LÉVY PROCESSES, THE WIENER-HOPF FACTORISATION AND APPLICATIONS: PART II

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1 The Wiener-Hopf factorisation for random walks

Suppose that $\{\xi_i : i = 1, 2, ...\}$ are a sequence of $\mathbb{R}$-valued identically and independently distributed random variables defined on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with common distribution function F. Let

$$S_0 = 0 \text{ and } S_n = \sum_{i=1}^{n} \xi_i.$$

The process $S = \{S_n : n \geq 0\}$ is called a (real valued) random walk. For convenience we shall make a number of assumptions on F. First,

$$\min\{F(0, \infty), F(-\infty, 0)\} > 0,$$

meaning that the random walk may experience both positive and negative jumps, and second, $F\{0\} = 0$.

We now turn out attention to the Wiener-Hopf factorization. Fix $0 < p < 1$ and define

$$G = \min\{k = 0, 1, ..., \mathbf{\Gamma}_p : S_k = \max_{j=0,1,...,\mathbf{\Gamma}_p} S_j\}$$

and

$$D = \max\{k = 0, 1, ..., \mathbf{\Gamma}_p : S_k = \min_{j=0,1,...,\mathbf{\Gamma}_p} S_j\}$$

where $\mathbf{\Gamma}_p$ is a geometrically distributed random variable with parameter p which is independent of the random walk S. In words, G is the first visit of S to its maximum over the time period $\{0, 1, ..., \mathbf{\Gamma}_p\}$. Now define

$$N = \inf\{n > 0 : S_n > 0\}.$$

In words, the first visit of S to $(0, \infty)$ *after time* 0.

The next theorem is one of many statements that could be made under the heading of 'The Wiener-Hopf Factorisation', however we consider it the most fundamental of them all.

Theorem 1.1 *Assume all of the notation and conventions above.* (G, S_G) *is independent of* $(\mathbf{\Gamma}_p - G, S_{\mathbf{\Gamma}_p} - S_G)$ *and both pairs are infinitely divisible. Moreover, the latter has the same law as* (D, S_D).

Other statements that can be made in the spirit of the Wiener-Hopf factorisation go further in specifying additional distributional information about the pairs (G, S_G) and (D, S_D). However, we refrain here from pursuing those more detailed results.

Crucial to the proof of Theorem 1.1 is the *ladder height process* of the random walk S. The latter is the bivariate random walk $(T, H) := \{(T_n, H_n) : n = 0, 1, 2, ...\}$ where $(T_0, H_0) = (0, 0)$ and otherwise for $n = 1, 2, 3, ...,$

$$T_n = \begin{cases} \min\{k = 1, 2, ... : S_{T_{n-1}+k} > H_{n-1}\} & \text{if } T_{n-1} < \infty \\ \infty & \text{if } T_{n-1} = \infty \end{cases}$$

and

$$H_n = \begin{cases} S_{T_n} & \text{if } T_n < \infty \\ \infty & \text{if } T_n = \infty. \end{cases}$$

That is to say, the process (T, H), until becoming infinite in value, represents the times and positions of the running maxima of S; the so-called ladder times and ladder heights. It is not difficult to see that T_n is a stopping time for each $n = 0, 1, 2, ..$ and hence thanks to the i.i.d. increments of S, the increments of (T, H) are independent and identically distributed with the same law as the pair (N, S_N).

Proof of Theorem 1.1. The path of the random walk may be broken into $\nu \in \{0, 1, 2, ...\}$ finite (or completed) excursions from the maximum followed by an additional excursion which straddles the random time $\mathbf{\Gamma}_p$. Note that if $T_n < \infty$ then the $(n+1)$-th excursion from the maximum is identified as the path segment $\{S_n - H_n : n = T_n, ..., T_{n+1}\}$. Moreover, we understand the use of the word straddle to mean that if k is the index of the left end point of the straddling excursion then $k \leq \mathbf{\Gamma}_p$. By the Strong Markov Property for random walks and lack of memory, the completed excursions must have the same law; namely that of a random walk sampled on the time points $\{1, 2, ..., N\}$ conditioned on the event that $\{N \leq \mathbf{\Gamma}_p\}$ and hence ν is geometrically distributed with parameter $1 - P(N \leq \mathbf{\Gamma}_p)$. Mathematically we write

$$(G, S_G) = \sum_{i=1}^{\nu} (N^{(i)}, H^{(i)})$$

where the pairs $\{(N^{(i)}, H^{(i)}) : i = 1, 2, ...\}$ are independent having the same distribution as (N, S_N) conditioned on $\{N \leq \mathbf{\Gamma}_p\}$. Note also that G is the sum of the lengths of the latter conditioned excursions and S_G is the sum of the respective increment of the terminal value over the initial value of each excursion. In other words, (G, S_G) is the component-wise sum of ν independent copies of (N, S_N) (with $(G, S_G) = (0,0)$ if $\nu = 0$). Infinite divisibility follows as a consequence of the fact that (G, S_G) is a geometric sum of i.i.d. random variables; see for example Exercise 1. The independence of (G, S_G) and $(\mathbf{\Gamma}_p - G, S_{\mathbf{\Gamma}_p} - S_G)$ is immediate from the decomposition described above.

Feller's classic Duality Lemma (cf. Feller (1971)) for random walks says that for any $n = 0, 1, 2...$ (which may later be randomized with an independent Geometric distribution), the independence and common distribution of increments implies that $\{S_{n-k} - S_n : k = 0, 1, ..., n\}$ has the same law as $\{-S_k : k = 0, 1, ..., n\}$. In the current context, the Duality Lemma also implies that the pair $(\mathbf{\Gamma}_p - G, S_{\mathbf{\Gamma}_p} - S_G)$ is equal in distribution to (D, S_D). ■

2 The Wiener-Hopf factorisation for Lévy processes

It is reasonably clear that if one is able to decompose the path of a Lévy process into excursions from its maximum in a similar way to the case of random walks, then one might expect a similar independence result to hold such as we have established in Theorem 1.1. Indeed this turns out to be the case and this constitutes the following main result. Some notation first.

As usual, we shall write X for a general Lévy process and moreover we shall understand $\mathbf{e}_p$ to be an independent random variable which is exponentially distributed with mean $1/p$. Further, we define

$$\overline{X}_t = \sup_{s \leq t} X_s \text{ and } \underline{X}_t = \inf_{s \leq t} X_s.$$

Moreover we let

$$\overline{G}_t = \inf\{s < t : X_s = \overline{X}_t\} \text{ and } \underline{G}_t = \sup\{s < t : X_s = \underline{X}_t\}.$$

Theorem 2.1 *Suppose that X is any Lévy process. The pairs*

$$(\overline{G}_{\mathbf{e}_p}, \overline{X}_{\mathbf{e}_p}) \text{ and } (\mathbf{e}_p - \overline{G}_{\mathbf{e}_p}, \overline{X}_{\mathbf{e}_p} - X_{\mathbf{e}_p})$$

are independent and infinitely divisible. Moreover the latter is equal in distribution to $(\underline{G}_{\mathbf{e}_p}, -\underline{X}_{\mathbf{e}_p})$.

