

# Super-Brownian motion: $L^p$ -convergence of martingales through the pathwise spine decomposition

A.E. Kyprianou\* and A. Murillo-Salas†

*On the occasion of the 60th birthday of Sergei Kuznetsov*

March 31, 2011

## Abstract

Evans [7] described the semi-group of a superprocess with quadratic branching mechanism under a martingale change of measure in terms of the semi-group of an immortal particle and the semigroup of the superprocess prior to the change of measure. This result, commonly referred to as the spine decomposition, alludes to a pathwise decomposition in which independent copies of the original process ‘immigrate’ along the path of the immortal particle. For branching particle diffusions the analogue of this decomposition has already been demonstrated in the pathwise sense, see for example [11, 10]. The purpose of this short note is to exemplify a new *pathwise* spine decomposition for supercritical super-Brownian motion with general branching mechanism (cf. [13]) by studying  $L^p$  convergence of naturally underlying additive martingales in the spirit of analogous arguments for branching particle diffusions due to Harris and Hardy [10]. Amongst other ingredients, the Dynkin-Kuznetsov N-measure plays a pivotal role in the analysis.

KEY WORDS AND PHRASES: Super-Brownian motion, additive martingales, N-measure, spine decomposition,  $L^p$ -convergence.

MSC 2010 subject classifications: 60J68, 60F25.

---

\*UNIVERSITY OF BATH, UK. E-mail: a.kyprianou@bath.ac.uk

†UNIVERSIDAD DE GUANAJUATO, MÉXICO. E-mail: amurillos@ugto.mx

# 1 Introduction

Suppose that  $X = \{X_t : t \geq 0\}$  is a (one-dimensional)  $\psi$ -super-Brownian motion with general branching mechanism  $\psi$  taking the form

$$\psi(\lambda) = -\alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x)\nu(dx), \quad (1)$$

for  $\lambda \geq 0$  where  $\alpha = -\psi'(0+) \in (0, \infty)$ ,  $\beta \geq 0$  and  $\nu$  is a measure concentrated on  $(0, \infty)$  which satisfies  $\int_{(0,\infty)} (x \wedge x^2)\nu(dx) < \infty$ . Let  $\mathcal{M}_F(\mathbb{R})$  be the space of finite measures on  $\mathbb{R}$  and note that  $X$  is a  $\mathcal{M}_F(\mathbb{R})$ -valued Markov process under  $\mathbb{P}_\mu$  for each  $\mu \in \mathcal{M}_F(\mathbb{R})$ , where  $\mathbb{P}_\mu$  is the law of  $X$  with initial configuration  $\mu$ . We shall use standard inner product notation, for  $f \in C_b^+(\mathbb{R})$ , the space of positive, uniformly bounded, continuous functions on  $\mathbb{R}$ , and  $\mu \in \mathcal{M}_F(\mathbb{R})$ ,

$$\langle f, \mu \rangle = \int_{\mathbb{R}} f(x)\mu(dx).$$

Accordingly we shall write  $\|\mu\| = \langle 1, \mu \rangle$ . Recall that the total mass of the process  $X$ ,  $\{\|X_t\| : t \geq 0\}$  is a continuous-state branching process with branching mechanism  $\psi$ . Such processes may exhibit explosive behaviour, however, under the conditions assumed above,  $\|X\|$  remains finite at all times. We insist moreover that  $\psi(\infty) = \infty$  which means that with positive probability the event  $\lim_{t \uparrow \infty} \|X_t\| = 0$  will occur. Equivalently this means that the total mass process does not have monotone increasing paths; see for example the summary in Chapter 10 of Kyprianou [12]. The existence of these superprocesses processes is guaranteed by [1, 3, 4].

The following standard result from the theory of superprocesses describes the evolution of  $X$  as a Markov process. For all  $f \in C_b^+(\mathbb{R})$  and  $\mu \in \mathcal{M}_F(\mathbb{R})$ ,

$$-\log \mathbb{E}_\mu(e^{-\langle f, X_t \rangle}) = \int_{\mathbb{R}} u_f(x, t)\mu(dx), \quad \mu \in \mathcal{M}_F(\mathbb{R}), t \geq 0, \quad (2)$$

where  $u_f(x, t)$  is the unique positive solution to the evolution equation for  $x \in \mathbb{R}$  and  $t > 0$

$$\frac{\partial}{\partial t} u_f(x, t) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u_f(x, t) - \psi(u_f(x, t)), \quad (3)$$

with initial condition  $u_f(x, 0) = f(x)$ . The reader is referred to Theorem 1.1 of Dynkin [2], Proposition 2.3 of Fitzsimmons [8] and Proposition 2.2 of Watanabe [15] for further details; see also Dynkin [3, 4] and Engländer and Pinsky [6] for a general overview.

