  $a \in \mathbb{R}$ ,  $\sigma \geq 0$  and measure  $\Pi$  concentrated on  $\mathbb{R} \setminus \{0\}$  satisfying*

$$\int_{\mathbb{R}} (1 \wedge x^2) \Pi(dx) < \infty,$$

*there exists a probability space on which three independent Lévy processes exist,  $X^{(1)}$ ,  $X^{(2)}$  and  $X^{(3)}$  where  $X^{(1)}$  is a linear Brownian motion given by (3.2),  $X^{(2)}$  is a compound Poisson process given by (3.3) and  $X^{(3)}$  is a square integrable martingale with characteristic exponent given by  $\Psi^{(3)}$ . By taking  $X = X^{(1)} + X^{(2)} + X^{(3)}$  we see that the conclusion of Theorem 1.2 holds, that there exists a probability space on which a Lévy process is defined with characteristic exponent*

$$\Psi(\theta) = ai\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbf{1}_{(|x| < 1)}) \Pi(dx) \quad (3.5)$$

*for  $\theta \in \mathbb{R}$ .*

## 4 The Lévy-Itô decomposition rigorously

Fix a time horizon  $T > 0$ . Let us assume that  $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, P)$  is a filtered probability space in which the filtration  $\{\mathcal{F}_t : t \geq 0\}$  is complete with respect to the null sets of  $P$  and right continuous in the sense that  $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ .

**Definition 4.1** Fix  $T > 0$ . Define  $\mathcal{M}_T^2 = \mathcal{M}_T^2(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, P)$  to be the space of real valued, zero mean, almost surely right-continuous, square integrable  $P$ -martingales with respect to the given filtration over the finite time period  $[0, T]$ .

One luxury that follows from the assumptions on  $\{\mathcal{F}_t : t \geq 0\}$  is that any zero mean square integrable martingale with respect to this filtration has a right continuous modification<sup>4</sup> which is also a member of  $\mathcal{M}_T^2$ .

If we quotient out the equivalent classes of versions<sup>5</sup> of each martingale, it is straightforward to deduce that  $\mathcal{M}_T^2$  is a vector space over the real numbers with zero element  $M_t = 0$  for all  $t \in [0, T]$  and all  $\omega \in \Omega$ . In fact, as we shall shortly see,  $\mathcal{M}_T^2$  is a Hilbert space<sup>6</sup> with respect to the inner product

$$\langle M, N \rangle = E(M_T N_T),$$

where  $M, N \in \mathcal{M}_T^2$ . It is left to the reader to verify the fact that  $\langle \cdot, \cdot \rangle$  forms an inner product. The only mild technical difficulty in this verification is showing that for  $M \in \mathcal{M}_T^2$ ,  $\langle M, M \rangle = 0$  implies that  $M = 0$ , the zero element. Note however that if  $\langle M, M \rangle = 0$  then by Doob's Maximal Inequality, which says that for  $M \in \mathcal{M}_T^2$ ,

$$E\left(\sup_{0 \leq s \leq T} M_s^2\right) \leq 4E(M_T^2),$$

we have that  $\sup_{0 \leq t \leq T} |M_t| = 0$  almost surely. It follows necessarily that  $M_t = 0$  for all  $t \in [0, T]$  with probability one.

As alluded to above, we can show without too much difficulty that  $\mathcal{M}_T^2$  is a Hilbert space. To do that we are required to show that given  $\{M^{(n)} : n = 1, 2, \dots\}$  is a Cauchy sequence of martingales taken from  $\mathcal{M}_T^2$  then there exists an  $M \in \mathcal{M}_T^2$  such that

$$\|M^{(n)} - M\| \rightarrow 0$$

<sup>4</sup>Recall that  $M' = \{M'_t : t \in [0, T]\}$  is a modification of  $M$  if for every  $t \geq 0$ , we have  $P(M'_t = M_t) = 1$ .

<sup>5</sup>Recall that  $M' = \{M'_t : t \in [0, T]\}$  is a version of  $M$  if it is defined on the same probability space and  $\{\exists t \in [0, T] : M'_t \neq M_t\}$  is measurable with zero probability. Note that if  $M'$  is a modification of  $M$  then it is not necessarily a version of  $M$ . However, it is obviously the case that if  $M'$  is a version of  $M$  then it also fulfils the requirement of being a modification.

<sup>6</sup>Recall that  $\langle \cdot, \cdot \rangle : L \times L \rightarrow \mathbb{R}$  is an inner product on a vector space  $L$  over the reals if it satisfies the following properties for  $f, g \in L$  and  $a, b \in \mathbb{R}$ ; (i)  $\langle af + bg, h \rangle = a \langle f, h \rangle + b \langle g, h \rangle$  for all  $h \in L$ , (ii)  $\langle f, g \rangle = \langle g, f \rangle$ , (iii)  $\langle f, f \rangle \geq 0$  and (iv)  $\langle f, f \rangle = 0$  if and only if  $f = 0$ .

For each  $f \in L$  let  $\|f\| = \langle f, f \rangle^{1/2}$ . The pair  $(L, \langle \cdot, \cdot \rangle)$  are said to form a Hilbert space if all sequences,  $\{f_n : n = 1, 2, \dots\}$  in  $L$  that satisfy  $\|f_n - f_m\| \rightarrow 0$  as  $m, n \rightarrow \infty$ , so called Cauchy sequences, have a limit which exists in  $L$ .

as  $n \uparrow \infty$  where  $\|\cdot\| := \langle \cdot, \cdot \rangle^{1/2}$ . To this end let us assume that the sequence of processes  $\{M^{(n)} : n = 1, 2, \dots\}$  is a Cauchy sequence, in other words,

$$\left\{ \mathbb{E} \left[ \left( M_T^{(m)} - M_T^{(n)} \right)^2 \right] \right\}^{1/2} \rightarrow 0 \text{ as } m, n \uparrow \infty.$$