Note that the geometric distribution in Theorem 1.1 has been replaced by its natural continuous-time analogue, the exponential distribution. Note also that Theorem 2.1 (and by analogy Theorem 1.1) may be associated with the terminology '*factorisation*' in the sense that, for all $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E}(\mathrm{e}^{\mathrm{i}\alpha \mathbf{e}_p + \mathrm{i}\beta X_{\mathbf{e}_p}}) &= \mathbb{E}(\mathrm{e}^{\mathrm{i}\alpha \overline{G}_{\mathbf{e}_p} + \mathrm{i}\beta \overline{X}_{\mathbf{e}_p}}) \mathbb{E}(\mathrm{e}^{\mathrm{i}(\mathbf{e}_p - \overline{G}_{\mathbf{e}_p}) + \mathrm{i}\beta (X_{\mathbf{e}_p} - \overline{X}_{\mathbf{e}_p})}) \\ &= \mathbb{E}(\mathrm{e}^{\mathrm{i}\alpha \overline{G}_{\mathbf{e}_p} + \mathrm{i}\beta \overline{X}_{\mathbf{e}_p}}) \mathbb{E}(\mathrm{e}^{\mathrm{i}\underline{G}_{\mathbf{e}_p} + \mathrm{i}\beta \underline{X}_{\mathbf{e}_p}}). \end{aligned}$$

As alluded to above, the proof of Theorem 2.1 will require us understand how we can decompose the path of our Lévy process into excursions from the maximum and from this the proof will follow in an analogous way to the proof of Theorem 1.1. However this turns out to be technically much more demanding than one might anticipate. We are going to make things easier for ourselves by proving the result for the case of spectrally negative Lévy processes. Even so we still need to make a digression into the theory of Poisson random measures.

3 Poisson Random Measures

With a view to describing the Wiener-Hopf factorisation, we would like to decompose the paths of a general Lévy process in a different way, through its so-called excursions from the maximum. For this we will need to review some classical theory on Poisson random measure.

Definition 3.1 (Poisson random measure) *In what follows we shall assume that $(S, \mathcal{S}, \eta)$ is an arbitrary σ-finite measure space. Let $N : \mathcal{S} \to \{0, 1, 2, ...\} \cup \{\infty\}$ in such a way that the family $\{N(A) : A \in \mathcal{S}\}$ are random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. Then N is called a Poisson random measure on $(S, \mathcal{S}, \eta)$ (or sometimes a Poisson random measure on S with intensity η) if*

(i) *for mutually disjoint $A_1, ..., A_n$ in $\mathcal{S}$, the variables $N(A_1), ..., N(A_n)$ are independent,*

(ii) *for each $A \in \mathcal{S}$, $N(A)$ is Poisson distributed with parameter $\eta(A)$ (here we allow $0 \leq \eta(A) \leq \infty$),*

(iii) *$N(\cdot)$ is a measure P-almost surely.*

In the second condition we note that if $\eta(A) = 0$ then it is understood that $N(A) = 0$ with probability one and if $\eta(A) = \infty$ then $N(A)$ is infinite with probability one.

Theorem 3.1 *There exists a Poisson random measure $N(\cdot)$ as in Definition 3.1.*

Proof. First suppose that S is such that $0 < \eta(S) < \infty$. There exists a standard construction of an infinite product space, say $(\Omega, \mathcal{F}, P)$ on which the independent random variables

$$\mathtt{N} \text{ and } \{\upsilon_1, \upsilon_2, ...\}$$

are collectively defined such that $\mathtt{N}$ has a Poisson distribution with parameter $\eta(S)$ and each of the variables υ_i have distribution $\eta(\mathrm{d}x)/\eta(S)$ on S. Define for each $A \in \mathcal{S}$,

$$N(A) = \sum_{i=1}^{\mathtt{N}} \mathbf{1}_{(\upsilon_i \in A)}$$

so that $\mathtt{N} = N(S)$. As for each $A \in \mathcal{S}$ and $i \geq 1$, the random variables $\mathbf{1}_{(\upsilon_i \in A)}$ and $\mathtt{N}$ are $\mathcal{F}$-measurable, then so are the random variables $N(A)$.

Fix $k \geq 2$. Suppose now that $A_1, ..., A_k$ are disjoint sets of $\mathcal{S}$ and let $A_0 = S \backslash (A_1 \cup \cdots \cup A_k)$. Suppose that $n_1, n_2, \cdots, n_k$ are non-negative integers such that $\sum_{i=1}^{k} n_i \leq n$ and accordingly define $n_0 = n - \sum_{i \leq k} n_i$. By conditioning on the even $\{\mathtt{N} = n\}$, we have by classical 'balls-in-boxes' combinatorial probability[1] that

$$P(N(A_1) = n_1, ..., N(A_k) = n_k | \mathtt{N} = n) = \frac{n!}{n_0! n_1! \cdots n_k!} \prod_{i=0}^{k} \left(\frac{\eta(A_i)}{\eta(S)} \right)^{n_i}$$

It follows that

$$\begin{aligned}
& P(N(A_1) = n_1, ..., N(A_k) = n_k) \\
&= \sum_{n \geq \Sigma_{i=1}^{k} n_i} \mathrm{e}^{-\eta(S)} \frac{(\eta(S))^n}{n!} \frac{n!}{n_0! n_1! ... n_k!} \prod_{i=0}^{k} \left(\frac{\eta(A_i)}{\eta(S)} \right)^{n_i} \\
&= \sum_{n \geq \Sigma_{i=1}^{k} n_i} \mathrm{e}^{-\eta(A_0)} \frac{\eta(A_0)^{(n - \Sigma_{i=1}^{k} n_i)}}{(n - \Sigma_{i=1}^{k} n_i)!} \left(\prod_{i=1}^{k} \mathrm{e}^{-\eta(A_i)} \frac{(\eta(A_i))^{n_i}}{n_i!} \right) \\
&= \prod_{i=1}^{k} \mathrm{e}^{-\eta(A_i)} \frac{\eta(A_i)^{n_i}}{n_i!}
\end{aligned}$$

Returning to Definition 3.1 it is now clear from the previous calculation that conditions (i)–(iii) are met by $N(\cdot)$. In particular, the third condition is automatic as $N(\cdot)$ is a counting measure by definition.

Next we turn to the case that $(S, \mathcal{S}, \eta)$ is a σ-finite measure space. The meaning of σ-finite is that there exists a countable disjoint exhaustive sequence of sets $B_1, B_2, ...$ in S such that $0 < \eta(B_i) < \infty$ for each $i \geq 1$. Define the measures $\eta_i(\cdot) = \eta(\cdot \cap B_i)$ for each $i \geq 1$. The first part of this proof shows that for each $i \geq 1$ there exists some probability space $(\Omega_i, \mathcal{F}_i, P_i)$ on which we can define a Poisson random measure, say $N_i(\cdot)$, in $(B_i, \mathcal{S} \cap B_i, \eta_i)$ where $\mathcal{S} \cap B_i = \{A \cap B_i : A \in \mathcal{S}\}$ (the reader should verify easily that $\mathcal{S} \cap B_i$ is indeed a sigma algebra on B_i). The idea is now to show that

$$N(\cdot) = \sum_{i \geq 1} N_i(\cdot \cap B_i)$$

is a Poisson random measure on S with intensity η defined on the product space

$$(\Omega, \mathcal{F}, P) := \prod_{i \geq 1} (\Omega_i, \mathcal{F}_i, P_i).$$

First note that it is again immediate from its definition that $N(\cdot)$ is P-almost surely a measure. In particular with the help of Fubini's Theorem, for disjoint

[1] We want to put n balls in the $k+1$ boxes $A_0, A_1, \cdots, A_k$. Note the importance of including $A_0 = S \backslash (A_1 \cup \cdots \cup A_k)$ as a box.

$A_1, A_2, ...$, we have

$$\begin{aligned} N\left(\bigcup_{j\geq 1} A_j\right) &= \sum_{i\geq 1} N_i\left(\bigcup_j A_j \cap B_i\right) = \sum_{i\geq 1}\sum_{j\geq 1} N(A_j\cap B_i) \\ &= \sum_{j\geq 1}\sum_{i\geq 1} N(A_j\cap B_i) \\ &= \sum_{j\geq 1} N(A_j). \end{aligned}$$

Next, for each $i \geq 1$. we have that $N_i(A\cap B_i)$ is Poisson distributed with parameter $\eta_i(A)$; Exercise 2 tells us that under P the random variable $N(A)$ is Poisson distributed with parameter $\eta(A)$. The proof is complete once we show that for disjoint $A_1, ..., A_k$ in $\mathcal{S}$ the variables $N(A_1), ..., N(A_k)$ are all independent under P. However this is obvious since the double array of variables

$$\{N_i(A_j\cap B_i) : i = 1, 2, ... \text{ and } j = 1, ..., k\}$$

is also an independent sequence of variables. ■

From the construction of the Poisson random measure, the following two corollaries should be clear.