Associated to the process  $X$  is the following martingale  $Z(\lambda) = \{Z_t(\lambda), t \geq 0\}$ , where

$$Z_t(\lambda) := e^{\lambda c_\lambda t} \langle e^{\lambda \cdot}, X_t \rangle, \quad t \geq 0, \quad (4)$$

where  $c_\lambda = \psi'(0+)/\lambda - \lambda/2$  and  $\lambda \in \mathbb{R}$  (cf. [13] Lemma 2.2). To see why this is a martingale note the following steps. Define for each  $x \in \mathbb{R}$ ,  $g \in C_b^+(\mathbb{R})$  and  $\theta, t \geq 0$ ,  $u_g^\theta(x, t) =$

$-\log \mathbb{E}_{\delta_x}(e^{-\theta \langle g, X_t \rangle})$ . With limits understood as  $\theta \downarrow 0$ , we have  $u_g(x, t)|_{\theta=0} = 0$ , moreover,  $v_g(x, t) := \mathbb{E}_{\delta_x}(\langle g, X_t \rangle) = \partial u_g^\theta(x, t)/\partial \theta|_{\theta=0}$ . Differentiating in  $\theta$  in (3) shows that  $v_g$  solves the equation

$$\frac{\partial}{\partial t} v_g(x, t) = \frac{1}{2} \frac{\partial^2}{\partial x^2} v_g(x, t) - \psi'(0+) v_g(x, t), \quad (5)$$

with  $v_g(x, 0) = g(x)$ . Classical Feynman-Kac theory tells us that (5) has a unique solution which is necessarily equal to  $\Pi_x(e^{-\psi'(0+)t} g(\xi_t))$  where  $\{\xi_t : t \geq 0\}$  is a Brownian motion issued from  $x \in \mathbb{R}$  under the measure  $\Pi_x$ . The above procedure also works for  $g(x) = e^{\lambda x}$  in which case we easily conclude that for all  $x \in \mathbb{R}$  and  $t \geq 0$ ,  $e^{\lambda c_\lambda t} \mathbb{E}_{\delta_x}(\langle e^\lambda, X_t \rangle) = e^{\lambda x}$ . Finally, the martingale property follows using the previous equality together with the Markov branching property associated with  $X$ . Note that as a positive martingale, it is automatic that

$$\lim_{t \uparrow \infty} Z_t(\lambda) = Z_\infty(\lambda)$$

$\mathbb{P}_\mu$ -almost surely for all  $\mu \in \mathcal{M}_F(\mathbb{R})$  such that  $\langle e^\lambda, \mu \rangle < \infty$ .

The purpose of this note is to demonstrate the robustness of a new path decomposition of our  $\psi$ -super-Brownian motion by studying the  $L^p$ -convergence of the martingales  $Z(\lambda)$ . Specifically we shall prove the following theorem.

**Theorem 1.1.** *Assume that  $p \in (1, 2]$ ,  $\int_{(0, \infty)} r^p \nu(dr) < \infty$  and  $p\lambda^2 < -2\psi'(0+)$ . Then  $Z_t(\lambda)$  converges to  $Z_\infty(\lambda)$  in  $L^p(\mathbb{P}_\mu)$ , for all  $\mu \in \mathcal{M}_F(\mathbb{R})$  such that  $\langle e^\lambda, \mu \rangle$  and  $\langle e^{\lambda p}, \mu \rangle$  are finite.*

The method of proof we use is quite similar to the one used in Harris and Hardy [10] for branching Brownian motion, where a pathwise spine decomposition functions as the key instrument of the proof. Roughly speaking, in that setting, the spine decomposition says that under a change of measure, the law of the branching Brownian motion has the same law as an immortal Brownian diffusion (with drift) along the path of which independent copies of the original branching Brownian motion immigrate at times which form a Poisson process. Until recently such a spine decomposition for superdiffusions was only available in the literature in a weak form; meaning that it takes the form of a semi-group decomposition. See the original paper of Evans [7] as well as, for example amongst others, Engländer and Kyprianou [9]. Recently however Kyprianou et al. [13] give a pathwise spine decomposition which provides a natural analogue to the pathwise spine decomposition for branching Brownian motion. Amongst other ingredients, the Dynkin-Kuznetsov  $\mathbb{N}$ -measure plays a pivotal role in describing the immigration off an immortal particle. We give a description of this new spine decomposition in the next section and thereafter we proceed to the proof of Theorem 1.1 in Section 3.

