Lévy processes, the Wiener-Hopf factorisation and applications: part III

Andreas E. Kyprianou,
Department of Mathematical Sciences,
University of Bath,
Claverton Down,
Bath, BA2 7AY.
a.kyprianou@bath.ac.uk

# 1 The $M/G/1$ Queue

Recall that an $M/G/1$ queue consists of a single server who receives customers at the times of a Poisson process at rate $\lambda > 0$ that wait in line to be served on a sequential basis in order of arrival. The $i$-th customer comes with a workload for the server given by the random variable $\xi_i$. The quantities $\{\xi_i : i \geq 1\}$ are i.i.d. The server processes incoming work at a constant (unit) rate as long as the queue is not empty. When all work has been processes and there are no customers, the server remains idle until the next arrival of a customer.

It turns out that one can model the workload of the server via a Lévy process. Indeed, suppose that

$$X_t = t - \sum_{i=1}^{N_t} \xi_i$$

where $\{N_t : t \geq 0\}$ is the Poisson process of arrivals. We know that the process $X$ has Laplace exponent

$$\psi(\beta) = \beta - \int_{(0,\infty)} (1 - \mathrm{e}^{-\beta x})\lambda F(\mathrm{d}x), \ \beta \geq 0,$$

where $F$ is the distribution function of $\xi_1$. For this elementary example of a spectrally negative Lévy process, by writing $S_t = \sum_{i=1}^{N_t} \xi_i$, $t \geq 0$, we have that

$$\begin{aligned}
\overline{X}_t &= \int_0^t \mathbf{1}_{(\overline{X}_s = X_s)} \mathrm{d}X_s \\
&= \int_0^t \mathbf{1}_{(\overline{X}_s = X_s)} \mathrm{d}s - \int_0^t \mathbf{1}_{(\overline{X}_s = X_s)} \mathrm{d}S_s \\
&= \int_0^t \mathbf{1}_{(\overline{X}_s = X_s)} \mathrm{d}s
\end{aligned}$$

almost surely where the final equality follows as a consequence of the fact that

$$\int_0^t \mathbf{1}_{(\overline{X}_s = X_s)} \mathrm{d}S_s \leq \int_0^t \mathbf{1}_{(\Delta S_s = 0)} \mathrm{d}S_s = 0.$$

Then writing $W_t^y$ for the workload at time $t \geq 0$ when the initial workload is $y \geq 0$. It is straightforward to check that

$$W_t^y = (y \vee \overline{X}_t) - X_t, \ t \geq 0,$$

where $\overline{X}_t = \sup_{s \leq t} X_s$.

Let us introduce the parameter

$$\rho := \lambda \mathbb{E}(\xi_1),$$

also known as the *traffic intensity*. Note that regimes $0 < \rho < 1$, $\rho = 1$ and $\rho > 1$ correspond precisely to the regimes $\psi'(0+) > 0$, $\psi'(0+) = 0$ and $\psi'(0+) < 0$, respectively. The first two of these cases thus imply that $\Phi(0) = 0$ and the third case implies $\Phi(0) > 0$ where we recall that, for $q \geq 0$,

$$\Phi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}.$$

We are interested in characterising two main quantities associated with the workload; that is, where appropriate, the distribution of the idle period and the stationary distribution of the workload.

**Theorem 1.1** *Fix $y \geq 0$. Suppose that $\rho > 1$ then the total time that the M/G/1 queue spends idle,*

$$I := \int_0^\infty \mathbf{1}_{(W_t^y = 0)} \mathrm{d}t,$$

*has the following distribution*

$$\mathbb{P}\left(I \in \mathrm{d}x\right) = (1 - \mathrm{e}^{-\Phi(0)y})\delta_0\left(\mathrm{d}x\right) + \Phi(0)\mathrm{e}^{-\Phi(0)(y+x)}\mathrm{d}x.$$

*Otherwise if $0 < \rho \leq 1$ then $I$ is infinite with probability one.*

**Proof.** Now recall that $\overline{X}_\infty$ is exponentially distributed with parameter $\Phi(0)$. When $\Phi(0) = 0$ then the previous statement is understood to mean that $\mathbb{P}(\overline{X}_\infty = \infty) = 1$. When $y = 0$ we have that

$$\overline{X}_\infty = \int_0^\infty \mathbf{1}_{(\overline{X}_s = X_s)} \mathrm{d}s = \int_0^\infty \mathbf{1}_{(W_s^0 = 0)} \mathrm{d}s. \quad (1.1)$$

Hence we see that $I$ is exponentially distributed with parameter $\Phi(0)$. Recalling which values of $\rho$ imply that $\Phi(0) > 0$ we see that the statement of the theorem follows for the case $y = 0$.

In general however, when $y > 0$ the equality (1.1) is not valid. Instead we have that

$$\begin{aligned}
\int_0^\infty \mathbf{1}_{(W_s^y = 0)} \mathrm{d}s &= \int_0^{\tau_y^+} \mathbf{1}_{(W_s^y = 0)} \mathrm{d}s + \int_{\tau_y^+}^\infty \mathbf{1}_{(W_s^y = 0)} \mathrm{d}s \\
&= \mathbf{1}_{(\tau_y^+ < \infty)} \int_{\tau_y^+}^\infty \mathbf{1}_{(W_s^y = 0)} \mathrm{d}s \\
&= \mathbf{1}_{(\overline{X}_\infty \geq y)} I^*, \quad (1.2)
\end{aligned}$$

where $I^*$ is independent of $\mathcal{F}_{\tau_y^+}$ on $\{\tau_y^+ < \infty\}$ and equal in distribution to $\int_0^\infty \mathbf{1}_{(W_s^y=0)}\mathrm{d}s$ with $y=0$. Note that the first integral in the right-hand side of the first equality disappears on account of the fact that $W_s^y > 0$ for all $s < \tau_y^+$. The statement of the theorem now follows for $0 < \rho \leq 1$ by once again recalling that in this regime $\Phi(0) = 0$ and hence from (1.2) $\overline{X}_\infty = \infty$ with probability one, which, in turn, implies that $I = I^*$. This quantity has previously been shown to be infinite with probability one. On the other hand, when $\rho > 1$, we see from (1.2) that there is an atom at zero corresponding to the event $\{\overline{X}_\infty < y\}$ with probability $1 - \mathrm{e}^{-\Phi(0)y}$. Otherwise, with independent probability $\mathrm{e}^{-\Phi(0)y}$, the integral $I$ has the same distribution as $I^*$. Again from previous calculations for the case $w = 0$ we have seen that this is exponential with parameter $\Phi(0)$ and the proof is complete. ∎