Corollary 3.1 *Suppose that $N(\cdot)$ is a Poisson random measure on $(S, \mathcal{S}, \eta)$. Then for each $A\in\mathcal{S}$, $N(\cdot\cap A)$ is a Poisson random measure on $(S\cap A, \mathcal{S}\cap A, \eta(\cdot\cap A))$. Further, if $A, B\in\mathcal{S}$ and $A\cap B = \emptyset$ then $N(\cdot\cap A)$ and $N(\cdot\cap B)$ are independent.*

Corollary 3.2 *Suppose that $N(\cdot)$ is a Poisson random measure on $(S, \mathcal{S}, \eta)$, then the support of $N(\cdot)$ is P-almost surely countable. If in addition, η is a finite measure, then the support is P-almost surely finite.*

Finally, note that if η is a measure with an atom at, say, the singleton $s\in S$ and $\{s\}\in\mathcal{S}$, then it is intuitively obvious from the construction of $N(\cdot)$ in the proof of Theorem 3.1 that $P(N(\{s\})\geq 1) > 0$. Conversely, if η has no atoms then $P(N(\{s\}) = 0) = 1$ for all singletons $s\in S$ such that $\{s\}\in\mathcal{S}$. For further discussion on this point, the reader is referred to Kingman [2].

It should be clear from the proof of Theorem 3.1 that the Poisson random measure N is supported by a countable number of points in S. One often refers to supp$N$ as simply a *Poisson random field*. In the special case that $S = [0,\infty)\times\mathcal{E}$ for some space $\mathcal{E}$ and $\eta = \mathrm{d}t\times\mathrm{d}n$ where n is a measure on $\mathcal{E}$, then we can think of the Poisson random field as a process in time, say $\{\epsilon_t : t\geq 0\}$ where $\epsilon_t\in\mathcal{E}$ if $(t,\epsilon_t)\in$ supp$N$ and otherwise we introduce an isolated point, say ∂, such that $\epsilon = \partial$ when $(t,\epsilon_t)\notin$ supp$N$. In that case we call the process $\epsilon := \{\epsilon_t : t\geq 0\}$ a *Poisson point process* on $\mathcal{E}$ with intensity n.

There is a lot of structure in the definition of a Poisson point process which leads us to the following classical result for Poisson point processes.

Theorem 3.2 (Thinning Theorem) *Suppose that $\{\epsilon_t : t \geq 0\}$ is a Poisson point process on $\mathcal{E}$ with intensity n. Suppose that $A \in \mathcal{E}$ is an n-measurable set such that $n(A) < \infty$ and let*

$$\sigma_A := \inf\{t > 0 : \epsilon_t \in A\}.$$

Then,

(i) *σ_A is a stopping time with respect to the natural filtration of the underlying Poisson point process and is exponentially distributed with parameter $n(A)$,*

(ii) *ϵ_{σ_A} is independent of $\{\epsilon_t : t < \sigma_A\}$ and has law given by*

$$\frac{n(\mathrm{d}\epsilon; A)}{n(A)},$$

(iii) *$\{\epsilon_t : t < \sigma_A\}$ is equal in law to a Poisson point process with intensity $\mathrm{d}t \times n(\mathrm{d}\epsilon; A)$ stopped at an independent and exponentially distributed time with parameter $n(A)$.*

Proof. We give a sketch proof. Suppose that $f : [0, \infty) \times \mathcal{E}$ is any non-negative, bounded measurable function with respect to $\mathrm{d}t \times n(\mathrm{d}\epsilon; A)$. We start by noting that for any n-measurable $B \subseteq A$,

$$\begin{aligned}
&E\left[\mathrm{e}^{-\int_0^t \int_{\mathcal{E}} f(s,\epsilon) N(\mathrm{d}s, \mathrm{d}\epsilon)}; \sigma_A > t; \epsilon_{\sigma_A} \in B\right] \\
&\quad = E\left[\mathrm{e}^{-\int_0^t \int_{\mathcal{E}\setminus A} f(s,\epsilon) N(\mathrm{d}s, \mathrm{d}\epsilon)}; N([0,t] \times A) = 0; \widetilde{\epsilon}_{\widetilde{\sigma}_A} \in B\right]
\end{aligned}$$

where $\{\widetilde{\epsilon}_s : s \geq 0\} = \{\epsilon_{t+s} : s \geq 0\}$ and $\widetilde{\sigma}_A = \inf\{s > 0 : \widetilde{\epsilon}_s \in A\}$. Note in particular that on the event $\{\sigma_{\epsilon_A} > t\} = \{N([0,t] \times A) = 0\}$ (this equality justifies the first statement of the theorem) we have that $\widetilde{\sigma}_A = \sigma_A - t$. Note also that $\{\widetilde{\epsilon}_s : s \geq 0\}$ is independent of $\{\epsilon_s : s \leq t\}$ and hence, making use of Corollary 3.1, we have that

$$\begin{aligned}
&E\left[\mathrm{e}^{-\int_0^t \int_{\mathcal{E}} f(s,\epsilon) N(\mathrm{d}s, \mathrm{d}\epsilon)}; \sigma_A > t; \epsilon_{\sigma_A} \in B\right] \\
&\quad = E\left[\mathrm{e}^{-\int_0^t \int_{\mathcal{E}\setminus A} f(s,\epsilon) N(\mathrm{d}s, \mathrm{d}\epsilon)}\right] E\left[N([0,t] \times A) = 0\right] E\left[\widetilde{\epsilon}_{\widetilde{\sigma}_A} \in B\right] \\
&\quad = E\left[\mathrm{e}^{-\int_0^t \int_{\mathcal{E}\setminus A} f(s,\epsilon) N(\mathrm{d}s, \mathrm{d}\epsilon)}\right] \mathrm{e}^{-n(A)t} E\left[\epsilon_{\sigma_A} \in B\right] \\
&\quad = E\left[\mathrm{e}^{-\int_0^t \int_{\mathcal{E}\setminus A} f(s,\epsilon) N(\mathrm{d}s, \mathrm{d}\epsilon)}\right] \mathrm{e}^{-n(A)t} P\left[\sigma_B < \sigma_{A\setminus B}\right] \qquad (3.1)
\end{aligned}$$

where we have used obvious notation for σ_B and $\sigma_{A\setminus B}$.

At this point in the proof, if one is prepared to accept that $E\left[\mathrm{e}^{-\int_0^t \int_{\mathcal{E}} f(s,\epsilon) N(\mathrm{d}s, \mathrm{d}\epsilon)}\right]$ characterises the law of $\{\epsilon_s : s \leq t\}$, then the penultimate equality of (3.1) already indicates that $\{\epsilon_t : t \leq \sigma_A\}$, ϵ_{σ_A} and σ_A, the latter of which is necessarily exponentially distributed with parameter $n(A)$. Since A is arbitrary, the latter fact may also be used in turn to easily deduce that

$$P\left[\sigma_B < \sigma_{A\backslash B}\right] = \frac{n(B)}{n(B) + n(A\backslash B)} = \frac{n(B)}{n(A)}.$$

This justifies part (ii) and (iii) of the theorem. ■

There is one glaringly obvious example of a Poisson point process that we have already encountered thus far in this course. That is the the case that $\mathcal{E} = \mathbb{R}$ and $n(\mathrm{d}x) = \Pi(\mathrm{d}x)$ where Π is a Lévy measure. Indeed this Poisson random measure describes the jumps in any Lévy process with Lévy measure Π.