## 2 Spine decomposition

For each  $\lambda \in \mathbb{R}$  and  $\mu \in \mathcal{M}(\mathbb{R})$  satisfying  $\langle e^\lambda, \mu \rangle$ , we introduce the following martingale change of measure

$$\frac{d\mathbb{P}_\mu^\lambda}{d\mathbb{P}_\mu} \Big|_{\mathcal{F}_t} = \frac{Z_t(\lambda)}{\langle e^\lambda, \mu \rangle}, \quad t \geq 0, \quad (6)$$

where  $\mathcal{F}_t := \sigma(X_s, s \leq t)$ . The preceding change of measure induces the *spine decomposition* of  $X$  alluded to above. To describe it in detail we need some more ingredients.

According to Dynkin and Kuznetsov [5] there exists a collection of measures  $\{\mathbb{N}_x, x \in \mathbb{R}\}$ , defined on the same probability space as  $X$ , such that

$$\mathbb{N}_x(1 - e^{-\langle f, X_t \rangle}) = u_f(x, t), \quad x \in \mathbb{R}, t \geq 0. \quad (7)$$

Roughly speaking, the branching property tells us that for each  $n \in \mathbb{N}$ , the measures  $\mathbb{P}_{\delta_x}$  can be written as the  $n$ -fold convolution of  $\mathbb{P}_{\frac{1}{n}\delta_x}$  which indicates that, on the trajectory space of the superprocess,  $\mathbb{P}_x$  is infinitely divisible. Hence the role of  $\mathbb{N}_x$  in (7) is analogous to that of the Lévy measure for positive real-valued random variables.

From identity (7) and equation (2), it is straightforward to deduce that

$$\mathbb{N}_x(\langle f, X_t \rangle) = \mathbb{E}_{\delta_x}(\langle f, X_t \rangle), \quad (8)$$

whenever  $f \in C_b^+(\mathbb{R})$ .

For each  $x \in \mathbb{R}$ , let  $\Pi_x$  be the law of a Brownian motion  $\xi := \{\xi_t : t \geq 0\}$  issued from  $x$ . If  $\Pi_x^\lambda$  is the law under which  $\xi$  is a Brownian motion with drift  $\lambda \in \mathbb{R}$  and issued from  $x \in \mathbb{R}$ , then for each  $t \geq 0$ ,

$$\frac{d\Pi_x^\lambda}{d\Pi_x} \Big|_{\mathcal{G}_t} = e^{\lambda(\xi_t - x) - \frac{1}{2}\lambda^2 t}, \quad t \geq 0, \quad (9)$$

where  $\mathcal{G}_t := \sigma(\xi_s, s \leq t)$ . For convenience we shall also introduce the measure

$$\Pi_\mu^\lambda(\cdot) := \frac{1}{\langle e^\lambda, \mu \rangle} \int e^{\lambda x} \mu(dx) \Pi_x^\lambda(\cdot), \quad (10)$$

for all  $\lambda \in \mathbb{R}$ . In other words,  $\Pi_\mu^\lambda$  has the law of a Brownian motion with drift at rate  $\lambda$  with an initial position which has been independently randomised in a way that is determined by  $\mu$ .

Now fix  $\mu \in \mathcal{M}_F(\mathbb{R})$  and  $x \in \mathbb{R}$  and let us define a measure-valued process  $\Lambda := \{\Lambda_t, t \geq 0\}$  as follows:

- (i) Take a copy of the process  $\xi = \{\xi_t, t \geq 0\}$  under  $\Pi_x^\lambda$ , we shall refer to this process as the *spine*.