The next result looks at the stationary distribution of the workload. To this end, let us note from the Wiener-Hopf factorisation that, since $\overline{X}_{\mathbf{e}_q}$ is exponentially distributed with parameter $\Phi(q)$ when $q > 0$, we have for $\theta \in \mathbb{R}$

$$\begin{aligned}\frac{q}{q+\Psi(\theta)} &= \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta X_{\mathbf{e}_q}}) \\ &= \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta \overline{X}_{\mathbf{e}_q}})\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta \underline{X}_{\mathbf{e}_q}}) \\ &= \frac{\Phi(q)}{\Phi(q)-\mathrm{i}\theta}\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta \underline{X}_{\mathbf{e}_q}}).\end{aligned}$$

where we recall that $\Psi(\theta)$ is the characteristic exponent of $X$. Rewriting we have

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta \underline{X}_{\mathbf{e}_q}}) = \frac{q}{\Phi(q)}\frac{\Phi(q)-\mathrm{i}\theta}{q+\Psi(\theta)}.$$

Note that both left and right hand sides of the above identity can be analytically extended to the complex half plane which admits a non-negative real part. Indeed, for the right hand side, the existence of a Laplace exponent $\psi(\beta)$ for $\beta \geq 0$ means that we can identify $\psi(\beta) = -\Psi(-\mathrm{i}\beta)$ and moreover,

$$\mathbb{E}(\mathrm{e}^{\beta \underline{X}_{\mathbf{e}_q}}) = \frac{q}{\Phi(q)}\frac{\Phi(q)-\beta}{q-\psi(\beta)}. \tag{1.3}$$

Assuming that $\mathbb{E}(X_1) > 0$ so that $\Phi(0) = 0$ we have, noting that $\psi(\Phi(q)) = q$,

$$\mathbb{E}(\mathrm{e}^{\beta \underline{X}_\infty}) = \lim_{q\downarrow 0}\frac{q}{\Phi(q)}\frac{\Phi(q)-\beta}{q-\psi(\beta)} = \psi'(0+)\frac{\beta}{\psi(\beta)} = \mathbb{E}(X_1).\frac{\beta}{\psi(\beta)} \tag{1.4}$$

On the other hand, suppose that $\mathbb{E}(X_1) \leq 0$ so that $\Phi(0) > 0$. In that case it is trivial to see that

$$\mathbb{E}(\mathrm{e}^{\beta \underline{X}_\infty}) = 0$$

showing that

$$\mathbb{P}(-\underline{X}_\infty = \infty) = 1.$$

These results lead us to the following conclusion.

**Theorem 1.2** *Fix $y \geq 0$.*

*(i) Suppose that $\mathbb{E}(X_1) > 0$ then $W^y$ has a stationary distribution equal to that of the random variable $W^y_\infty$ with Laplace transform*

$$\mathbb{E}(\mathrm{e}^{-\beta W^y_\infty}) = \mathbb{E}(X_1)\frac{\beta}{\psi(\beta)}$$

*(ii) Suppose that $\mathbb{E}(X_1) \leq 0$ then $W^y$ does not converge in distribution.*

**Proof.** First suppose that $\mathbb{E}(X_1) > 0$ which implies that $\Phi(0) = 0$. This implies that $\mathbb{P}(\overline{X}_\infty = \infty) = 1$ and accordingly $\mathbb{P}(\tau^+_y < \infty) = 1$. It follows that for all sufficiently large $t$, $W^y_t = \overline{X}_t - X_t$ and hence, for $\beta \geq 0$, by the Duality Lemma,

$$\lim_{t\uparrow\infty} \mathbb{E}(\mathrm{e}^{-\beta W^y_t}) = \lim_{t\uparrow\infty} \mathbb{E}(\mathrm{e}^{-\beta(\overline{X}_t - X_t)}) = \lim_{t\uparrow\infty} \mathbb{E}(\mathrm{e}^{\beta \underline{X}_t}) = \mathbb{E}(X_1)\frac{\beta}{\psi(\beta)}.$$

A similar argument establishes the second part of the theorem. ■

It is worth noting that the previous theorem does not require the underlying spectrally negative Lévy process to have paths with a compound Poisson jump structure. The computations work verbatim for any spectrally negative Lévy process. It is on this basis that we move forward to the theory of scale functions and exit problems.

# 2 Scale functions and insurance risk

Consider the following model of the revenue of an insurance company as a process in time proposed by [8]. The insurance company collects premiums at a fixed rate $c > 0$ from its customers. At times of a Poisson process, a customer will make a claim causing the revenue to jump downwards. The size of claims is independent and identically distributed. If we call $X_t$ the capital of the company at time $t$, or the *surplus process*, then the above description amounts to

$$X_t = x + ct - \sum_{i=1}^{N_t} \xi_i, \ t \geq 0,$$

where $x > 0$ is the initial capital of the company, $N = \{N_t : t \geq 0\}$ is a Poisson process with rate $\lambda > 0$, and $\{\xi_i : i \geq 1\}$ is a sequence of positive, independent and identically distributed random variables also independent of $N$. The process $X = \{X_t : t \geq 0\}$ is nothing more than a compound Poisson process with drift of rate $c$, initiated from $x \geq 0$. Denote its law by $\mathbb{P}_x$ and for convenience write $\mathbb{P}$ instead of $\mathbb{P}_0$.