To see why, recall that we earlier described the latter as the superposition of a number of (compensated) compound Poisson processes whose jumps are respectively concentrated on the annuli $\{x : |x| \geq 1\}$, $\{x : 2^{-1} \leq |x| < 1\}$, $\{x : 2^{-2} \leq |x| < 2^{-1}\}$, ... , $\{x : 2^{-(n+1)} \leq |x| < 2^{-n}\}$, ... For the n-th of these compound Poisson processes, we could define $N_n((s,t] \times A)$ be the the count of the number of jumps in the time interval $(s,t]$ whose magnitude belongs to $A \subseteq [2^{-(n+1)}, 2^{-n})$. A little thought reveals that N_n is a Poisson point process with intensity $\Pi(\mathrm{d}x)|_{\{x:2^{-(n+1)} \leq |x| < 2^{-n}\}}$. By independence we can define

$$N(\cdot) = \sum_{n\geq 0} N_n(\cdot),$$

where $N_0((s,t] \times A)$ counts the the number of jumps in the time interval $(s,t]$ whose magnitude belongs to $A \subseteq [1,\infty)$, and note that by Corollary 3.1 we have that N is a Poisson point process with intensity Π. Moreover, N necessarily counts all the jumps in the Lévy process associated with Π.

A second important and less obvious example of a Poisson point process embedded within the paths of Lévy processes is discussed in the next section.

4 The ladder process

As our ultimate goal is to show how to decompose events concerning the path of a Lévy process according to the behaviour of the path in individual excursions, we need a way of indexing them. To this end we introduce the notion of *local time at the maximum.* To see that we are staying close in analogy with random walks, note that in the latter case a notion of local time at the maximum is perfectly visible in the discussion of Section 1. Indeed there we spoke freely of sequentially counting excursions from the maximum. Mathematically, this counting process can be described as

$$\ell_n = \max\{k : T_k \leq n\},\ n \geq 0.$$

To avoid trivialities we shall assume throughout this section that neither X nor $-X$ is a subordinator. Recall also the definition $\overline{X}_t = \sup_{s\leq t} X_s$. In the sequel we shall repeatedly refer to the process $\overline{X} - X = \{\overline{X}_t - X_t : t \geq 0\}$, which we also recall from Exercise 4, can be shown to be a strong Markov process.

Definition 4.1 (Local time at the maximum) *A continuous, non-decreasing, $[0, \infty)$-valued, $\mathbb{F}$-adapted process $L = \{L_t : t \geq 0\}$ is called a local time at the maximum (or just local time for short) if the following hold.*

(i) *The support of the Stieltjes measure $\mathrm{d}L_t$ is the closure of the (random) set of times $\{t \geq 0 : \overline{X}_t = X_t\}$ and is finite for each $t \geq 0$.*

(ii) *For every $\mathbb{F}$-stopping time T such that $\overline{X}_T = X_T$ on $\{T < \infty\}$ almost surely, the shifted process*

$$\{L_{T+t} - L_T : t \geq 0\}$$

is independent of $\mathcal{F}_T$ on $\{T < \infty\}$ and has the same law as L under $\mathbb{P}$.

(The process which is identically zero is excluded).

Let us make some remarks about the above definition. Firstly note that since X and $\overline{X} - X$ are strong Markov processes, it also follows from the requirement in part (ii) of the above definition that the shifted trivariate process

$$\{(X_{T+t} - X_T, \overline{X}_{T+t} - X_{T+t}, L_{T+t} - L_T) : t \geq 0\}$$

is independent of $\mathcal{F}_T$ on $\{T < \infty\}$ and has the same law as $\left(X, \overline{X} - X, L\right)$ under $\mathbb{P}$. Next note that if L is a local time then so is kL for any constant $k > 0$. Hence local times can at best be defined uniquely up to a multiplicative constant. On occasion we shall need to talk about both local time and the time scale on which the Lévy process itself is defined. In such cases we shall refer to it as *real time*. Finally, by applying this definition of local time to $-X$ it is clear that one may talk of a local time at the minimum. This will always be referred to as $\widehat{L}$.

Local times as defined above do not always exist on account of the requirement of continuity. Nonetheless, in such cases, it turns out that one may construct right continuous processes which satisfy conditions (i) and (ii) of Definition 4.1 and which serve their purpose equally well in the forthcoming analysis of the Wiener–Hopf factorisation. We leave these cases out in our forthcoming discussion and henceforth, unless otherwise indicated:

We assume that X is a spectrally negative Lévy process.

Theorem 4.1 When X is spectrally negative, we may always take for local time at the maximum, $L = \overline{X}$.

Proof. The claim can s be trivially checked against Definition 4.1. ■

Let us temporarily return to the case of the random walk and note an interesting relation between the ladder times $\{T_n : n \geq 0\}$ and the process $\{\ell_n : n \geq 0\}$. First recall that for each n, T_n is a stopping time and by the Strong Markov property, on the event $\{T_n < \infty\}$ we have that $T_{n+1} - T_n$ is independent of $\{S_k : k \leq T_n\}$ and has the same distribution as $N = \inf\{n > 0 : S_n > 0\}$. Note moreover the simple functional relation

$$T_n = \ell_n^{-1} := \inf\{k \geq 0 : \ell_k \geq n\}.$$

We are thus lead to the conclusion that $\ell^{-1} := \{\ell_n^{-1} : n \geq 0\}$ has the law of a random walk with positive increments, equal in distribution to the common law of T_1 conditional on $\{T_1 < \infty\}$, that jumps to $+\infty$ at the first n for which $T_{n+1} - T_n = \infty$. Said another way, the process ℓ^{-1} is an increasing random walk killed at an independent and geometrically distributed random time with parameter equal to the probability that $T_1 = \infty$. With this in mind, the following result should be thought of as very natural for the inverse process $L^{-1} := \{L_x^{-1} : x \geq 0\}$ where

$$L_x^{-1} = \inf\{t > 0 : L_t > x\}.$$

Theorem 4.2 *If $\mathbb{E}(X_1) \geq 0$ then the process L^{-1} is a subordinator and otherwise it is equal in law to a subordinator killed (i.e. sent to $+\infty$) at an independent and exponentially distributed time.*

Note in particular that

$$L_x^{-1} = \inf\{t > 0 : L_t > x\} = \inf\{t > 0 : \overline{X}_t > x\} = \inf\{t > 0 : X_t > x\},$$

making it the first passage time above level x and, as such, is a stopping time. Hence the above theorem is as much a result about first passage times as it is local time at the maximum. This concurrence is unique to the case of spectrally negative Lévy processes. We also refer to the process L^{-1} as the *ladder time process.*

Before we can prove this theorem we need to introduce the Laplace exponent of a spectrally negative Lévy process. If we revisit the construction of a general spectrally negative Lévy process through the Lévy-Itô decomposition, in particular the use of Theorem 4.1 in Part I of the notes, we see that the Laplace exponent of X,

$$\psi(\lambda) := \frac{1}{t} \log \mathbb{E}(\mathrm{e}^{\lambda X_t}),$$

is well defined for all $\lambda \geq 0$ as soon as the Laplace exponent (in the above sense) of a compound Poisson process with negative jumps is well defined for all $\lambda \geq 0$. However the latter is easily verified. Moreover, again working through the Lévy-Itô decomposition with Laplace exponents instead of characteristic exponents, one easily finds that

$$\psi(\lambda) = -a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} (\mathrm{e}^{\lambda x} - 1 - \lambda x \mathbf{1}_{(|x|<1)})\Pi(\mathrm{d}x).$$

Figure 1: Two examples of ψ, the Laplace exponent of a spectrally negative Lévy process, and the relation to Φ.

for $\lambda \geq 0$.