- (ii) Suppose that  $\mathbf{n}$  is a Poisson point process such that, for  $t \geq 0$ , given the spine  $\xi$ ,  $\mathbf{n}$  issues superprocess  $X^{\mathbf{n},t}$  at space-time position  $(\xi_t, t)$  with rate  $dt \times 2\beta d\mathbb{N}_{\xi_t}$ .
- (iii) Suppose that  $\mathbf{m}$  is a Poisson point process such that, independently of  $\mathbf{n}$ , given the spine  $\xi$ ,  $\mathbf{m}$  issues a superprocess  $X^{\mathbf{m},t}$  at space-time point  $(\xi_t, t)$  with initial mass  $r$  at rate  $dt \times r\nu(dr) \times d\mathbb{P}_{r\delta_{\xi_t}}$ .

Note in particular that, when  $\beta > 0$ , the rate of immigration under the process  $\mathbf{n}$  is infinite and moreover, each process that immigrates is issued with zero mass. One may therefore think of  $\mathbf{n}$  as a process of continuous immigration. In contrast, when  $\nu$  is a non-zero measure, processes that immigrate under  $\mathbf{m}$  have strictly positive initial mass and therefore contribute to path discontinuities of  $\|X\|$ .

Now, for each  $t \geq 0$ , we define

$$\Lambda_t = X'_t + X_t^{(\mathbf{n})} + X_t^{(\mathbf{m})}, \quad (11)$$

where  $\{X'_t : t \geq 0\}$  is an independent copy of  $(X, \mathbb{P}_\mu)$ ,

$$X_t^{(\mathbf{n})} = \sum_{s \leq t: \mathbf{n}} X_{t-s}^{\mathbf{n},s}, \quad t \geq 0 \quad \text{and} \quad X_t^{(\mathbf{m})} = \sum_{s \leq t: \mathbf{m}} X_{t-s}^{\mathbf{m},s}, \quad t \geq 0.$$

In the last two equalities we understand the first sum to be over times for which  $\mathbf{n}$  experiences points and the second sum is understood similarly. Note that since the processes  $X^{(\mathbf{n})}$  and  $X^{(\mathbf{m})}$  are initially zero valued it is clear that since  $X'_0 = \mu$  then  $\Lambda_0 = \mu$ . In that case, we use the notation  $\tilde{\mathbb{P}}_{\mu,x}^\lambda$  to denote the law of the pair  $(\Lambda, \xi)$ . Note also that the pair  $(\Lambda, \xi)$  is a time-homogenous Markov process. We are interested in the case that the initial position of the spine  $\xi$  is randomised using the measure  $\mu$  via (10). In that case we shall write

$$\tilde{\mathbb{P}}_\mu^\lambda(\cdot) = \frac{1}{\langle e^\lambda, \mu \rangle} \int_{\mathbb{R}} e^{\lambda x} \mu(dx) \tilde{\mathbb{P}}_{\mu,x}^\lambda(\cdot)$$

for short. The next theorem identifies the process  $\Lambda$  as the *pathwise* spine decomposition of  $(X, \mathbb{P}_\mu^\lambda)$  and in particular it shows that as a process on its own  $\Lambda$  is Markovian.

**Theorem 2.1.** (Theorem 5.1, [13]) *For all  $\mu \in \mathcal{M}_F(\mathbb{R})$  such that  $\langle e^\lambda, \mu \rangle < \infty$ ,  $(X, \mathbb{P}_\mu^\lambda)$  and  $(\Lambda, \tilde{\mathbb{P}}_\mu^\lambda)$  are equal in law.*

### 3 Proof of Theorem 1.1

From the last section we have the following spine decomposition of the martingale (4),

$$Z_t^\Lambda(\lambda) = Z'_t(\lambda) + \sum_{s \leq t: \mathbf{n}} e^{\lambda c_\lambda s} Z_{t-s}^{\mathbf{n},s}(\lambda) + \sum_{s \leq t: \mathbf{m}} e^{\lambda c_\lambda s} Z_{t-s}^{\mathbf{m},s}(\lambda), \quad (12)$$

where  $Z'(\lambda)$  is an independent copy of  $Z(\lambda)$  under  $\mathbb{P}_\mu$ ,

$$Z_{t-s}^{\mathbf{n},s} := e^{\lambda c_\lambda(t-s)} \langle e^{\lambda \cdot}, X_{t-s}^{\mathbf{n},s} \rangle,$$

and

$$Z_{t-s}^{\mathbf{m},s} := e^{\lambda c_\lambda(t-s)} \langle e^{\lambda \cdot}, X_{t-s}^{\mathbf{m},s} \rangle.$$

Since  $\{Z_t(\lambda), t \geq 0\}$  is a martingale and we assume that  $p \in (1, 2]$ , then Doob's submartingale inequality tells us that  $Z(\lambda)$  converges in  $L^p(\mathbb{P}_\mu)$  as soon as we can show that  $\sup_{t \geq 0} \mathbb{E}_\mu(Z_t(\lambda)^p) < \infty$ . To this end, and with the above pathwise spine decomposition in hand we may now proceed to address the analogue of the proof for branching Brownian motion given in [10].