Financial ruin in this model (or just *ruin* for short) will occur if the surplus of the insurance company drops below zero. Since this will happen with probability one if $\mathbb{P}(\liminf_{t\uparrow\infty} X_t = -\infty) = 1$, an additional assumption imposed on the model is that

$$\lim_{t\uparrow\infty} X_t = \infty.$$

A sufficient condition to guarantee this is that the distribution of $\xi$ has finite mean, say $\mu > 0$, and that

$$\frac{\lambda\mu}{c} < 1,$$

the so-called *net profit condition.* To see why this presents a sufficient condition, note that the Strong Law of Large Numbers, $\lim_{t\uparrow\infty} N_t/t = \lambda$, and the obvious fact that $\lim_{t\uparrow\infty} N_t = \infty$ imply that

$$\lim_{t\uparrow\infty} \frac{X_t}{t} = \lim_{t\uparrow\infty} \left( \frac{x}{t} + c - \frac{N_t}{t} \frac{\sum_{i=1}^{N_t} \xi_i}{N_t} \right) = c - \lambda\mu > 0,$$

Under the net profit condition it follows that ruin will occur only with probability less than one. Fundamental quantities of interest in this model thus become the distribution of the time to ruin and the deficit at ruin; otherwise identified as

$$\tau_0^- := \inf\{t > 0 : X_t < 0\} \text{ and } X_{\tau_0^-} \text{ on } \{\tau_0^- < \infty\}$$

when the process $X$ drifts to infinity.

Of course the surplus process is nothing more than a spectrally negative Lévy process and it therefore makes sense to look at similar problems for the general class of spectrally negative processes. Therefore we shall henceforth understand $X$ to be a spectrally negative Lévy process.

In this section we shall develop semi-explicit identities concerning exiting from a half line and a strip. Recall that $\mathbb{P}_x$ and $\mathbb{E}_x$ are shorthand for $\mathbb{P}(\cdot|X_0 = x)$ and $\mathbb{E}(\cdot|X_0 = x)$ and for the special case that $x = 0$ we keep with our old notation, so that $\mathbb{P}_0 = \mathbb{P}$ and $\mathbb{E}_0 = \mathbb{E}$, unless we wish to emphasise the fact that $X_0 = 0$. Recall also

$$\tau_x^+ = \inf\{t > 0 : X_t > x\} \text{ and } \tau_x^- = \inf\{t > 0 : X_t < x\}$$

for all $x \in \mathbb{R}$.

A key element of our analysis will be the following change of measure. it is easy to deduce under this assumption that $\mathcal{E}(\beta) = \{\mathcal{E}_t(\beta)\colon t \geq 0\}$ is a $\mathbb{P}$-martingale with respect to $\mathbb{F}$ where

$$\mathcal{E}_t(\beta) = \mathrm{e}^{\beta X_t - \psi(\beta)t},\ t \geq 0. \tag{2.1}$$

Since it has mean one, it may be used to perform a change of measure via

$$\left.\frac{\mathrm{d}\mathbb{P}^\beta}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}_t(\beta), \tag{2.2}$$

for any $\beta \geq 0$. The change of measure above, known as the *Esscher transform*, is a natural generalisation of the Cameron–Martin–Girsanov change of measure. The proof of the following result is left as an exercise.

**Theorem 2.1** *Fix $\beta \geq 0$. Under $\mathbb{P}^\beta$, $X$ remains within the class of spectrally negative Lévy processes. The Laplace exponent $\psi_\beta$ of $X$ under $\mathbb{P}^\beta$ satisfies*

$$\psi_\beta(\theta) = \psi(\theta + \beta) - \psi(\beta) \tag{2.3}$$

*for all $\theta \geq -\beta$.*

The main results of this section are the following.

**Theorem 2.2 (One- and two-sided exit formulae)** *Assume that $X$ is a spectrally negative Lévy process that satisfies $\psi'(0+) > 0$.*[1] *There exist a family of functions $W^{(q)} : \mathbb{R} \to [0, \infty)$ and*

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y)\mathrm{d}y, \text{ for } x \in \mathbb{R}$$

*defined for each $q \geq 0$ such that the following hold (for short we shall write $W^{(0)} = W$).*

**(i)** *For any $q \geq 0$, we have $W^{(q)}(x) = 0$ for $x < 0$ and $W^{(q)}$ is characterised on $[0, \infty)$ as a non-decreasing function whose Laplace transform satisfies*

$$\int_0^\infty \mathrm{e}^{-\beta x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\beta) - q} \text{ for } \beta > \Phi(q). \tag{2.4}$$

**(ii)** *For any $x \in \mathbb{R}$ and $q \geq 0$,*

$$\mathbb{E}_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{(\tau_0^- < \infty)}\right) = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x), \tag{2.5}$$

*where we understand $q/\Phi(q)$ in the limiting sense for $q = 0$, so that*

$$\mathbb{P}_x(\tau_0^- < \infty) = 1 - \psi'(0+)W(x) \tag{2.6}$$

**(iii)** *For any $x \leq a$ and $q \geq 0$,*

$$\mathbb{E}_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{(\tau_0^- > \tau_a^+)}\right) = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{2.7}$$

*and*

$$\mathbb{E}_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{(\tau_0^- < \tau_a^+)}\right) = Z^{(q)}(x) - Z^{(q)}(a)\frac{W^{(q)}(x)}{W^{(q)}(a)}. \tag{2.8}$$

---

[1] This condition is unnecessary and in fact the theorem holds for any spectrally negative Lévy process. It has been imposed for convenience.

In keeping with existing literature we will refer to the functions $W^{(q)}$ and $Z^{(q)}$ as the $q$-scale functions.