Henceforth we shall denote its right inverse by

$$\Phi(q) = \sup\{\lambda \geq 0 : \psi(\lambda) = q\}.$$

Exercise 7 shows that on $[0, \infty)$, ψ is infinitely differentiable, strictly convex and that $\psi(0) = 0$ whilst $\psi(\infty) = \infty$. As a particular consequence of these facts, it follows that $\mathbb{E}(X_1) = \psi'(0+) \in [-\infty, \infty)$. In the case that $\mathbb{E}(X_1) \geq 0$, $\Phi(q)$ is the unique solution to $\psi(\theta) = q$ in $[0, \infty)$. When $\mathbb{E}(X_1) < 0$ the latter statement is true only when $q > 0$ and when $q = 0$ there are two roots to the equation $\psi(\theta) = 0$, one of them being $\theta = 0$ and the other being $\Phi(0) > 0$. See Fig. 1 for further clarification.

An immediate consequence of the existence of a Laplace exponent together with stationary and independent increments is the fact that

$$\mathcal{E}_t(\beta) := \mathrm{e}^{\beta X_t - \psi(\beta)t}, \qquad t \geq 0$$

is a martingale with respect to $\{\mathcal{F}_t : t \geq 0\}$ the natural filtration generated by X. When X is a Brownian motion, this martingale is the same as the exponential martingale that is used in the context of the Girsanov change of measure.

As a first step to proving Theorem 4.2 we shall first establish the following lemma.

Lemma 4.1 *For any spectrally negative Lévy process, with $q \geq 0$,*

$$\mathbb{E}(\mathrm{e}^{-qL_x^{-1}} \mathbf{1}_{(L_x^{-1} < \infty)}) = \mathrm{e}^{-\Phi(q)x}.$$

Proof. Fix $q > 0$. Using spectral negativity to write $x = X_{L_x^{-1}}$ on $\{\tau_x^+ < \infty\}$, note with the help of the Strong Markov Property that

$$\begin{aligned}
&\mathbb{E}(\mathrm{e}^{\Phi(q)X_t - qt}|\mathcal{F}_{L_x^{-1}})\\
&= \mathbf{1}_{(L_x^{-1}\geq t)}\mathrm{e}^{\Phi(q)X_t - qt} + \mathbf{1}_{(L_x^{-1}<t)}\mathrm{e}^{\Phi(q)x - qL_x^{-1}}\mathbb{E}(\mathrm{e}^{\Phi(q)(X_t - X_{L_x^{-1}}) - q(t - L_x^{-1})}|\mathcal{F}_{L_x^{-1}}),\\
&= \mathrm{e}^{\Phi(q)X_{t\wedge L_x^{-1}} - q(t\wedge L_x^{-1})}
\end{aligned}$$

where in the final equality we have used the fact that $\mathbb{E}(\mathcal{E}_t(\Phi(q))) = 1$ for all $t \geq 0$. Taking expectations again we have

$$\mathbb{E}(\mathrm{e}^{\Phi(q)X_{t\wedge L_x^{-1}} - q(t\wedge L_x^{-1})}) = 1.$$

Noting that the expression in the latter expectation is bounded above by $\mathrm{e}^{\Phi(q)x}$, an application of dominated convergence yields

$$\mathbb{E}(\mathrm{e}^{\Phi(q)x - qL_x^{-1}}\mathbf{1}_{(L_x^{-1}<\infty)}) = 1$$

which is equivalent to the statement of the theorem.

The case $q = 0$ can be achieved by taking limits as $q \downarrow 0$. ∎

Note in particular that the above result tells us that

$$\mathbb{P}(L_x^{-1} < \infty) = \mathrm{e}^{-\Phi(0)x}$$

for all $x \geq 0$.

Proof of Theorem 4.2. First we claim that $\Phi(q) - \Phi(0)$ is the Laplace exponent of a non-negative infinitely divisible random variable. To see this, note that for all $x \geq 0$,

$$\mathbb{E}(\mathrm{e}^{-qL_x^{-1}}|L_x^{-1} < \infty) = \mathrm{e}^{-(\Phi(q)-\Phi(0))x} = \mathbb{E}(\mathrm{e}^{-qL_1^{-1}}|L_1^{-1} < \infty)^x,$$

and hence in particular

$$\mathbb{E}(\mathrm{e}^{-qL_1^{-1}}|L_1^{-1} < \infty) = \mathbb{E}(\mathrm{e}^{-qL_{1/n}^{-1}}|L_{1/n}^{-1} < \infty)^n$$

showing that $\mathbb{P}(L_1^{-1} \in \mathrm{d}z|L_1^{-1} < \infty)$ for $z \geq 0$ is the law of an infinitely divisible random variable. Next, using the Strong Markov Property, spatial homogeneity and again the special feature of spectral negativity that $\{X_{L_x^{-1}} = x\}$ on the event $\{L_x^{-1} < \infty\}$, we have for $x, y \geq 0$ and $q \geq 0$,

$$\begin{aligned}
&\mathbb{E}(\mathrm{e}^{-q(L_{x+y}^{-1} - L_x^{-1})}\mathbf{1}_{(L_{x+y}^{-1}<\infty)}|\mathcal{F}_{L_x^{-1}})\mathbf{1}_{(L_x^{-1}<\infty)}\\
&= \mathbb{E}(\mathrm{e}^{-qL_y^{-1}}\mathbf{1}_{(L_y^{-1}<\infty)})\mathbf{1}_{(L_x^{-1}<\infty)}\\
&= \mathrm{e}^{-(\Phi(q)-\Phi(0))y}\mathrm{e}^{-\Phi(0)y}\mathbf{1}_{(L_x^{-1}<\infty)}.
\end{aligned}$$

In the first equality we have used standard notation for Markov processes, $\mathbb{E}_x(\cdot) = \mathbb{E}(\cdot|X_0 = x)$. We see then that the increment $L^{-1}_{x+y} - L^{-1}_x$ is independent of $\mathcal{F}_{L^{-1}_x}$ on $\{L^{-1}_x < \infty\}$ and has the same law as the subordinator with Laplace exponent $\Phi(q) - \Phi(0)$ but killed at an independent exponential time with parameter $\Phi(0)$.

When $\mathbb{E}(X_1) \geq 0$ we have that $\Phi(0) = 0$ and hence the concluding statement of the previous paragraph indicates that $\{L^{-1}_x : x \geq 0\}$ is a subordinator (without killing). On the other hand, if $\mathbb{E}(X_1) < 0$, or equivalently $\Phi(0) > 0$ then the second statement of the corollary follows. In particular the rate at which the subordinator is killed is $\Phi(0)$. ■

Corollary 4.1 *Suppose that $q > 0$ and let $\mathbf{e}_q$ be an exponentially distributed random variable which is independent of the spectrally negative Lévy process X. Then $\overline{X}_{\mathbf{e}_q}$ is exponentially distributed with parameter $\Phi(q)$. When $\mathbb{E}(X_1) < 0$ (equivalently $\Phi(0) > 0$), and with the understanding that $\mathbf{e}_0 = \infty$, the previous conclusion still holds when $q = 0$.*

Proof. The result is an easy consequence of the fact that

$$\mathbb{P}(\overline{X}_{\mathbf{e}_q} > x) = \mathbb{P}(\tau^+_x < \mathbf{e}_q) = \mathbb{E}(e^{-q\tau^+_x}\mathbf{1}_{(\tau^+_x<\infty)})$$

together with the conclusion of Theorem 4.1. ■

Note that the last corollary tells us that in the case $\mathbb{E}(X_1) < 0$, the maximum (equivalently the local time at the maximum) increases to a terminal value which is exponentially distributed with parameter $\Phi(0)$.

We conclude this section by considering the analogue of the *ladder height process* for random walks. Quite simply we can now define for each $t < L_\infty$

$$H_t = X_{L^{-1}_t}.$$

Moreover, a special consequence of the fact that X is spectrally negative, and that L^{-1}_t the time to first passage over level t, is that $H_t = t$ on $t < L_\infty$.