Necessarily then the sequence of *random variables*  $\{M_T^{(k)} : k \geq 1\}$  is a Cauchy sequence in the Hilbert space of zero mean, square integrable random variables defined on  $(\Omega, \mathcal{F}_T, \mathbb{P})$ , say  $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ , endowed with the inner product  $\langle M, N \rangle = \mathbb{E}(MN)$ . Hence there exists a limiting variable, say  $M_T$  in  $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$  satisfying

$$\left\{ \mathbb{E} \left[ \left( M_T^{(n)} - M_T \right)^2 \right] \right\}^{1/2} \rightarrow 0$$

as  $n \uparrow \infty$ . Define the martingale  $M$  to be the right continuous version<sup>7</sup> of

$$\mathbb{E} (M_T | \mathcal{F}_t^*) \text{ for } t \in [0, T]$$

and note that by definition

$$\|M^{(n)} - M\| \rightarrow 0$$

as  $n$  tends to infinity. Clearly it is an  $\mathcal{F}_t^*$ -adapted process and by Jensen's inequality

$$\begin{aligned} \mathbb{E} (M_t^2) &= \mathbb{E} \left( \mathbb{E} (M_T | \mathcal{F}_t^*)^2 \right) \\ &\leq \mathbb{E} \left( \mathbb{E} (M_T^2 | \mathcal{F}_t^*) \right) \\ &= \mathbb{E} (M_T^2) \end{aligned}$$

which is finite. Hence Cauchy sequences converge in  $\mathcal{M}_T^2$  and we see that  $\mathcal{M}_T^2$  is indeed a Hilbert space.

Having reminded ourselves of some relevant properties of the space of square integrable martingales, let us consider a special class of martingale within the latter class that are also Lévy processes and which are key to the proof of the Lévy-Itô decomposition.

Henceforth, we shall suppose that  $\{\xi_i : i \geq 1\}$  is a sequence of i.i.d. random variables with common law  $F$  (which does not assign mass to the origin) and that  $N = \{N_t : t \geq 0\}$  is a Poisson process with rate  $\lambda > 0$ .

**Lemma 4.1** *Suppose that  $\int_{\mathbb{R}} |x| F(dx) < \infty$ .*

(i) *The process  $M = \{M_t : t \geq 0\}$  where*

$$M_t := \sum_{i=1}^{N_t} \xi_i - \lambda t \int_{\mathbb{R}} x F(dx)$$

*is a martingale with respect to its natural filtration.*

---

<sup>7</sup>Here we use the fact that  $\{\mathcal{F}_t^* : t \in [0, T]\}$  is complete and right continuous.

(ii) If moreover  $\int_{\mathbb{R}} x^2 F(dx) < \infty$  then  $M$  is a square integrable martingale such that

$$\mathbb{E}(M_t^2) = \lambda t \int_{\mathbb{R}} x^2 F(dx).$$

**Proof.** For the first part, note that  $M$  has stationary and independent increments (in fact, as seen earlier, it is a Lévy process). Hence if we define  $\mathcal{F}_t = \sigma(M_s : s \leq t)$  then for  $t \geq s \geq 0$

$$\mathbb{E}(M_t | \mathcal{F}_s) = M_s + \mathbb{E}(M_t - M_s | \mathcal{F}_s) = M_s + \mathbb{E}(M_{t-s}).$$

Hence it suffices to prove that for all  $u \geq 0$ ,  $\mathbb{E}(M_u) = 0$ . However the latter is evident since, by a classical computation (left to the reader) which utilizes the independence of  $N$  and  $\{\xi_i : i \geq 0\}$ ,

$$\mathbb{E} \left( \sum_{i=1}^{N_t} \xi_i \right) = \lambda t \mathbb{E}(\xi_1) = \lambda t \int_{\mathbb{R}} x F(dx).$$

Note that this computation also shows that  $\mathbb{E}|M_t| < \infty$  for each  $t \geq 0$ .

For the second part, using the independence and distributional properties of  $N$  and  $\{\xi_i : i \geq 1\}$  we have that

$$\begin{aligned} \mathbb{E}(M_t^2) &= \mathbb{E} \left[ \left( \sum_{i=1}^{N_t} \xi_i \right)^2 \right] - \lambda^2 t^2 \left( \int_{\mathbb{R}} x F(dx) \right)^2 \\ &= \mathbb{E} \left( \sum_{i=1}^{N_t} \xi_i^2 \right) + \mathbb{E} \left( \sum_{i=1}^{N_t} \sum_{j=1}^{N_t} \mathbf{1}_{\{j \neq i\}} \xi_i \xi_j \right) - \lambda^2 t^2 \left( \int_{\mathbb{R}} x F(dx) \right)^2 \\ &= \lambda t \int_{\mathbb{R}} x^2 F(dx) + \mathbb{E}(N_t^2 - N_t) \left( \int_{\mathbb{R}} x F(dx) \right)^2 - \lambda^2 t^2 \left( \int_{\mathbb{R}} x F(dx) \right)^2 \\ &= \lambda t \int_{\mathbb{R}} x^2 F(dx) + \lambda^2 t^2 \left( \int_{\mathbb{R}} x^2 F(dx) \right)^2 - \lambda^2 t^2 \left( \int_{\mathbb{R}} x F(dx) \right)^2 \\ &= \lambda t \int_{\mathbb{R}} x^2 F(dx) \end{aligned}$$

as required. ■

If we recall the intuition in the discussion following (3.4) then our interest in the type of martingales described in Lemma 4.1 (which are also Lévy processes) as far as the Lévy-Itô decomposition is concerned is tantamount to understanding how to superimpose a countably infinite number of such processes together so that the resulting sum converges in an appropriate sense. Precisely this point is tackled in the next theorem.