**Proof of Theorem 2.2 (i).** Define

$$W(x) = \frac{1}{\psi'(0+)}\mathbb{P}_x(\underline{X}_\infty \geq 0) = \frac{1}{\psi'(0+)}\mathbb{P}(-\underline{X}_\infty \leq x), \tag{2.9}$$

which, as a distribution function, is right continuous and non-decreasing Recall from (1.4)

$$\mathbb{E}\left(\mathrm{e}^{\beta \underline{X}_\infty}\right) = \psi'(0+)\frac{\beta}{\psi(\beta)}$$

for $\beta > 0$. Integrating by parts, we also see that

$$\begin{aligned}
\mathbb{E}\left(\mathrm{e}^{\beta \underline{X}_\infty}\right) &= \int_{[0,\infty)} \mathrm{e}^{-\beta x}\mathbb{P}\left(-\underline{X}_\infty \in \mathrm{d}x\right) \\
&= \mathbb{P}\left(-\underline{X}_\infty = 0\right) + \int_{(0,\infty)} \mathrm{e}^{-\beta x}\, \mathrm{d}\mathbb{P}\left(-\underline{X}_\infty \in (0,x]\right) \\
&= \int_0^\infty \mathbb{P}\left(-\underline{X}_\infty = 0\right)\beta\, \mathrm{e}^{-\beta x}\, \mathrm{d}x + \beta\int_0^\infty \mathrm{e}^{-\beta x}\mathbb{P}\left(-\underline{X}_\infty \in (0,x]\right)\mathrm{d}x \\
&= \beta\int_0^\infty \mathrm{e}^{-\beta x}\mathbb{P}\left(-\underline{X}_\infty \leq x\right)\mathrm{d}x \\
&= \beta\int_0^\infty \mathrm{e}^{-\beta x}\mathbb{P}_x\left(\underline{X}_\infty \geq 0\right)\mathrm{d}x,
\end{aligned}$$

and hence

$$\int_0^\infty \mathrm{e}^{-\beta x}W(x)\, \mathrm{d}x = \frac{1}{\psi(\beta)} \tag{2.10}$$

for all $\beta > 0 = \Phi(0)$.

Now for the case that $q > 0$ take as before $W^{(q)}(x) = \mathrm{e}^{\Phi(q)x}W_{\Phi(q)}(x)$. As remarked earlier, $X$ under $\mathbb{P}^{\Phi(q)}$ drifts to infinity and hence using the conclusion from the previous paragraph together with (2.3) we have

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-\beta x}W^{(q)}(x)\mathrm{d}x &= \int_0^\infty \mathrm{e}^{-(\beta-\Phi(q))x}W_{\Phi(q)}(x)\mathrm{d}x \\
&= \frac{1}{\psi_{\Phi(q)}\left(\beta - \Phi\left(q\right)\right)} \\
&= \frac{1}{\psi\left(\beta\right) - q}
\end{aligned} \tag{2.11}$$

provided $\beta - \Phi\left(q\right) > 0$.

It is not immediately clear that $W^{(q)}$ is non-decreasing, however this will transpire in the proof of the remaining parts of the theorem. ■

**Proof of Theorem 2.2 (iii) equation (2.7).** A simple argument using the law of total probability and the Strong Markov Property now yields for $x \in [0, a)$

$$\begin{aligned}
\mathbb{P}_x(\underline{X}_\infty \geq 0) &= \mathbb{E}_x\left(\mathbb{P}_x(\underline{X}_\infty \geq 0 | \mathcal{F}_{\tau_a^+})\right) \\
&= \mathbb{E}_x\left(\mathbf{1}_{(\tau_a^+ < \tau_0^-)}\mathbb{P}_a(\underline{X}_\infty \geq 0)\right) + \mathbb{E}_x\left(\mathbf{1}_{(\tau_a^+ > \tau_0^-)}\mathbb{P}_{X_{\tau_0^-}}(\underline{X}_\infty \geq 0)\right) \\
&= \mathbb{P}_a(\underline{X}_\infty \geq 0)\mathbb{P}_x(\tau_a^+ < \tau_0^-).
\end{aligned}$$

To justify that the second term in the second equality disappears we have to use the following subtle facts that we have note proved. If $X$ has no Gaussian component then it cannot cross below zero continuously, i.e. $X_{\tau_0^-} < 0$, and then we use that $\mathbb{P}_x(\underline{X}_\infty \geq 0) = 0$ for $x < 0$. If $X$ has a Gaussian component then $X_{\tau_0^-} \leq 0$ and we need to know that $W(0) = 0$. However, in the presence of a Gaussian component, the Lévy process will be visit either side of the origin instantaneously and consequently we have that $\underline{X}_\infty < 0$ $\mathbb{P}$-almost surely which is the same as $W(0) = 0$.

We now have

$$\mathbb{P}_x(\tau_a^+ < \tau_0^-) = \frac{W(x)}{W(a)}. \tag{2.12}$$

It is trivial, but nonetheless useful for later use, to note that the same equality holds even when $x < 0$ since both sides are equal to zero.