5 Excursions

Now that we have established the concept of local time at the maximum for any Lévy process we can give the general decomposition of the path of a Lévy process in terms of its excursions from the maximum.

Definition 5.1 *For each moment of local time $t > 0$ we define*

$$\epsilon_t = \begin{cases} \{X_{L^{-1}_{t-}+s} - X_{L^{-1}_{t-}} : 0 < s \leq L^{-1}_t - L^{-1}_{t-}\} & \text{if } L^{-1}_{t-} < L^{-1}_t \\ \partial & \text{if } L^{-1}_{t-} = L^{-1}_t \end{cases}$$

where we take $L_{0-}^{-1} = 0$ and ∂ is some "dummy" state. Note that for each fixed $t > 0$, when $L_{t-}^{-1} < L_t^{-1}$, the object ϵ_t is a stochastic process and hence is double indexed with $\epsilon_t(s) = X_{L_{t-}^{-1}+s} - X_{L_{t-}^{-1}}$ for $0 < s \leq L_t^{-1} - L_{t-}^{-1}$. When $\epsilon_t \neq \partial$ we refer to it as the excursion (from the maximum) associated with local time t.

Note also that for t such that $\epsilon_t \neq \partial$, ϵ_t has paths that are right continuous with left limits and, with the exception of its terminal value (in the case that $L_t^{-1} < \infty$), is valued in $(-\infty, 0)$.

Definition 5.2 *Let $\mathcal{E}$ be the space of excursions of X from its running supremum. That is the space of mappings which are right continuous with left limits satisfying*

$$\epsilon : (0, \zeta] \to (-\infty, 0] \quad \text{for some } \zeta \in (0, \infty]$$

where $\zeta = \zeta(\epsilon)$ is the excursion length. Finally let $\overline{\epsilon} = -\inf_{s\in(0,\zeta)} \epsilon(s)$ for the excursion height.

Theorem 5.1 *There exists a σ-algebra Σ and σ-finite measure n such that $(\mathcal{E}, \Sigma, n)$ is a measure space and Σ is rich enough to contain sets of the form*

$$\{\epsilon \in \mathcal{E} : \zeta(\epsilon) \in A,\ \overline{\epsilon} \in B\}$$

where, for a given $\epsilon \in \mathcal{E}$, $\zeta(\epsilon)$ and $\overline{\epsilon}$ were all given in Definition 5.2. Further, A and B are Borel sets of $[0, \infty]$.

(i) *If $\mathbb{E}(X_1) \geq 0$ then $\{(t, \epsilon_t) : t \geq 0 \text{ and } \epsilon_t \neq \partial\}$ is a Poisson point process on $([0, \infty) \times \mathcal{E}, \mathcal{B}[0, \infty) \times \Sigma, \mathrm{d}t \times \mathrm{d}n)$.*

(ii) *If $\mathbb{E}(X_1) < 0$ then $\{(t, \epsilon_t) : t \leq L_\infty \text{ and } \epsilon_t \neq \partial\}$ is a Poisson point process on $([0, \infty) \times \mathcal{E}, \mathcal{B}[0, \infty) \times \Sigma, \mathrm{d}t \times \mathrm{d}n)$ stopped at the first arrival of an excursion in $\mathcal{E}_\infty := \{\epsilon \in \mathcal{E} : \zeta(\epsilon) = \infty\}$.*

We offer no proof for this result. We refer instead to Bertoin (1996) who gives a rigorous treatment. However, the intuition behind this theorem lies with the observation that for each $t > 0$, L_{t-}^{-1} is a stopping time. To see why note that

$$\{L_{t-}^{-1} < s\} = \bigcap_{n\geq 1} \{L_{t-1/n}^{-1} < s\} \in \mathcal{F}_s.$$

The Strong Markov Property for Lévy processes thus tells us that the progression of $X_{L_{t-}^{-1}+s} - X_{L_{t-}^{-1}}$ in the time interval $(L_{t-}^{-1}, L_t^{-1}]$ is independent of $\mathcal{F}_{L_{t-}^{-1}}$. As alluded to earlier, this means that the paths of X may be decomposed into the juxtaposition of independent excursions from the maximum. Excursions from the maximum are interlaced by moments of real time where X can be described as *drifting at its maximum.* That is to say, moments of real time which contribute to a strict increase in the Lebesgue measure of the real time the process spends at its maximum. The aforesaid Lebesgue measure may be identically zero for some processes. If there is a last maximum, then the process of excursions is stopped at the first arrival of an excursion with infinite length; i.e. the first arrival of an excursion in $\mathcal{E}_\infty$.

Theorem 5.1 generalises the statement of Theorem 4.2. To see why, suppose that we write

$$\Lambda(\mathrm{d}x) = n(\zeta(\epsilon) \in \mathrm{d}x). \tag{5.1}$$

On $\{t < L_\infty\}$ the jumps of the ladder time process L^{-1} form a Poisson process on $[0,\infty) \times (0,\infty)$ with intensity measure $\mathrm{d}t \times \Lambda(\mathrm{d}x)$. We can write L_t^{-1} as the sum of the Lebesgue measure of the time X spends drifting at the maximum (if at all) plus the jumps L^{-1} makes due to excursions from the maximum. Hence, if N is the counting measure associated with the Poisson point process of excursions, then on $\{L_\infty > t\}$,

$$\begin{aligned} L_t^{-1} &= \int_0^{L_t^{-1}} \mathbf{1}_{(\epsilon_s = \partial)} \mathrm{d}s + \int_{[0,t]} \int_{\mathcal{E}} \zeta(\epsilon)\, N(\mathrm{d}s \times \mathrm{d}\epsilon) \\ &= \int_0^{L_t^{-1}} \mathbf{1}_{(\overline{X}_s = X_s)} \mathrm{d}s + \int_{[0,t]} \int_{\mathcal{E}} \zeta(\epsilon)\, N(\mathrm{d}s \times \mathrm{d}\epsilon) \\ &= \mathtt{a}t + + \int_{[0,t]} \int_{\mathcal{E}} \zeta(\epsilon)\, N(\mathrm{d}s \times \mathrm{d}\epsilon), \end{aligned} \tag{5.2}$$

where $\mathtt{a} \geq 0$ is necessarily a constant on account of the fact that L^{-1} is a (possibly killed) subordinator.

We can also see that $\mathbb{P}(L_\infty > t)$ is the probability that in the process of excursions the first arrival in $\mathcal{E}_\infty$ is after time t. Written in terms of the Poisson point process of excursions we see that

$$\mathbb{P}(L_\infty > t) = \mathbb{P}(N([0,t] \times \mathcal{E}_\infty) = 0) = \mathrm{e}^{-n(\mathcal{E}_\infty)t}.$$

This reinforces the earlier conclusion that L_∞ is exponentially distributed and we equate the parameters

$$\Phi(0) = n(\mathcal{E}_\infty). \tag{5.3}$$

6 Proof of Theorem 2.1

The crux of the first part of the Wiener–Hopf factorisation lies with the following important observation. Consider the Poisson point process of *marked excursions* on

$$([0,\infty) \times \mathcal{E} \times [0,\infty), \mathcal{B}[0,\infty) \times \Sigma \times \mathcal{B}[0,\infty), \mathrm{d}t \times \mathrm{d}n \times \mathrm{d}\eta)$$

where $\eta(\mathrm{d}x) = p\mathrm{e}^{-px}\mathrm{d}x$ for $x \geq 0$. That is to say, a Poisson point process whose points are described by $\{(t, \epsilon_t, \mathbf{e}_p^{(t)}) : t \leq L_\infty \text{ and } \epsilon_t \neq \partial\}$ where $\mathbf{e}_p^{(t)}$ is an independent copy of an exponentially distributed random variable if t is such that $\epsilon_t \neq \partial$, and otherwise $\mathbf{e}_p^{(t)} := \partial$. The Poisson point process of unmarked excursions is then obtained as a projection on to $[0,\infty) \times \mathcal{E}$. Sampling the Lévy process X up to an independent exponentially distributed random time