We need to introduce some notation first. Suppose that for each  $n = 1, 2, 3, \dots$  the process  $N^{(n)} = \{N_t^{(n)} : t \geq 0\}$  is a Poisson process with rate  $\lambda_n \geq 0$ . Here we understand the process  $N_t^{(n)} = 0$  for all  $t \geq 0$  if  $\lambda_n = 0$ . We shall also

assume that the processes  $N^{(n)}$  are mutually independent. Moreover, for each  $n = 1, 2, 3, \dots$  let us introduce the i.i.d. sequences  $\{\xi_i^{(n)} : i = 1, 2, 3, \dots\}$  (which are themselves mutually independent sequences) with common distribution  $F_n$  which does not assign mass to the origin and which satisfies  $\int x^2 F_n(dx) < \infty$ . Associated with each pair  $(\lambda_n, F_n)$  is the square integrable martingale described in Lemma 4.1 which we denote by  $M^{(n)} = \{M_t^{(n)} : t \geq 0\}$ . Suppose that we denote by  $\{\mathcal{F}_t^{(n)} : t \geq 0\}$  as the natural filtration generated by the process  $M^{(n)}$ , then we can put all processes  $\{M^{(n)} : n \geq 1\}$  on the same probability space and take them as martingales with respect to the common filtration

$$\mathcal{F}_t := \sigma\left(\bigcup_{n \geq 1} \mathcal{F}_t^{(n)}\right)$$

which we may also assume without loss of generality is taken in its completed and right continuous form<sup>8</sup>.

**Theorem 4.1** *If*

$$\sum_{n \geq 1} \lambda_n \int_{\mathbb{R}} x^2 F_n(dx) < \infty \quad (4.1)$$

*then there exists a Lévy process  $X = \{X_t : t \geq 0\}$  defined on the same probability space as the processes  $\{M^{(n)} : n \geq 1\}$  which is also a square integrable martingale and whose characteristic exponent is given by*

$$\Psi(\theta) = \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x) \sum_{n \geq 1} \lambda_n F_n(dx)$$

*for all  $\theta \in \mathbb{R}$  such that the for each fixed  $T > 0$ ,*

$$\lim_{k \uparrow \infty} \mathbb{E} \left[ \sup_{t \leq T} \left( X_t - \sum_{n=1}^k M_t^{(n)} \right)^2 \right] = 0.$$

**Proof.** First note that by linearity of conditional expectation,  $\sum_{n=1}^k M^{(k)}$  is a square integrable martingale. In fact, since, by independence and the martingale property, for  $i \neq j$  we have  $\mathbb{E}(M_t^{(i)} M_t^{(j)}) = \mathbb{E}(M_t^{(i)})\mathbb{E}(M_t^{(j)}) = 0$ , it follows that

$$\mathbb{E} \left[ \left( \sum_{n=1}^k M_t^{(n)} \right)^2 \right] = \sum_{n=1}^k \mathbb{E} \left[ (M_t^{(n)})^2 \right] = t \sum_{n=1}^k \lambda_n \int_{\mathbb{R}} x^2 F_n(dx) < \infty \quad (4.2)$$

where the last equality follows by the assumption (4.1).

Fix  $T > 0$ . We now claim that the sequence  $\{X^{(k)} : k \geq 1\}$  is a Cauchy sequence with respect to  $\|\cdot\|$  where  $X^{(k)} = \{X_t^{(k)} : 0 \leq t \leq T\}$  and

$$X_t^{(k)} = \sum_{n=1}^k M_t^{(n)}.$$

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<sup>8</sup>That is to say, if  $\mathcal{F}_t$  is not right continuous then we may work instead with  $\mathcal{F}_t^* = \bigcap_{s > t} \mathcal{F}_s$ .

To see why, note that for  $k \geq l$ , similar calculations to those in (4.2) show that

$$\|X^{(k)} - X^{(l)}\|^2 = \mathbb{E}[(X_T^{(k)} - X_T^{(l)})^2] = T \sum_{n=l}^k \lambda_n \int_{\mathbb{R}} x^2 F_n(dx)$$

which tends to zero as  $k, l \uparrow \infty$  by the assumption (4.1). It follows that there exists a martingale, say  $X = \{X_t : 0 \leq t \leq T\}$  with respect to the filtration  $\{\mathcal{F}_t : 0 \leq t \leq T\}$  such that  $\|X - X^{(k)}\| \rightarrow 0$  as  $k \uparrow \infty$ . Thanks to Doob's maximal inequality (??), it follows moreover that

$$\lim_{k \uparrow \infty} \mathbb{E} \left( \sup_{0 \leq t \leq T} (X_t - X_t^{(k)})^2 \right) = 0. \quad (4.3)$$

From (4.3) it is implicit one-dimensional and hence finite-dimensional distributions of  $X^{(k)}$  converge to those of  $X$ . Consequently since the processes  $X^{(k)}$  are all Lévy processes, for all  $0 \leq s \leq t \leq T$ ,

$$\begin{aligned} \mathbb{E}(e^{i\theta(X_t - X_s)}) &= \lim_{k \uparrow \infty} \mathbb{E}(e^{i\theta(X_t^{(k)} - X_s^{(k)})}) \\ &= \lim_{k \uparrow \infty} \mathbb{E}(e^{i\theta X_{t-s}^{(k)}}) \\ &= \mathbb{E}(e^{i\theta X_{t-s}}) \end{aligned}$$

showing that  $X$  has stationary and independent increments. From Section 2.2 we see that for all  $t \geq 0$

$$\mathbb{E}(e^{i\theta X_t^{(k)}}) = \prod_{n=1}^k \mathbb{E}(e^{i\theta M_t^{(n)}}) = \exp \left\{ - \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x) \sum_{n=1}^k \lambda_n F_n(dx) \right\}$$

which converges to  $\exp\{-\Psi(\theta)\}$  as  $k \uparrow \infty$ . Note in particular that the last integral converges thanks to the assumption (4.1).