First note that, for all  $p \in (1, 2]$ ,

$$\mathbb{E}_\mu(Z_t(\lambda)^p) = \mathbb{E}_\mu^\lambda(Z_t(\lambda)^q) = \tilde{\mathbb{E}}_\mu^\lambda(Z_t^\Lambda(\lambda)^q), \text{ for all } t \geq 0, \quad (13)$$

where  $q = p - 1$ . By Jensen's inequality we have that, for all  $q \in (0, 1]$

$$\begin{aligned} & \tilde{\mathbb{E}}_\mu^\lambda(Z_t^\Lambda(\lambda)^q \mid \xi) \\ & \leq \left[ \tilde{\mathbb{E}}_\mu^\lambda(Z_t^\Lambda(\lambda) \mid \xi) \right]^q \\ & \leq \langle e^{\lambda \cdot}, \mu \rangle^q + \left[ \tilde{\mathbb{E}}_\mu^\lambda \left( \sum_{s \leq t: \mathbf{n}} e^{\lambda c_\lambda s} Z_{t-s}^{\mathbf{n},s}(\lambda) \mid \xi \right) \right]^q + \left[ \tilde{\mathbb{E}}_\mu^\lambda \left( \sum_{s \leq t: \mathbf{m}} e^{\lambda c_\lambda s} Z_{t-s}^{\mathbf{m},s}(\lambda) \mid \xi \right) \right]^q, \end{aligned} \quad (14)$$

to get the last inequality we have used the fact that  $(\sum_i u_i)^q \leq \sum_i u_i^q$  with  $u_i \geq 0$ . On the one hand, recalling from (8) that  $\mathbb{N}_{\xi_s}[Z_{t-s}(\lambda)] = \mathbb{E}_{\xi_s}[Z_{t-s}(\lambda)]$ , we obtain

$$\begin{aligned} \tilde{\mathbb{E}}_\mu^\lambda \left( \sum_{s \leq t: \mathbf{n}} e^{\lambda c_\lambda s} Z_{t-s}^{\mathbf{n},s}(\lambda) \mid \xi \right) &= \int_0^t e^{\lambda c_\lambda s} \mathbb{N}_{\xi_s}[Z_{t-s}(\lambda)] ds \\ &= \int_0^t e^{\lambda(\xi_s + c_\lambda s)} ds. \end{aligned} \quad (15)$$

On the other hand, we have that

$$\begin{aligned} \tilde{\mathbb{E}}_\mu^\lambda \left( \sum_{s \leq t: \mathbf{m}} e^{\lambda c_\lambda s} Z_{t-s}^{\mathbf{m},s}(\lambda) \mid \xi \right) &= \tilde{\mathbb{E}}_\mu^\lambda \left[ \tilde{\mathbb{E}}_\mu^\lambda \left( \sum_{s \leq t: \mathbf{m}} e^{\lambda c_\lambda s} Z_{t-s}^{\mathbf{m},s}(\lambda) \mid \xi, \mathbf{m} \right) \mid \xi \right] \\ &= \tilde{\mathbb{E}}_\mu^\lambda \left( \sum_{s \leq t: \mathbf{m}} m_s e^{\lambda(\xi_s + c_\lambda s)} \mid \xi \right) \\ &= \sum_{s \leq t: \mathbf{m}} m_s e^{\lambda(\xi_s + c_\lambda s)}, \end{aligned} \quad (16)$$

where for  $s \geq 0$ ,  $m_s = \|X_0^{\mathbf{m},s}\|$ . In particular note that the process  $\{m_t : t \geq 0\}$  is a Poisson point process on  $(0, \infty)^2$ , independent of  $\xi$ , with intensity  $dt \times r\nu(dr)$ . Then, putting (15)

and (16) into (14), making use again of the inequality  $(\sum_i u_i)^q \leq \sum_i u_i^q$  where  $u_i \geq 0$  for all  $i$ , we obtain