$\mathbf{e}_p$ corresponds to sampling the Poisson process of excursions up to time $L_{\mathbf{e}_p}$; that is $\{(t, \epsilon_t) : t \leq L_{\mathbf{e}_p}$ and $t \neq \partial\}$. In turn, we claim that this process is equal in law to the projection on to $[0, \infty) \times \mathcal{E}$ of

$$\{(t, \epsilon_t, \mathbf{e}_p^{(t)}) : t \leq \sigma_1 \wedge \sigma_2 \text{ and } \epsilon_t \neq \partial\} \tag{6.1}$$

where

$$\sigma_1 := \inf\{t > 0 : \int_0^{L_t^{-1}} \mathbf{1}_{(\overline{X}_s = X_s)} \mathrm{d}s > \mathbf{e}_p\}$$

and

$$\sigma_2 := \inf\{t > 0 : \zeta(\epsilon_t) > \mathbf{e}_p^{(t)}\}$$

where we recall that $\zeta(\epsilon_t)$ is the duration of the excursion indexed by local time t. Note that in the case that the constant $\mathtt{a}$ in (5.2) is zero, in other words $\int_0^{\cdot} \mathbf{1}_{(\overline{X}_s = X_s)} \mathrm{d}s = 0$, we have simply that $\sigma_1 = \infty$. A formal proof of this claim would require the use of some additional mathematical tools. However, for the sake of brevity, we shall lean instead on an intuitive explanation as follows.

We recall that the path of the Lévy process up to time $\mathbf{e}_p$ is the independent juxtaposition of excursions interlaced with moments of real time when $X = \overline{X}$ (which accumulate positive Lebesgue measure when $\mathtt{a} > 0$). The event $\{t < L_{\mathbf{e}_p}\}$ corresponds to the event that there are at least t units of local time for a given stretch of $\mathbf{e}_p$ units of real time. By the lack of memory property this is equivalent to the event that the total amount of real time accumulated at the maximum by local time t has survived independent exponential killing at rate p as well as each of the excursion lengths up to local time t have survived independent exponential killing at rate p.

The times σ_1 and σ_2 are independent and further σ_2 is of the type of stopping time considered in Theorem 3.2 with $A = \{\zeta(\epsilon) > \mathbf{e}_p\}$ when applied to the Poisson point process (6.1). From each of the statements given in Lemma 3.2 we deduce three facts concerning the Poisson point process (6.1).

(1) Since $\int_0^{L_t^{-1}} \mathbf{1}_{(\overline{X}_s = X_s)} \mathrm{d}s = \mathtt{a}t$ we have

$$\mathbb{P}(\sigma_1 > t) = \mathbb{P}\left(\int_0^{L_t^{-1}} \mathbf{1}_{(\overline{X}_s = X_s)} \mathrm{d}s < \mathbf{e}_p\right) = \mathrm{e}^{-\mathtt{a}pt}.$$

As mentioned earlier, if the constant $\mathtt{a} = 0$ then we have that $\sigma_1 = \infty$. Further, we also have that

$$\mathbb{P}(\sigma_2 > t) = \exp\left\{-t \int_0^{\infty} p\mathrm{e}^{-px} \mathrm{d}x \cdot n(\zeta(\epsilon) > x)\right\},$$

i.e. σ_2 is exponentially distributed. As σ_1 and σ_2 are independent it follows that $\sigma_1 \wedge \sigma_2$ is independent with the sum of their rates, say

$$\kappa = \mathtt{a}p + \int_0^{\infty} p\mathrm{e}^{-px} \mathrm{d}x \cdot n(\zeta(\epsilon) > x)$$

(2) The Poisson point process (6.1) is equal in law to a Poisson point process on $[0, \infty) \times \mathcal{E} \times [0, \infty)$ with intensity

$$\mathrm{d}t \times n(\mathrm{d}\epsilon; \zeta(\epsilon) < x) \times \eta(\mathrm{d}x) \tag{6.2}$$

which is stopped at an independent time which is exponentially distributed with parameter κ.

(3) On the event $\sigma_2 < \sigma_1$, the process

$$\{(t, \epsilon_t, \mathbf{e}_p^{(t)}) : t < \sigma_1 \wedge \sigma_2 \text{ and } \epsilon_t \neq \partial\} \tag{6.3}$$

is independent of $\epsilon_{\sigma_2} = \epsilon_{\sigma_1 \wedge \sigma_2}$. On the other hand, when $\sigma_1 < \sigma_2$ since at the local time σ_1 we have $\partial = \epsilon_{\sigma_1} = \epsilon_{\sigma_1 \wedge \sigma_2}$ We conclude that $\epsilon_{\sigma_1 \wedge \sigma_2}$ is independent of (6.3).

Now note that

$$\overline{G}_{\mathbf{e}_p} \stackrel{d}{=} L^{-1}_{(\sigma_1 \wedge \sigma_2)-} = \mathtt{a}(\sigma_1 \wedge \sigma_2) + \int_{[0, \sigma_1 \wedge \sigma_2)} \int_{\mathcal{E}} \zeta(\epsilon_t) N(\mathrm{d}t \times \mathrm{d}\epsilon) \tag{6.4}$$

and

$$\overline{X}_{\mathbf{e}_p} \stackrel{d}{=} H_{(\sigma_1 \wedge \sigma_2)-} = \sigma_1 \wedge \sigma_2. \tag{6.5}$$

From point (3) above, the random variables on the right hand sides of (6.4) and (6.5) are independent of the excursion $\epsilon_{\sigma_1 \wedge \sigma_2}$. Moreover, the pair $(\mathbf{e}_p - \overline{G}_{\mathbf{e}_p}, X_{\mathbf{e}_p} - \overline{X}_{\mathbf{e}_p})$ are equal in law to

$$(\mathbf{e}_p^{(\sigma_1 \wedge \sigma_2)}, \epsilon_{\sigma_1 \wedge \sigma_2}(\mathbf{e}_p^{(\sigma_1 \wedge \sigma_2)}))\mathbf{1}_{(\sigma_2 < \sigma_1)} + (0, 0)\mathbf{1}_{(\sigma_1 < \sigma_2)}$$

. In conclusion $(\overline{G}_{\mathbf{e}_p}, \overline{X}_{\mathbf{e}_p})$ is independent of $(\mathbf{e}_p - \overline{G}_{\mathbf{e}_p}, \overline{X}_{\mathbf{e}_p} - X_{\mathbf{e}_p})$.

From point (2), the process $\{L_t^{-1} : t < \sigma_1 \wedge \sigma_2\}$ behaves like a subordinator with characteristic measure

$$\int_0^\infty p\mathrm{e}^{-pt} \mathrm{d}t \cdot n(\zeta(\epsilon) \in \mathrm{d}x, x < t) = \mathrm{e}^{-px} \Lambda(\mathrm{d}x)$$

and drift $\mathtt{a}$ which is stopped at an independent exponentially distributed time with parameter κ. Suppose that we denote this subordinator $\mathbb{L}^{-1} = \{\mathbb{L}_t^{-1} : t \geq 0\}$. Then

$$(\mathbb{L}^{-1}_{\mathbf{e}_\kappa}, \mathbf{e}_\kappa) \stackrel{d}{=} (\overline{G}_{\mathbf{e}_p}, \overline{X}_{\mathbf{e}_p})$$

where $\mathbf{e}_\kappa$ is an independent exponential random variable with parameter κ.

Whilst it is clear that $\mathbf{e}_\kappa$ is infinitely divisible, we leave it as an exercise for the reader to show that any subordinator sampled at an independent and exponentially distributed random time is also infinitely divisible. See Exercise 8.