Now assume that $q > 0$. In this case, by the convexity of $\psi$, we know that $\Phi(q) > 0$ and hence $\psi'_{\Phi(q)}(0) = \psi'(\Phi(q)) > 0$ (again by convexity) which implies that under $\mathbb{P}^{\Phi(q)}$, the process $X$ drifts to infinity. For $(X, \mathbb{P}^{\Phi(q)})$ we have already established the existence of a 0-scale function $W_{\Phi(q)}(x) = \mathbb{P}_x^{\Phi(q)}(\underline{X}_\infty \geq 0)$ which fulfils the relation

$$\mathbb{P}_x^{\Phi(q)}(\tau_a^+ < \tau_0^-) = \frac{W_{\Phi(q)}(x)}{W_{\Phi(q)}(a)}. \tag{2.13}$$

However by definition of $\mathbb{P}^{\Phi(q)}$, we also have that

$$\begin{aligned}
\mathbb{P}_x^{\Phi(q)}(\tau_a^+ < \tau_0^-) &= \mathbb{E}_x(\mathrm{e}^{\Phi(q)(X_{\tau_a^+} - x) - q\tau_a^+}\mathbf{1}_{(\tau_a^+ < \tau_0^-)}) \\
&= \mathrm{e}^{\Phi(q)(a-x)}\mathbb{E}_x(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+ < \tau_0^-)}).
\end{aligned} \tag{2.14}$$

Combining (2.13) and (2.14) gives

$$\mathbb{E}_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+ < \tau_0^-)}\right) = \mathrm{e}^{-\Phi(q)(a-x)}\frac{W_{\Phi(q)}(x)}{W_{\Phi(q)}(a)} = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{2.15}$$

where $W^{(q)}(x) = \mathrm{e}^{\Phi(q)x}W_{\Phi(q)}(x)$. Clearly $W^{(q)}$ is identically zero on $(-\infty, 0)$ and non-decreasing.

■

Note that the definition of $W^{(q)}$ we have given above may be taken up to any multiplicative constant without affecting the validity of the arguments. This also justifies the terminology 'scale function'.

**Proof of Theorem 2.2 (ii).** As $W^{(q)}$ is non-decreasing we may also treat it as a distribution function of a measure, in which case, integrating (2.11) we find

$$\int_{[0,\infty)} \mathrm{e}^{-\beta x} W^{(q)}(\mathrm{d}x) = \frac{\beta}{\psi(\beta) - q} \tag{2.16}$$

Using the Laplace transform of $W^{(q)}(x)$ (given in (2.4)) as well as the Laplace–Stieltjes transform (2.16), we can interpret the Wiener–Hopf factor in (1.3) as saying that for $x \geq 0$,

$$\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \in \mathrm{d}x) = \frac{q}{\Phi(q)} W^{(q)}(\mathrm{d}x) - qW^{(q)}(x)\mathrm{d}x, \tag{2.17}$$

and hence for $x \geq 0$,

$$\begin{aligned} \mathbb{E}_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{(\tau_0^- < \infty)}\right) &= \mathbb{P}_x(\mathbf{e}_q > \tau_0^-) \\ &= \mathbb{P}_x(\underline{X}_{\mathbf{e}_q} < 0) \\ &= \mathbb{P}(-\underline{X}_{\mathbf{e}_q} > x) \\ &= 1 - \mathbb{P}(-\underline{X}_{\mathbf{e}_q} \leq x) \\ &= 1 + q\int_0^x W^{(q)}(y)\mathrm{d}y - \frac{q}{\Phi(q)} W^{(q)}(x) \\ &= Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x). \end{aligned} \tag{2.18}$$

Note that since $Z^{(q)}(x) = 1$ and $W^{(q)}(x) = 0$ for all $x \in (-\infty, 0)$, the statement is valid for all $x \in \mathbb{R}$. The proof is now complete for the case that $q > 0$.

Finally we have that $\lim_{q\downarrow 0} q/\Phi(q) = \lim_{q\downarrow 0} \psi(\Phi(q))/\Phi(q)$ which equal to $\psi'(0+)$. The proof is thus completed by taking the limit in $q$ in (2.5). ■

**Proof of Theorem 2.2 (iii) equation (2.8).** Fix $q > 0$. We have for $x \geq 0$,

$$\mathbb{E}_x(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{(\tau_0^- < \tau_a^+)}) = \mathbb{E}_x(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{(\tau_0^- < \infty)}) - \mathbb{E}_x(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{(\tau_a^+ < \tau_0^-)}).$$

Applying the Strong Markov Property at $\tau_a^+$ and using the fact that $X$ creeps upwards, we also have that

$$\mathbb{E}_x(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{(\tau_a^+ < \tau_0^-)}) = \mathbb{E}_x(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+ < \tau_0^-)})\mathbb{E}_a(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{(\tau_0^- < \infty)}).$$

Appealing to (2.5) and (2.7) we now have that

$$\mathbb{E}_x(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{(\tau_0^-<\tau_a^+)}) = Z^{(q)}(x) - \frac{q}{\Phi(q)}W^{(q)}(x) - \frac{W^{(q)}(x)}{W^{(q)}(a)}\left(Z^{(q)}(a) - \frac{q}{\Phi(q)}W^{(q)}(a)\right)$$

and the required result follows in the case that $q > 0$. The case that $q = 0$ is again dealt with by taking limits as $q \downarrow 0$. ■

These identities would carry much greater practical value if they the Laplace transform (2.4) could be inverted. In general this a much more difficult problem than one might imagine. This has been a topic of recent interest. Below we describe a method that can be used to generate examples of the scale function. Unfortunately it has its limitations and it is quite hard to generate examples of $W^{(q)}$.

An auxiliary result we need first is the following. We remind the reader that the Laplace exponent of a spectrally negative Lévy process takes the form

$$\psi(\lambda) = -a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)}(\mathrm{e}^{\lambda x} - 1 - \lambda x\mathbf{1}_{(|x|<1)})\Pi(\mathrm{d}x).$$

for $\lambda \geq 0$.