To show that  $X$  is a Lévy process we need only to deduce the paths of  $X$  are right continuous with left limits almost surely. However this is a direct result of the fact that, if  $D[0, T]$  is the space of functions  $f : [0, T] \rightarrow \mathbb{R}$  which are right continuous with left limits, then  $D[0, T]$  is a closed space under the metric  $d(f, g) = \sup_{t \in [0, T]} |f(t) - g(t)|$  for  $f, g \in D[0, 1]$ . See Exercise 8 for a proof of this fact.

There is one outstanding issue that needs to be dealt with to complete the proof. In principle the limiting process  $X$  depends on  $T$  and we need to dismiss this eventuality. Suppose we index  $X$  by the time horizon  $T$ , say  $X^T$ . Using that for any two sequences of real numbers  $\{a_n\}$  and  $\{b_n\}$ ,  $\sup_n a_n^2 = (\sup_n |a_n|)^2$  and  $\sup_n |a_n + b_n| \leq \sup_n |a_n| + \sup_n |b_n|$ , we have together with an application of Minkowski's inequality that, when  $T_1 \leq T_2$ ,

$$\mathbb{E}[\sup_{s \leq T_1} (X_s^{T_1} - X_s^{T_2})^2]^{1/2} \leq \mathbb{E}[\sup_{s \leq T_1} (X_s^{T_1} - X_s^{(k)})^2]^{1/2} + \mathbb{E}[\sup_{s \leq T_1} (X_s^{T_2} - X_s^{(k)})^2]^{1/2}.$$

Hence taking limits as  $k \uparrow \infty$  we may appeal to (4.3) to deduce that the right hand side tends to 0. Consequently the two processes agree almost surely on the time horizon  $[0, T_1]$  and we may comfortably say that the limit  $X$  does indeed not depend on the time horizon  $T$ . ■

## 4.1 Proof of the Lévy-Itô decomposition

We are now in a position to prove the Lévy-Itô decomposition.

**Proof of Theorem 3.1.** Recalling the decomposition (3.1) and the discussion thereafter, it suffices to prove the existence of the process  $X^{(3)}$  and that it has a countable number of discontinuities which are less than unity in magnitude over each finite time horizon. However this follows immediately from Theorem 4.1 when we take  $\lambda_n = \Pi(\{x : 2^{-(n+1)} \leq |x| < 2^{-n}\})$  and

$$F_n(dx) = \lambda_n^{-1} \Pi(dx)|_{\{x: 2^{-(n+1)} \leq |x| < 2^{-n}\}}.$$

Note in particular that

$$\sum_{n=0}^{\infty} \lambda_n \int_{\mathbb{R}} x^2 F_n(dx) = \int_{(-1,1)} x^2 \Pi(dx) < \infty$$

where the last inequality follows by assumption.

The fact that the processes  $X^{(1)}$ ,  $X^{(2)}$  and  $X^{(3)}$  may be constructed on the same probability space is simply a matter of constructing an appropriate product space which supports all the associated independent processes. ■

**Remark 4.1** In the case that  $\Pi(\mathbb{R}) < \infty$  the Lévy-Itô decomposition is somewhat overkill as in that case it is clear that  $(1 - e^{ix})$  is integrable on its own against  $\Pi$  so we may write instead

$$\begin{aligned} \Psi(\theta) &= \left\{ i\theta \left( a + \int_{0 < |x| < 1} x \Pi(dx) \right) + \frac{1}{2} \sigma^2 \theta^2 \right\} \\ &\quad + \left\{ \Pi(\mathbb{R}) \int_{|x| \geq 1} (1 - e^{i\theta x}) \frac{\Pi(dx)}{\Pi(\mathbb{R})} \right\} \end{aligned}$$

showing that  $X$  is the independent sum of a linear Brownian motion and a compound Poisson process.

**Remark 4.2** Note that the Lévy-Itô decomposition indicates that when  $\Pi(\mathbb{R}) = \infty$  the compound Poisson martingales associated with  $(\lambda_n, F_n)$  is such that the rate of arrival of the jumps increases and the size of the jump decreases as  $n$  tends to infinity. The contribution of these compound Poisson processes is thus to a countable infinity of arbitrarily small jumps. Note also that the use of the interval  $(-1, 1)$  is completely arbitrary in the construction. One could equally have constructed a sequence of compound Poisson processes martingales from the concentration of the measure  $\Pi$  on  $(-\alpha, \beta)$  for any  $\alpha, \beta > 0$ . The effect of this change would be that the constant  $a$  in the Lévy-Khintchine formula changes.

## 5 Path variation

The proof of the Lévy-Itô decomposition reveals a little more than the decomposition in Theorem 3.1. It provides us the necessary point of view to characterize precisely the almost sure path variation of the process.

Suppose that  $X$  is any Lévy process whose characteristic exponent we shall always refer to in the form (3.5). Moreover we shall talk about  $X^{(1)}$ ,  $X^{(2)}$  and  $X^{(3)}$  as the three components of the Lévy-Itô decomposition mentioned in Theorem 3.1.

If  $\sigma > 0$ , since it is known that Brownian motion has unbounded variation and since the continuous and discontinuous elements of the path of  $X$  make independent contributions to its variation, the latter must also have unbounded variation. In the case that  $\sigma = 0$  however, the Lévy process may have either bounded or unbounded variation. Since the process  $X^{(2)}$  is a compound Poisson process, which obviously has paths of bounded variation, the variation of  $X$  boils down to the variation of  $X^{(3)}$  in this case.