Finally to conclude that $(\mathbf{e}_p - \overline{G}_{\mathbf{e}_p}, \overline{X}_{\mathbf{e}_p} - X_{\mathbf{e}_p})$ and $(\underline{G}_{\mathbf{e}_p}, -\underline{X}_{\mathbf{e}_p})$ are equal in distribution, we use exactly the same argument as in the random walk case. Specifically we need the analogue of Feller's Duality Lemma which we give below, the proof of which is essentially the same as in the random walk case.

Figure 2: Duality of the processes $X = \{X_s : s \leq t\}$ and $Y = \{X_{(t-s)-} - X_t : s \leq t\}$. The path of Y is a reflection of the path of X with an adjustment of continuity at jump times.

Lemma 6.1 (Duality Lemma) *For each fixed $t > 0$, define the reversed process*

$$\{X_{(t-s)-} - X_t : 0 \leq s \leq t\}$$

and the dual process,

$$\{-X_s : 0 \leq s \leq t\}.$$

Then the two processes have the same law under $\mathbb{P}$.

Proof. Define the time reversed process $Y_s = X_{(t-s)-} - X_t$ for $0 \leq s \leq t$ and note that under $\mathbb{P}$ we have $Y_0 = 0$ almost surely as t is a jump time with probability zero. (For the last statement recall that jumps appear as the countable superposition of compensated Poisson point processes). As can be seen from Fig. 2 (which is to be understood symbolically), the paths of Y are obtained from those of X by a reflection about the vertical axis with an adjustment of the continuity at the jump times so that its paths are almost surely right continuous with left limits. Further, the stationary independent increments of X imply directly the same as is true of Y. Further, for each $0 \leq s \leq t$, the distribution of $X_{(t-s)-} - X_t$ is identical to that of $-X_s$ and hence, since the finite time distributions of Y determine its law, the proof is complete. ■

Exercises

Exercise 1 Suppose that $S = \{S_n : n \geq 0\}$ is any random walk and $\mathbf{\Gamma}_p$ is an independent random variable with a geometric distribution on $\{0, 1, 2, ...\}$ with parameter p.

(i) Show that $\mathbf{\Gamma}_p$ is infinitely divisible.

(ii) Show that $S_{\mathbf{\Gamma}_p}$ is infinitely divisible.

Exercise 2 The object of this exercise is to give a reminder of the additive property of Poisson distributions (which is also the reason why they belong to the class of infinite divisible distributions). Suppose that $\{N_i : i = 1, 2, ...\}$ is an independent sequence of random variables defined on $(\Omega, \mathcal{F}, P)$ which are Poisson distributed with parameters λ_i for $i = 1, 2, ...$, respectively. Let $S = \sum_{i\geq 1} N_i$. Show that

(i) if $\sum_{i\geq 1} \lambda_i < \infty$ then S is Poisson distributed with parameter $\sum_{i\geq 1} \lambda_i$ and hence in particular $P(S < \infty) = 1$,

(ii) if $\sum_{i\geq 1} \lambda_i = \infty$ then $P(S = \infty) = 1$.

Exercise 3 Denote by $\{T_i : i \geq 1\}$ the arrival times in the Poisson process $N = \{N_t : t \geq 0\}$ with parameter λ.

(i) By recalling that inter-arrival times are independent and exponential, show that for any $A \in \mathcal{B}([0, \infty)^n)$,

$$P((T_1, ..., T_n) \in A | N_t = n) = \int_A \frac{n!}{t^n} \mathbf{1}_{(0\leq t_1 \leq ... \leq t_n \leq t)} \mathrm{d}t_1 \times ... \times \mathrm{d}t_n.$$

(ii) Deduce that the distribution of $(T_1, ..., T_n)$ conditional on $N_t = n$ has the same law as the distribution of an ordered independent sample of size n taken from the uniform distribution on $[0, t]$.

Exercise 4 Show that for any $y \geq 0$,

$$\{(y \vee \overline{X}_t) - X_t : t \geq 0\} \text{ and } \{X_t - (\underline{X}_t \wedge (-y)) : t \geq 0\}$$

are $[0, \infty)$-valued strong Markov process.

Exercise 5 Let X be a Lévy process with Lévy measure Π. Denote by N the Poisson random measure associated with its jumps.

(i) Show that

$$\mathbb{P}(\sup_{0<s\leq t} |X_s - X_{s-}| \geq a) = 1 - \mathrm{e}^{-t\Pi(\mathbb{R}\backslash(-a,a))}$$

for $a > 0$.

(ii) Show that the paths of X are continuous if and only if $\Pi = 0$.

(iii) Show that the paths of X are piece-wise linear if and only if it is a compound Poisson process with drift if and only if $\sigma = 0$ and $\Pi(\mathbb{R}) < \infty$. [Recall that a function $f : [0, \infty) \to \mathbb{R}$ is right continuous and piece-wise linear if there exist sequence of times $0 = t_0 < t_1 < ... < t_n < ...$ with $\lim_{n\uparrow\infty} t_n = \infty$ such that on $[t_{j-1}, t_j)$ the function f is linear].

(iv) Now suppose that $\Pi(\mathbb{R}) = \infty$. Argue by contradiction that for each positive rational $q \in \mathbb{Q}$ there exists a decreasing sequence of jump times for X, say $\{T_n(\omega) : n \geq 0\}$, such that $\lim_{n\uparrow\infty} T_n = q$. Hence deduce that the set of jump times are dense in $[0, \infty)$.

Exercise 6 Suppose that X is a compound Poisson process with drift $\delta \geq 0$. Just by considering the piece-wise linearity of the paths of these processes, one has obviously that over any finite time horizon, the time spent at the maximum has strictly positive Lebesgue measure with probability one. Hence the quantity

$$L_t := \int_0^t \mathbf{1}_{(\overline{X}_s = X_s)} \mathrm{d}s, \quad t \geq 0 \tag{E.1}$$

is almost surely positive.

(i) Show that in fact $\{L_t : t \geq 0\}$ is a local time according to Definition 4.1.

(ii) If we only allow negative jumps and $\delta > 0$ (in particular, X is a spectrally negative Lévy process), show that the local time in (i) agrees with the $\overline{X}$ up to a multiplicative constant. Specifically, show that

$$\overline{X}_t = \delta \int_0^t \mathbf{1}_{(\overline{X}_s = X_s)} \mathrm{d}s$$

for all $t \geq 0$.

Exercise 7 Suppose that ψ is the Laplace exponent of a spectrally negative Lévy process. By considering explicitly the formula

$$\psi(\beta) = -a\beta + \frac{1}{2}\sigma^2\beta^2 + \int_{(-\infty,0)} (\mathrm{e}^{\beta x} - 1 - \beta x \mathbf{1}_{(x>-1)})\Pi(\mathrm{d}x)$$

show that on $[0, \infty)$, ψ is infinitely differentiable, strictly convex and that $\psi(0) = 0$ whilst $\psi(\infty) = \infty$.

Exercise 8 *Suppose that X is any Lévy process with characteristic exponent Ψ. Let $\mathbf{e}_p$ be an independent and exponentially distributed random variable.*

(i) *Show that, on the one hand*

$$\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\theta X_{\mathbf{e}_p}}\right) = \frac{p}{p + \Psi(\theta)}.$$

(ii) *Show on the other hand, using the Frullani integral that*

$$\exp\left\{-\int_0^\infty \int_{\mathbb{R}} (1 - \mathrm{e}^{\mathrm{i}\theta x}) \frac{1}{t} \mathrm{e}^{-pt} \mathbb{P}(X_t \in \mathrm{d}x)\, \mathrm{d}t\right\}$$
$$= \frac{p}{p + \Psi(\theta)}.$$

(iii) *Deduce that $X_{\mathbf{e}_p}$ is infinitely divisible.*

References

[1] Bertoin, J. (1996) *Lévy Processes*. Cambridge University Press, Cambridge.

[2] Kingman, J.F.C. (1993) *Poisson Processes*. Oxford University Press, Oxford.