$$\begin{aligned} \tilde{\mathbb{E}}_\mu^\lambda (Z_t^\Lambda(\lambda)^q | \xi) &\leq \langle e^{\lambda \cdot}, \mu \rangle^q + \left( \int_0^t e^{\lambda(\xi_s + c_\lambda s)} ds \right)^q + \sum_{s \leq t: \mathbf{m}} m_s^q e^{q\lambda(\xi_s + c_\lambda s)} \\ &\leq \langle e^{\lambda \cdot}, \mu \rangle^q + \left( \int_0^\infty e^{\lambda(\xi_s + c_\lambda s)} ds \right)^q + \sum_{s \geq 0: \mathbf{m}} m_s^q e^{q\lambda(\xi_s + c_\lambda s)}. \end{aligned} \quad (17)$$

Taking expectations again in (17) gives us that, for all  $t \geq 0$ ,

$$\tilde{\mathbb{E}}_\mu^\lambda (Z_t^\Lambda(\lambda)^q) \leq \langle e^{\lambda \cdot}, \mu \rangle^q + \Pi_\mu^\lambda \left[ \left( \int_0^\infty e^{\lambda(\xi_s + c_\lambda s)} ds \right)^q \right] + \tilde{\mathbb{E}}_\mu^\lambda \left( \sum_{s \geq 0: \mathbf{m}} m_s^q e^{q\lambda(\xi_s + c_\lambda s)} \right). \quad (18)$$

We know that, under  $\Pi_\mu^\lambda$ , the process  $\xi$  is a Brownian motion with drift  $\lambda$ . Thus, with respect to the same measure,  $\xi_s + c_\lambda s$  is a Brownian motion with drift  $\lambda + c_\lambda$  which is strictly negative for  $\lambda \in (0, \sqrt{-2\psi'(0+)})$ . Note that this latter condition in particular holds under assumption that  $p\lambda^2 < -2\psi'(0+)$  and  $p > 1$ . From Section 2 of Maulik and Zwart [14] we can conclude that

$$\Pi_0^\lambda \left( \int_0^\infty e^{\lambda(\xi_s + c_\lambda s)} ds \right) < \infty,$$

which in turn implies that, for all  $q \in (0, 1]$ ,

$$\Pi_0^\lambda \left[ \left( \int_0^\infty e^{\lambda(\xi_s + c_\lambda s)} ds \right)^q \right] < \infty,$$

and hence

$$\begin{aligned} \Pi_\mu^\lambda \left[ \left( \int_0^\infty e^{\lambda(\xi_s + c_\lambda s)} ds \right)^q \right] &= \frac{1}{\langle e^{\lambda \cdot}, \mu \rangle} \int e^{\lambda x} \mu(dx) \Pi_0^\lambda \left[ \left( \int_0^\infty e^{\lambda(x + \xi_s + c_\lambda s)} ds \right)^q \right] \\ &= \frac{\langle e^{\lambda p \cdot}, \mu \rangle}{\langle e^{\lambda \cdot}, \mu \rangle} \Pi_0^\lambda \left[ \left( \int_0^\infty e^{\lambda(\xi_s + c_\lambda s)} ds \right)^q \right] < \infty. \end{aligned} \quad (19)$$

It remains to prove that the last term in (18) is finite. This can be done by computing the expectation directly. We obtain,

$$\begin{aligned} \tilde{\mathbb{E}}_\mu^\lambda \left( \sum_{s \geq 0: \mathbf{m}} m_s^q e^{q\lambda(\xi_s + c_\lambda s)} \right) &= \int_0^\infty ds \int_0^\infty r \nu(dr) r^q \Pi_\mu^\lambda (e^{q\lambda(\xi_s + c_\lambda s)}) \\ &= \int_0^\infty ds \int_0^\infty r^p \nu(dr) \frac{1}{\langle e^{\lambda \cdot}, \mu \rangle} \int e^{\lambda x} \mu(dx) \Pi_0^\lambda (e^{q\lambda(x + \xi_s + c_\lambda s)}) \\ &= e^{q\lambda x} \frac{\langle e^{\lambda p \cdot}, \mu \rangle}{\langle e^{\lambda \cdot}, \mu \rangle} \int_0^\infty r^p \nu(dr) \int_0^\infty \Pi_0^\lambda (e^{q\lambda(\xi_s + c_\lambda s)}) ds. \end{aligned}$$