Recall that we may see the process  $X^{(3)}$  as the uniform  $L^2$ -limit over finite time horizons of a sequence of partial sums of compound Poisson martingales  $X_t^{(k)} = \sum_{n=0}^k M_t^{(n)}$  for  $t \geq 0$ . Suppose in the aforementioned analysis we split the compound Poisson martingales into two sub-processes containing the positive and negative jumps respectively. To be more precise, define

$$M_t^{(n,+)} = \sum_{i=1}^{N_t^{(n)}} \xi_i^{(n)} \mathbf{1}_{(\xi_i^{(n)} > 0)} - t\lambda_n \int_{(0,\infty)} x F_n(dx)$$

and

$$M_t^{(n,-)} = \sum_{i=1}^{N_t^{(n)}} |\xi_i^{(n)}| \mathbf{1}_{(\xi_i^{(n)} < 0)} - t\lambda_n \int_{(-\infty,0)} |x| F_n(dx)$$

for  $t \geq 0$  so that  $M_t^{(n,+)} = M_t^{(n)} - M_t^{(n,-)}$ . It is straightforward to check as before that the processes  $X^{(k,+)} := \sum_{n=0}^k M^{(n,+)}$  and  $X^{(k,-)} := \sum_{n=0}^k M^{(n,-)}$  are compound Poisson martingales that form Cauchy sequences which converge uniformly over finite time horizons in  $L^2$  to their limits which we denote by  $X^{(+)}$  and  $X^{(-)}$ . Moreover it is clear that necessarily  $X = X^{(+)} - X^{(-)}$ .

Next note that for each fixed  $t$

$$C_t^{(k,+)} := X_t^{(k,+)} + t \int_{(0,\infty)} x \sum_{n=0}^k \lambda_n F_n(dx) = \sum_{n=0}^k \sum_{i=1}^{N_t^{(n)}} \xi_i^{(n)} \mathbf{1}_{(\xi_i^{(n)} > 0)}$$

is increasing in  $k$  and therefore has a limit almost surely (which may be infinite). Since  $X_t^{(k,+)}$  converges in distribution to a non-degenerate random variable, it follows that

$$\lim_{k \uparrow \infty} C_t^{(k,+)} < \infty \text{ a.s.} \Leftrightarrow \int_{(0,\infty)} x \sum_{n=0}^{\infty} \lambda_n F_n(dx) < \infty$$



With  $C^{(k,-)}$  defined in the obvious way it also follows that

$$\lim_{k \uparrow \infty} C_t^{(k,-)} < \infty \text{ a.s.} \Leftrightarrow \int_{(-\infty,0)} |x| \sum_{n=0}^{\infty} \lambda_n F_n(dx) < \infty$$

and hence since

$$\sum_{n=0}^k \sum_{i=1}^{N_t^{(n)}} \xi_i^{(n)} = X_t^{(k)} + t \int_{\mathbb{R}} x \sum_{n=0}^k \lambda_n F_n(dx) = C_t^{(k,+)} - C_t^{(k,-)},$$

the left hand side above is almost surely absolutely convergent if and only if

$$\int_{\mathbb{R}} |x| \sum_{n=0}^{\infty} \lambda_n F_n(dx) < \infty.$$

The latter integral test is tantamount to saying

$$\int_{(-1,1)} |x| \Pi(dx) < \infty. \quad (5.1)$$

The importance of the previous calculations is that it is straightforward to check that the variation of  $X^{(3)}$  over  $[0, t]$  is at least  $C_t^{(\infty,+)} + C_t^{(\infty,-)}$  which explodes if and only if (5.1) fails and, when it is well defined, is equal to

$$C_t^{(\infty,+)} + C_t^{(\infty,-)} + t \int_{\mathbb{R}} x \sum_{n=0}^k \lambda_n F_n(dx)$$

which is almost surely finite if and only if (5.1) holds.

In conclusion we have the following result.

**Lemma 5.1** *Any Lévy process has paths of bounded variation if and only if*

$$\sigma = 0 \text{ and } \int_{(-1,1)} |x| \Pi(dx) < \infty.$$

In the case that  $X$  has bounded variation, we may express the process  $X^{(3)}$  in the, now meaningful, way

$$X_t^{(3)} = \sum_{s \leq t} \Delta X_s \mathbf{1}_{(|\Delta X_s| < 1)} - t \int_{(-1,1)} x \Pi(dx)$$

where  $\Delta X_s = X_s - X_{s-}$ . Note that there are only countably many times  $s \leq t$  for which  $\Delta X_s \neq 0$  on account of the fact that the paths are right continuous with left limits and hence the sum over a continuum of values  $s \leq t$  is in fact a sum over a countable set of times  $s \leq t$ . We may then write

$$X_t = -at + \sum_{s \leq t} \Delta X_s - t \int_{(-1,1)} x \Pi(dx). \quad (5.2)$$

Indeed the calculations above have shown that the sum  $\sum_{s \leq t} \Delta X_s$  is necessarily absolutely convergent almost surely.

Note that monotone functions necessarily have bounded variation and hence a Lévy process is a subordinator only if it has paths of bounded variation. In that case inspecting (5.2) tells us the following.

**Lemma 5.2** *A Lévy process is a subordinator if and only if it has paths of bounded variation,  $\Pi(-\infty, 0) = 0$  and*

$$\delta := -a - \int_{(0,1)} x\Pi(dx) \geq 0.$$

*In that case the characteristic exponent may be written*

$$\Psi(\theta) = -i\delta\theta + \int_{(0,\infty)} (1 - e^{i\theta x})\Pi(dx).$$

## Exercises

**Exercise 1** Using Definition 1.1, show that the sum of two (or indeed any finite number of) independent Lévy processes is again a Lévy process.

**Exercise 2** Suppose that  $S = \{S_n : n \geq 0\}$  is any random walk<sup>9</sup> and  $\Gamma_p$  is an independent random variable with a geometric distribution on  $\{0, 1, 2, \dots\}$  with parameter  $p$ .

(i) Show that  $\Gamma_p$  is infinitely divisible.

(ii) Show that  $S_{\Gamma_p}$  is infinitely divisible.