**Lemma 2.1** *Suppose that $X$ is a spectrally negative Lévy process such that $\psi'(0+) > 0$. Then we may always write for $\lambda \geq 0$,*

$$\psi(\lambda) = \lambda\phi(\lambda),$$

*where*

$$\phi(\lambda) = \kappa + \delta\lambda + \int_{(0,\infty)}(1 - \mathrm{e}^{-\lambda x})\Upsilon(\mathrm{d}x)$$

*such that $\kappa = \psi'(0+)$, $\delta = \sigma^2/2$ and*

$$\Upsilon(\mathrm{d}x) = \Pi(-\infty, -x)\mathrm{d}x,$$

*for $x > 0$.*

**Proof.** Since $\mathbb{E}(X_1) = \psi'(0+) > 0$, then necessarily it must also be finite. In that case it is a straightforward exercise to show that we can re-write

$$\psi(\lambda) = \psi'(0+)\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)}(\mathrm{e}^{\lambda x} - 1 - \lambda x)\Pi(\mathrm{d}x).$$

A straightforward integration by parts now yields

$$\psi(\lambda) = \psi'(0+)\lambda + \frac{1}{2}\sigma^2\lambda^2 + \lambda\int_0^\infty(1 - \mathrm{e}^{-\lambda x})\Pi(-\infty, -x)\mathrm{d}x \tag{2.19}$$

and the result follows. ■

There are two immediate consequences we should note from the above lemma. The first is that the factor $\phi(\lambda)$ is the Laplace exponent of a killed subordinator. In fact digging a little deeper into the Wiener-Hopf factorisation one will discover that this is the Laplace exponent of the descending ladder height process!

The second thing to notice is that from (2.16) that finding $W$ now boils down to inverting $1/\phi(\lambda)$.

Hubalek and Kyprianou [5] have the idea of choosing $\phi$ such that the inversion can be performed. This would yield an explicit example of a scale function so long as the product $\lambda\phi(\lambda)$ corresponds to the Laplace exponent of a spectrally negative Lévy process satisfying $\psi'(0+) > 0$. Inspection of (2.19) tells us that it would suffice for us to choose $\phi(\lambda)$ such that $\kappa > 0$ and the jump measure $\Upsilon$ to be absolutely continuous with non-increasing density, say $\upsilon$, which satisfies $\upsilon(x) < \infty$ for all $x > 0$ and $\int_0^1 \upsilon(x)\mathrm{d}x < \infty$. In that case we would necessarily have

$$\Pi(-\infty, -x) = \upsilon(x)$$

and the last two conditions would be ensure that $\Pi(-\infty, -x) < \infty$ for all $x > 0$ and (after an integration by parts) $\int_{(0,1)} x^2\Pi(\mathrm{d}x) < \infty$. Said another way, we have the following result.

**Lemma 2.2** *Suppose that*

$$\phi(\lambda) = \kappa + \delta\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda})\Upsilon(\mathrm{d}x)$$

*is the Laplace exponent of a killed subordinator. Then $\lambda\phi(\lambda)$ is the Laplace exponent of a spectrally negative Lévy process for $\lambda \geq 0$ satisfying $\mathbb{E}(X_1) > 0$ if and only if $\kappa > 0$ and $\Upsilon$ is absolutely continuous with non-increasing density, $\upsilon$. In that case we have $\sigma = \sqrt{2\delta}$ and $\Pi(-\infty, -x) = \upsilon(x)$.*

For any subordinator with Laplace exponent $\phi(\lambda)$ satisfying the conditions given in the above lemma, we say that the spectrally negative Lévy process with Laplace exponent $\psi(\lambda) = \lambda\phi(\lambda)$ is called the *parent process* to the subordinator

It turns out that there is something called the potential analysis of subordinators which is rich in examples of subordinators, equivalently Laplace exponents $\phi(\lambda)$, for which the inversion $1/\phi(\lambda)$ can be performed thereby offering new examples of scale functions for their parent processes. We give one such example below, however the interested reader can find out more about the robustness of this method in [5, 7, 6]; see also [1].

In this example we take

$$\phi(\lambda) = \kappa + (\lambda + \gamma)^\alpha - \gamma^\alpha,$$

where $\alpha \in (0,1)$. That is to say the descending ladder height process will be a tempered stable subordinator killed at rate $\kappa > 0$. [2]

The parent process has no Gaussian component, has a Lévy measure that satisfies

$$\Pi(\mathrm{d}x) = \frac{1}{-\Gamma(-\alpha)}\left(\frac{\gamma}{(-x)^{\alpha+1}}\mathrm{e}^{\gamma x} + \frac{(\alpha+1)}{(-x)^{\alpha+2}}\mathrm{e}^{\gamma x}\right)\mathrm{d}x$$

and Laplace exponent

$$\psi(\lambda) = \lambda\kappa + \lambda(\lambda+\gamma)^\alpha - \lambda\gamma^\alpha.$$

To compute the scale function we recall the definition of a Mittag-Leffler function

$$E_{\alpha,\beta}(x) := \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)},$$

for $\alpha, \beta > 0$, a natural generalisation of the exponential function. This family of special functions has a large base of analytical properties, one of which is the following. For $\alpha > 0$,

$$\int_0^\infty \mathrm{e}^{-\theta x} x^{\alpha-1} E_{\alpha,\alpha}(\lambda x^\alpha)\mathrm{d}x = \frac{1}{\theta^\alpha - \lambda}.$$

Using this transform, it is a straightforward exercise to deduce that

$$W(x) = \int_0^x \mathrm{e}^{-\gamma y} y^{\alpha-1} E_{\alpha,\alpha}\left((\gamma^\alpha - \kappa)y^\alpha\right)\mathrm{d}y.$$

# Exercises

**Exercise 1** Suppose that $X$ is a spectrally negative stable process with index $\alpha \in (1,2)$.

**(i)** Show that, up to a multiplicative constant, its Laplace exponent is given by $\psi(\theta) = \theta^\alpha$, $\theta \geq 0$.

**(ii)** Show that for $q > 0$ and $\beta > q^{1/\alpha}$,

$$\int_0^\infty \mathrm{e}^{-\beta x}\overline{W}^{(q)}(x)\mathrm{d}x = \frac{1}{\beta(\beta^\alpha - q)} = \sum_{n\geq 1} q^{n-1}\beta^{-\alpha n - 1},$$

where $\overline{W}^{(q)}(x) = \int_0^x W^{(q)}(y)\mathrm{d}y$.