Note that,

$$\begin{aligned} \Pi_0^\lambda (e^{q\lambda(\xi_s + c_\lambda s)}) &= \exp\{qs\lambda^2 + s(q\lambda)^2/2 + qs\psi'(0+) - qs\lambda^2/2\} \\ &= \exp\{qs(p\lambda^2/2 + \psi'(0+))\} \end{aligned}$$

for all  $s \geq 0$ . Moreover, this expectation has a negative exponent as soon as  $p\lambda^2 < -\psi'(0+)$ . Together with the assumption  $\int_0^\infty r^p \nu(dr) < \infty$  we conclude that

$$\tilde{\mathbb{E}}_\mu^\lambda \left( \sum_{s \geq 0: \mathbf{m}} m_s^q e^{q\lambda(\xi_s + c_\lambda s)} \right) < \infty. \quad (20)$$

Finally, from (18)-(20) we get that

$$\sup_{t \geq 0} \tilde{\mathbb{E}}_\mu^\lambda (Z_t^\Lambda(\lambda)^q) < \infty,$$

which, in combination with (13), completes the proof.  $\square$

## Acknowledgments

The second author would like to thank the University of Bath, where most of this research was done. He also acknowledges the financial support of CONACYT-Mexico grant number 129076.

## References

- [1] E.B. Dynkin (1991): Branching particle systems and superprocesses, *Ann. Probab.* **19(3)**, 1157–1194.
- [2] E.B. Dynkin (1991): A probabilistic approach to one class of non-linear differential equations. *Probab. Th. Rel. Fields.* **89**, 89–115.
- [3] E.B. Dynkin (1993): Superprocesses and Partial Differential Equations. *Ann. Probab.* **21**, 1185–1262.
- [4] E.B. Dynkin (2002): *Diffusions, Superdiffusions and Partial Differential Equations.* AMS, Providencem R.I.
- [5] E.B. Dynkin and S.E. Kuznetsov (2004):  $\mathbb{N}$ -measures for branching Markov exit systems and their applications to differential equations. *Probab. Th. Rel. Fields.* **130**, 135–150.
- [6] Engländer, J. and Pinsky, R. (1999): On the construction and support properties of measure-valued diffusions on  $D \subseteq R^d$  with spatially dependent branching. *Ann. Probab.* **27**, 684–730.



- [7] S.N. Evans (1993): Two representations of a superprocess. *Proc. Royal. Soc. Edin.* **123A**, 959–971.
- [8] P.J. Fitzsimmons (1988): Construction and regularity of measure-valued Markov branching processes. *Israeli J. Math.* **64**, 337–361.
- [9] J. Engländer and A.E. Kyprianou (2004): Local extinction versus local exponential growth for spatial branching processes. *Ann. Probab.* **32**, 78–99.
- [10] R. Hardy and S.C. Harris (2009): A spine approach to branching diffusions with applications to  $\mathcal{L}^p$ -convergence of martingales. *Séminaire de Probabilités*, **XLII**, 281–330.
- [11] Kyprianou, A.E. (2004): Travelling wave solutions to the K-P-P equation: alternatives to Simon Harris’ probabilistic analysis. *Ann. Inst. H. Poincaré.* **40**, 53–72.
- [12] Kyprianou, A.E. (2006): *Introductory lectures on fluctuations of Lévy processes with applications*. Springer.
- [13] A.E. Kyprianou, R.-L. Liu, A. Murillo-Salas and Y.-X. Ren. (2011): Supercritical super-Brownian motion with a general branching mechanism and travelling waves. *To appear in Ann. Inst. H. Poincaré.*
- [14] K. Maulik and B. Zwart (2006): Tail asymptotics for exponential functionals of Lévy processes. *Stoch. Proc. Appl.* **116**, 156–177.
- [15] S. Watanabe (1968): A limit theorem of branching processes and continuous-state branching processes. *J. Math. Kyoto Univ.* **8**, 141–167.

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF BATH, CLAVERTON DOWN,  
BATH BA2 7AY, UNITED KINGDOM.

*E-mail*: a.kyprianou@bath.ac.uk

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE GUANAJUATO, JALISCO S/N MIN-  
ERAL DE VALENCIANA, GUANAJUATO, GTO. C.P. 36240, MÉXICO.

*E-mail*: amurillos@ugto.mx