**Exercise 3** [Proof of Lemma 2.1] In this exercise we derive the Frullani identity.

(i) Show for any function  $f$  such that  $f'$  exists and is continuous and  $f(0)$  and  $f(\infty)$  are finite, that

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = (f(0) - f(\infty)) \log\left(\frac{b}{a}\right),$$

where  $b > a > 0$ .

(ii) By choosing  $f(x) = e^{-x}$ ,  $a = \alpha > 0$  and  $b = \alpha - z$  where  $z < 0$ , show that

$$\frac{1}{(1 - z/\alpha)^\beta} = e^{-\int_0^\infty (1 - e^{-zx}) \frac{\beta}{x} e^{-\alpha x} dx}$$

and hence by analytic extension show that the above identity is still valid for all  $z \in \mathbb{C}$  such that  $\Re z \leq 0$ .

**Exercise 4** Establishing formulae (2.5) and (2.6) from the Lévy measure given in (2.7) is the result of a series of technical manipulations of special integrals. In this exercise we work through them. In the following text we will use the gamma function  $\Gamma(z)$ , defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

for  $z > 0$ . Note the gamma function can also be analytically extended so that it is also defined on  $\mathbb{R} \setminus \{0, -1, -2, \dots\}$  (see [16]). Whilst the specific definition of the gamma function for negative numbers will not play an important role in this exercise, the following two facts that can be derived from it will. For  $z \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$  the gamma function observes the recursion  $\Gamma(1 + z) = z\Gamma(z)$  and  $\Gamma(1/2) = \sqrt{\pi}$ .

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<sup>9</sup>Recall that  $S = \{S_n : n \geq 0\}$  is a random walk if  $S_0 = 0$  and for  $n = 1, 2, 3, \dots$  the increments  $S_n - S_{n-1}$  are independent and identically distributed.

(i) Suppose that  $0 < \alpha < 1$ . Prove that for  $u > 0$ ,

$$\int_0^\infty (e^{-ur} - 1)r^{-\alpha-1}dr = \Gamma(-\alpha)u^\alpha$$

and show that the same equality is valid when  $-u$  is replaced by any complex number  $w \neq 0$  with  $\Re w \leq 0$ . Conclude by considering  $w = i$  that

$$\int_0^\infty (1 - e^{ir})r^{-\alpha-1}dr = -\Gamma(-\alpha)e^{-i\pi\alpha/2} \quad (\text{E.1})$$

as well as the complex conjugate of both sides being equal. Deduce (2.5) by considering the integral

$$\int_0^\infty (1 - e^{i\xi\theta r})r^{-\alpha-1}dr$$

for  $\xi = \pm 1$  and  $\theta \in \mathbb{R}$ . Note that you will have to take  $a = \eta - \int_{\mathbb{R}} x \mathbf{1}_{(|x|<1)} \Pi(dx)$ , which you should check is finite.

(ii) Now suppose that  $\alpha = 1$ . First prove that

$$\int_{|x|<1} e^{i\theta x}(1 - |x|)dx = 2 \left( \frac{1 - \cos \theta}{\theta^2} \right)$$

for  $\theta \in \mathbb{R}$  and hence by Fourier inversion,

$$\int_0^\infty \frac{1 - \cos r}{r^2}dr = \frac{\pi}{2}.$$

Use this identity to show that for  $z > 0$ ,

$$\int_0^\infty (1 - e^{irz} + izr \mathbf{1}_{(r<1)}) \frac{1}{r^2}dr = \frac{\pi}{2}z + iz \log z - ikz$$

for some constant  $k \in \mathbb{R}$ . By considering the complex conjugate of the above integral establish the expression in (2.6). Note that you will need a different choice of  $a$  to part (i).

(iii) Now suppose that  $1 < \alpha < 2$ . Integrate (E.1) by parts to reach

$$\int_0^\infty (e^{ir} - 1 - ir)r^{-\alpha-1}dr = \Gamma(-\alpha)e^{-i\pi\alpha/2}.$$

Consider the above integral for  $z = \xi\theta$ , where  $\xi = \pm 1$  and  $\theta \in \mathbb{R}$  and deduce the identity (2.5) in a similar manner to the proof in (i) and (ii).

**Exercise 5** Prove for any  $\theta \in \mathbb{R}$  that

$$\exp\{i\theta X_t + t\Psi(\theta)\}, \quad t \geq 0$$

is a martingale where  $\{X_t : t \geq 0\}$  is a Lévy process with characteristic exponent  $\Psi$ .

**Exercise 6** In this exercise we will work out in detail the features of the inverse Gaussian process discussed earlier on in this chapter. Recall that  $\tau = \{\tau_s : s \geq 0\}$  is a non-decreasing Lévy process defined by  $\tau_s = \inf\{t \geq 0 : B_t + bt > s\}$ ,  $s \geq 0$ , where  $B = \{B_t : t \geq 0\}$  is a standard Brownian motion and  $b > 0$ .

(i) Argue along the lines of Exercise 5 to show that for each  $\lambda > 0$ ,

$$e^{\lambda B_t - \frac{1}{2}\lambda^2 t}, \quad t \geq 0$$

is a martingale. Use Doob's Optional Sampling Theorem to obtain

$$\mathbb{E}(e^{-(\frac{1}{2}\lambda^2 + b\lambda)\tau_s}) = e^{-\lambda s}.$$

Use analytic extension to deduce further that  $\tau_s$  has characteristic exponent

$$\Psi(\theta) = s(\sqrt{-2i\theta + b^2} - b)$$

for all  $\theta \in \mathbb{R}$ .

(ii) Defining the measure  $\Pi(dx) = (2\pi x^3)^{-1/2} e^{-x b^2/2} dx$  on  $x > 0$ , check using (E.1) from Exercise 4 that

$$\int_0^\infty (1 - e^{i\theta x}) \Pi(dx) = \Psi(\theta)$$

for all  $\theta \in \mathbb{R}$ . Confirm that the triple  $(a, \sigma, \Pi)$  in the Lévy–Khintchine formula are thus  $\sigma = 0$ ,  $\Pi$  as above and  $a = -2sb^{-1} \int_0^b (2\pi)^{-1/2} e^{-y^2/2} dy$ .