[2] Note that tempered stable subordinators can be derived from stable subordinators by exponential tilting in the spirit of (2.2). In this case one uses the change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}^\beta}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \mathrm{e}^{-\gamma H_t + \gamma^\alpha t},$$

where $\{H_t : t \geq 0\}$ is a stable subordinator with index $\alpha \in (0,1)$.

**(iii)** Deduce that for $x \geq 0$

$$Z^{(q)}(x) = \sum_{n\geq 0} q^n \frac{x^{\alpha n}}{\Gamma(1+\alpha n)}.$$

Note that the right-hand side above is also equal to $E_\alpha(qx^\alpha)$ where $E_\alpha(\cdot)$ is the Mittag–Leffler function of parameter $\alpha$ (a generalisation of the exponential function with parameter $\alpha$).

**(iv)** Deduce that for $q \geq 0$,

$$W^{(q)}(x) = \alpha x^{\alpha-1} E'_\alpha(qx^\alpha)$$

for $x \geq 0$.

**(v)** Show that for standard Brownian motion that

$$W^{(q)}(x) = \sqrt{\frac{2}{q}} \sinh(\sqrt{2q}x) \text{ and } Z^{(q)}(x) = \cosh(\sqrt{2q}x)$$

for $x \geq 0$ and $q \geq 0$.

**(vi)** Suppose now that $X$ is a tempered stable spectrally negative Lévy process with Laplace exponent given by $\psi(\theta) = (\theta + c)^\alpha - c^\alpha$ where $c \geq 0$ and $\alpha \in (1, 2)$. Show that for $q \geq 0$,

$$W^{(q)}(x) = \mathrm{e}^{-cx} \alpha x^{\alpha-1} E'_\alpha((q + c^\alpha)x^\alpha).$$

**Exercise 2** Suppose that $X$ is a spectrally negative Lévy process of bounded variation such that $\lim_{t\uparrow\infty} X_t = \infty$. For convenience, write $X_t = \delta t - S_t$ where $S = \{S_t : t \geq 0\}$ is a subordinator with jump measure $\Pi$.

**(i)** Show that necessarily $\delta^{-1} \int_0^\infty \Pi(y, \infty)\mathrm{d}y < 1$.

**(ii)** Show that the scale function $W$ satisfies

$$\int_{[0,\infty)} \mathrm{e}^{-\beta x} W(\mathrm{d}x) = \frac{1}{\delta - \int_0^\infty \mathrm{e}^{-\beta y} \Pi(y, \infty)\mathrm{d}y}$$

and deduce that

$$W(\mathrm{d}x) = \frac{1}{\delta} \sum_{n\geq 0} \nu^{*n}(\mathrm{d}x),$$

where $\nu(\mathrm{d}x) = \delta^{-1}\Pi(x, \infty)\mathrm{d}x$ and as usual we understand $\nu^{*0}(\mathrm{d}x) = \delta_0(\mathrm{d}x)$.

**(iii)** Suppose that $S$ is a compound Poisson process with rate $\lambda > 0$ and jump distribution which is exponential with parameter $\mu > 0$. Show that

$$W(x) = \frac{1}{\delta}\left(1 + \frac{\lambda}{\delta\mu - \lambda}(1 - \mathrm{e}^{-(\mu - \delta^{-1}\lambda)x})\right).$$

**Exercise 3** Let $X$ be any spectrally negative Lévy process with Laplace exponent $\psi$.

**(i)** Use (2.8) and (2.5) to establish that for each $q \geq 0$,

$$\lim_{x\uparrow\infty} \frac{Z^{(q)}(x)}{W^{(q)}(x)} = \frac{q}{\Phi(q)},$$

where the right-hand side is understood in the limiting sense when $q = 0$. In addition, show that

$$\lim_{a\uparrow\infty} \frac{W^{(q)}(a-x)}{W^{(q)}(a)} = \mathrm{e}^{-\Phi(q)x}.$$

**(ii)** Taking account of a possible atom at the origin, write down the Laplace transform of $W^{(q)}(\mathrm{d}x)$ on $[0,\infty)$ and show that if $X$ has unbounded variation then $W^{(q)\prime}(0) = 2/\sigma^2$ where $\sigma$ is the Gaussian coefficient in the Lévy–Itô decomposition and it is understood that $1/0 = \infty$. If however, $X$ has bounded variation then

$$W^{(q)\prime}(0) = \frac{\Pi(-\infty,0) + q}{\delta^2},$$

where $\delta$ is the drift coefficient and it is understood that the right hand side is infinite if $\Pi(-\infty,0) = \infty$.

**Exercise 4** This exercise deals with first hitting of points below zero of spectrally negative Lévy processes following the work of [3]. For each $x > 0$ define

$$T(-x) = \inf\{t > 0 : X_t = -x\},$$

where $X$ is a spectrally negative Lévy process with Laplace exponent $\psi$ and right inverse $\Phi$.

**(i)** Show that for all $c \geq 0$ and $q \geq 0$,

$$\Phi_c(q) = \Phi(q + \psi(c)) - c.$$

**(ii)** Show for $x > 0$, $c \geq 0$ and $p \geq \psi(c) \vee 0$,

$$\mathbb{E}(\mathrm{e}^{-p\tau^-_{-x} + c(X_{\tau^-_{-x}} + x)} \mathbf{1}_{(\tau^-_{-x} < \infty)}) = \mathrm{e}^{cx}\left(Z_c^{(q)}(x) - \frac{q}{\Phi_c(q)} W_c^{(q)}(x)\right),$$

where $q = p - \psi(c)$. Use analytic extension to justify that the above identity is in fact valid for all $x > 0, c \geq 0$ and $p \geq 0$.