(iii) Taking

$$\mu_s(dx) = \frac{s}{\sqrt{2\pi x^3}} e^{sb} e^{-\frac{1}{2}(s^2 x^{-1} + b^2 x)} dx$$

on  $x > 0$  show that

$$\begin{aligned} \int_0^\infty e^{-\lambda x} \mu_s(dx) &= e^{bs - s\sqrt{b^2 + 2\lambda}} \int_0^\infty \frac{s}{\sqrt{2\pi x^3}} e^{-\frac{1}{2}(\frac{s}{\sqrt{x}} - \sqrt{(b^2 + 2\lambda)x})^2} dx \\ &= e^{bs - s\sqrt{b^2 + 2\lambda}} \int_0^\infty \sqrt{\frac{2\lambda + b^2}{2\pi u}} e^{-\frac{1}{2}(\frac{s}{\sqrt{u}} - \sqrt{(b^2 + 2\lambda)u})^2} du. \end{aligned}$$

Hence by adding the last two integrals together deduce that

$$\int_0^\infty e^{-\lambda x} \mu_s(dx) = e^{-s(\sqrt{b^2 + 2\lambda} - b)}$$

confirming both that  $\mu_s(dx)$  is a probability distribution as well as being the probability distribution of  $\tau_s$ .

**Exercise 7** Show that for a simple Brownian motion  $B = \{B_t : t > 0\}$  the first passage process  $\tau = \{\tau_s : s > 0\}$  (where  $\tau_s = \inf\{t \geq 0 : B_t \geq s\}$ ) is a stable process with parameters  $\alpha = 1/2$  and  $\beta = 1$ .

**Exercise 8** Recall that  $D[0, 1]$  is the space of functions  $f : [0, 1] \rightarrow \mathbb{R}$  which are right continuous with left limits.

- (i) Define the norm  $\|f\| = \sup_{x \in [0, 1]} |f(x)|$ . Use the triangle inequality to deduce that  $D[0, 1]$  is closed under uniform convergence with respect to the norm  $\|\cdot\|$ . That is to say, show that if  $\{f_n : n \geq 1\}$  is a sequence in  $D[0, 1]$  and  $f : [0, 1] \rightarrow \mathbb{R}$  such that  $\lim_{n \rightarrow \infty} \|f_n - f\| = 0$  then  $f \in D[0, 1]$ .
- (ii) Suppose that  $f \in D[0, 1]$  and let  $\Delta = \{t \in [0, 1] : |f(t) - f(t-)| \neq 0\}$  (the set of discontinuity points). Show that  $\Delta$  is countable if  $\Delta_c$  is countable for all  $c > 0$  where  $\Delta_c = \{t \in [0, 1] : |f(t) - f(t-)| > c\}$ . Next fix  $c > 0$ . By supposing for contradiction that  $\Delta_c$  has an accumulation point, say  $x$ , show that the existence of either a left or right limit at  $x$  fails as it would imply that there is no left or right limit of  $f$  at  $x$ . Deduce that  $\Delta_c$  and hence  $\Delta$  is countable.

**Exercise 9** The explicit construction of a Lévy process given in the Lévy–Itô decomposition begs the question as to whether one may construct examples of *deterministic* functions which have similar properties to those of the paths of Lévy processes. The objective of this exercise is to do precisely that. The reader is warned however, that this is purely an analytical exercise and one should not necessarily think of the paths of Lévy processes as being entirely similar to the functions constructed below in all respects.

- (i) Let us recall the definition of the Cantor function which we shall use to construct a deterministic function which has bounded variation and that is right continuous with left limits and whose points of discontinuity are dense in its domain. Take the interval  $C_0 := [0, 1]$  and perform the following iteration. For  $n \geq 0$  define  $C_n$  as the union of intervals which remain when removing the middle third of each of the intervals which make up  $C_{n-1}$ . The Cantor set  $C$  is the limiting object,  $\bigcap_{n \geq 0} C_n$  and can be described by

$$C = \{x \in [0, 1] : x = \sum_{k \geq 1} \frac{\alpha_k}{3^k} \text{ such that } \alpha_k \in \{0, 2\} \text{ for each } k \geq 1\}.$$

One sees then that the Cantor set is simply the points in  $[0, 1]$  which omits numbers whose tertiary expansion contain the digit 1. To describe the Cantor function, for each  $x \in [0, 1]$  let  $j(x)$  be the smallest  $j$  for which  $\alpha_j = 1$  in the tertiary expansion of  $\sum_{k \geq 1} \alpha_k / 3^k$  of  $x$ . If  $x \in C$  then  $j(x) = \infty$  and otherwise if  $x \in [0, 1] \setminus C$  then  $1 \leq j(x) < \infty$ . The Cantor function is defined as follows

$$f(x) = \frac{1}{2^{j(x)}} + \sum_{i=1}^{j(x)-1} \frac{\alpha_i}{2^{i+1}} \text{ for } x \in [0, 1].$$

Now consider the function  $g : [0, 1] \rightarrow [0, 1]$  given by  $g(x) = f^{-1}(x) - ax$  for  $a \in \mathbb{R}$ . Here we understand  $f^{-1}(x) = \inf\{\theta : f(\theta) > x\}$ . Note that  $g$  is

monotone if and only if  $a \leq 0$ . Show that  $g$  has only positive jumps and the value of  $x$  for which  $g$  jumps form a dense set in  $[0, 1]$ . Show further that  $g$  has bounded variation on  $[0, 1]$ .