**(iii)** By noting that $T(-x) \geq \tau^-_{-x}$, condition on $\mathcal{F}_{\tau^-_{-x}}$ to deduce that for $p$, $u \geq 0$,

$$\mathbb{E}(\mathrm{e}^{-pT(-x) - u(T(-x) - \tau^-_{-x})} \mathbf{1}_{(T(-x)<\infty)}) = \mathbb{E}(\mathrm{e}^{-p\tau^-_{-x} + \Phi(p+u)(X_{\tau^-_{-x}} + x)} \mathbf{1}_{(\tau^-_{-x} < \infty)}).$$

**(iv)** By taking a limit as $u \downarrow 0$ in part (iii) and making use of the identity in part (ii) deduce that

$$\mathbb{E}(\mathrm{e}^{-pT(-x)}\mathbf{1}_{(T(-x)<\infty)}) = \mathrm{e}^{\Phi(p)x} - \psi'(\Phi(p))W^{(p)}(x)$$

and hence by taking limits again as $x \downarrow 0$,

$$\mathbb{E}\left(\mathrm{e}^{-pT(0)}\mathbf{1}_{(T(0)<\infty)}\right) = \begin{cases} 1 - \psi'(\Phi(p))\frac{1}{\delta} & \text{if } X \text{ has bounded variation} \\ 1 & \text{if } X \text{ has unbounded variation,} \end{cases}$$

where $\delta$ is the drift term in the Laplace exponent if $X$ has bounded variation.

**Exercise 5** Again relying on [3] we shall make the following application of part (iii) of the previous exercise. Suppose that $B = \{B_t : t \geq 0\}$ is a Brownian motion. Denote

$$\sigma = \inf\{t > 0 : B_t = \overline{B}_t = t\}.$$

**(i)** Suppose that $X$ is a descending stable-$\frac{1}{2}$ subordinator with upward unit drift. Show that

$$\mathbb{P}(\sigma < \infty) = \mathbb{P}(T(0) < \infty),$$

where $T(0)$ is defined in Exercise 4.

**(ii)** Deduce from part (i) that $\mathbb{P}(\sigma < \infty) = \frac{1}{2}$.

**Exercise 6** This exercise is based on the results of [4]. Suppose that $X$ is any spectrally negative Lévy process with Laplace exponent $\psi$, satisfying $\lim_{t\uparrow\infty} X_t = \infty$. Recall that this necessarily implies that $\psi'(0+) > 0$. Define for each $x \in \mathbb{R}$,

$$\Lambda_0 = \sup\{t > 0 : X_t < 0\}.$$

Here we work with the definition $\sup \emptyset = 0$ so that the event $\{\Lambda_0 = 0\}$ corresponds to the event that $X$ never enters $(-\infty, 0)$.

**(i)** Using the equivalent events $\{\Lambda_0 < t\} = \{X_t \geq 0, \inf_{s\geq t} X_s \geq 0\}$ and the Markov Property, show that for each $q > 0$ and $y \in \mathbb{R}$

$$\mathbb{E}_y(\mathrm{e}^{-q\Lambda_0}) = q\int_0^\infty \theta^{(q)}(x-y)\mathbb{P}_x(\underline{X}_\infty \geq 0)\mathrm{d}x,$$

where $\theta^{(q)}$ is the $q$-potential density of $X$.

**(ii)** Hence show that for $y \leq 0$,

$$\mathbb{E}_y(\mathrm{e}^{-q\Lambda_0}) = \psi'(0+)\Phi'(q)\mathrm{e}^{\Phi(q)y},$$

where $\Phi$ is the right inverse of $\psi$ and in particular

$$\mathbb{P}(\Lambda_0 = 0) = \begin{cases} \psi'(0+)/\delta & \text{if } X \text{ has bounded variation with drift } \delta \\ 0 & \text{if } X \text{ has unbounded variation.} \end{cases}$$

**(iii)** Suppose now that $y > 0$. Use again the Strong Markov Property to deduce that for $q > 0$,

$$\mathbb{E}_y(\mathrm{e}^{-q\Lambda_0}\mathbf{1}_{(\Lambda_0>0)}) = \psi'(0+)\Phi'(q)\mathbb{E}_y(\mathrm{e}^{-q\tau_0^- + \Phi(q)X_{\tau_0^-}}\mathbf{1}_{(\tau_0^-<\infty)}).$$

**(iv)** Deduce that for $y > 0$ and $q > 0$,

$$\mathbb{E}_y(\mathrm{e}^{-q\Lambda_0}\mathbf{1}_{(\Lambda_0>0)}) = \psi'(0+)\Phi'(q)\mathrm{e}^{\Phi(q)y} - \psi'(0+)W^{(q)}(y).$$

## References


[1] Chazal, M., Kyprianou, A.E. and Patie, P. (2011) A transformation for Lévy processes with one-sided jumps and applications Authors: http://arxiv.org/abs/1010.3819

[2] Bertoin, J. (1996) *Lévy Processes.* Cambridge University Press, Cambridge.

[3] Doney, R.A. (1991) Hitting probabilities for spectrally positive Lévy processes. *J. London Math. Soc.* **44**, 566–576.

[4] Chiu, S.K. and Yin, C. (2005) Passage times for a spectrally negative Lévy process with applications to risk theory. *Bernoulli* **11**, 511–522.

[5] Hubalek, F. and Kyprianou, A.E. (2011) Old and new examples of scale functions for spectrally negative Lvy processes. In *Sixth Seminar on Stochastic Analysis, Random Fields and Applications, eds R. Dalang, M. Dozzi, F. Russo.* Progress in Probability, Birkhuser (2010).

[6] Kuznetsov, A., Kyprianou, A.E. and Rivero, V. (2011) The theory of scale functions for spectrally negative Lévy processes. http://arxiv.org/submit/226724

[7] Kyprianou, A.E. and Rivero, V. (2008) Special, conjugate and complete scale functions for spectrally negative Lvy processes. *Electronic Journal of Probability* Paper no 57, 167–1701.

[8] Lundberg, F. (1903) *Approximerad framställning av sannolikhetsfunktionen. Återförsäkring av kollektivrisker.* Akad. Afhandling. Almqvist. och Wiksell, Uppsala.