- (ii) Now let us construct an example of a deterministic function which has unbounded variation, whose points of discontinuity are dense in its domain and that is right continuous with left limits. Denote by  $\mathbb{Q}_2$  the dyadic rationals. Consider a function  $J : [0, \infty) \rightarrow \mathbb{R}$  as follows. For all  $x \geq 0$  which are not in  $\mathbb{Q}_2$ , set  $J(x) = 0$ . It remains to assign a value for each  $x = a/2^n$  where  $a = 1, 3, 5, \dots$  (even values of  $a$  cancel). Let

$$J(a/2^n) = \begin{cases} 2^{-n} & \text{if } a = 1, 5, 9, \dots \\ -2^{-n} & \text{if } a = 3, 7, 11, \dots \end{cases}$$

and define

$$f(x) = \sum_{s \in [0, x] \cap \mathbb{Q}_2} J(s).$$

Show that  $f$  is uniformly bounded on  $[0, 1]$ , is right continuous with left limits and has unbounded variation over  $[0, 1]$ .

**Exercise 10** Show that any Lévy process of bounded variation may be written as the difference of two independent subordinators.

**Exercise 11** Suppose that  $X = \{X_t : t \geq 0\}$  is a Lévy process with characteristic exponent  $\Psi$  and  $\tau = \{\tau_s : s \geq 0\}$  is an independent subordinator with characteristic exponent  $\Xi$ . Show that  $Y = \{X_{\tau_s} : s \geq 0\}$  is again a Lévy process with characteristic exponent  $\Xi \circ i\Psi$ .

**Exercise 12** This exercise gives another explicit example of a Lévy process; the variance gamma process, introduced by [21] for modelling financial data.

- (i) Suppose that  $\Gamma = \{\Gamma_t : t \geq 0\}$  is a gamma subordinator with parameters  $\alpha, \beta$  and that  $B = \{B_t : t \geq 0\}$  is an independent standard Brownian motion. Show that for  $c \in \mathbb{R}$  and  $\sigma > 0$ , the variance gamma process

$$X_t := c\Gamma_t + \sigma B_{\Gamma_t}, \quad t \geq 0$$

is a Lévy process with characteristic exponent

$$\Psi(\theta) = \beta \log\left(1 - i\frac{\theta c}{\alpha} + \frac{\sigma^2 \theta^2}{2\alpha}\right).$$

- (ii) Show that the variance gamma process is equal in law to the Lévy process

$$\Gamma^{(1)} - \Gamma^{(2)} = \{\Gamma_t^{(1)} - \Gamma_t^{(2)} : t \geq 0\},$$

where  $\Gamma^{(1)}$  is a Gamma subordinator with parameters

$$\alpha^{(1)} = \left( \sqrt{\frac{1}{4} \frac{c^2}{\alpha^2} + \frac{1}{2} \frac{\sigma^2}{\alpha}} + \frac{1}{2} \frac{c}{\alpha} \right)^{-1} \quad \text{and} \quad \beta^{(1)} = \beta$$

and  $\Gamma^{(2)}$  is a Gamma subordinator, independent of  $\Gamma^{(1)}$ , with parameters

$$\alpha^{(2)} = \left( \sqrt{\frac{1}{4} \frac{c^2}{\alpha^2} + \frac{1}{2} \frac{\sigma^2}{\alpha}} - \frac{1}{2} \frac{c}{\alpha} \right)^{-1} \quad \text{and } \beta^{(2)} = \beta.$$

**Exercise 13** Suppose that  $d$  is an integer greater than one. Choose  $\mathbf{a} \in \mathbb{R}^d$  and let  $\Pi$  be a measure concentrated on  $\mathbb{R}^d \setminus \{0\}$  satisfying

$$\int_{\mathbb{R}^d} (1 \wedge |\mathbf{x}|^2) \Pi(d\mathbf{x})$$

where  $|\cdot|$  is the standard Euclidian norm. Show that it is possible to construct a  $d$ -dimensional process  $\mathbf{X} = \{\mathbf{X}_t : t \geq 0\}$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  having the following properties.

- (i) The paths of  $\mathbf{X}$  are right continuous with left limits  $\mathbb{P}$ -almost surely in the sense that for each  $t \geq 0$ ,  $\mathbb{P}(\lim_{s \downarrow t} \mathbf{X}_s = \mathbf{X}_t) = 1$  and  $\mathbb{P}(\lim_{s \uparrow t} \mathbf{X}_s \text{ exists}) = 1$ .
- (ii)  $\mathbb{P}(\mathbf{X}_0 = \mathbf{0}) = 1$ , the zero vector in  $\mathbb{R}^d$ .
- (iii) For  $0 \leq s \leq t$ ,  $\mathbf{X}_t - \mathbf{X}_s$  is independent of  $\sigma(\mathbf{X}_u : u \leq s)$ .
- (iv) For  $0 \leq s \leq t$ ,  $\mathbf{X}_t - \mathbf{X}_s$  is equal in distribution to  $\mathbf{X}_{t-s}$ .
- (v) For any  $t \geq 0$  and  $\theta \in \mathbb{R}^d$ ,

$$\mathbb{E}(e^{i\theta \cdot \mathbf{X}_t}) = e^{-\Psi(\theta)t}$$

and

$$\Psi(\theta) = i\mathbf{a} \cdot \theta + \frac{1}{2} \theta \cdot \mathbf{A} \theta + \int_{\mathbb{R}^d} (1 - e^{i\theta \cdot \mathbf{x}} + i(\theta \cdot \mathbf{x}) \mathbf{1}_{(|\mathbf{x}| < 1)}) \Pi(d\mathbf{x}), \quad (\text{E.2})$$

where for any two vectors  $\mathbf{x}$  and  $\mathbf{y}$  in  $\mathbb{R}^d$ ,  $\mathbf{x} \cdot \mathbf{y}$  is the usual inner product and  $\mathbf{A}$  is a  $d \times d$  matrix whose eigenvalues are all non-negative.



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