Scaling limits of inhomogeneous trees and graphs

Part-I. Scaling limits of random walks

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Main questions

Random walk: we say $(S_k)_{k\geq 0}$ is a random walk if

$$\forall\, k \geq 0: \quad S_k = \sum_{i=1}^{k} \Delta_i, \qquad \text{where } (\Delta_i)_{i\geq 1} \text{ i.i.d. } \mathbb{R}\text{-valued random variables.}$$

Assume that

$$(\mathrm{Criticality}) \qquad \mathbb{E}|\Delta_1| < \infty \;\text{ and }\; \mathbb{E}[\Delta_1] = 0.$$

For each $n \geq 1$, let $S^{(n)} = (S_k^{(n)})_{k\geq 0}$ be a centred random walk. We are interested in the convergence of the type

$$(\mathrm{CV_S}) \qquad a_n \to \infty, \qquad \left(\tfrac{1}{a_n} S^{(n)}_{\lfloor nt \rfloor} : t \geq 0\right) \xrightarrow[n\to\infty]{(d)} (X_t : t \geq 0)$$

Main questions

- What will be $(X_t)_{t\geq 0}$?
- When does $(\mathrm{CV_S})$ take place?

Infinitely divisible distributions

Let us first consider the convergence of the marginal distributions in $(\mathrm{CV_S})$. Take $t = 1$, then

$$(\mathrm{CV}_1) \qquad \frac{1}{a_n}(\Delta_1^{(n)} + \cdots + \Delta_n^{(n)}) \to X_1.$$

Infinitely divisible distribution: an $\mathbb{R}$-valued random variable X is said to have ∞-divisible distribution if for all $n \in \mathbb{N}$, we can write

$$X = Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)},$$

where $Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_n^{(n)}$ are some i.i.d. random variables.

Proposition

X has infinitely divisible distribution if and only if for all n, $(Y_i^{(n)})_{1 \le i \le n}$ are i.i.d. and

$$Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)} \xrightarrow[n \to \infty]{(d)} X$$

Proof: See e.g. [Feller '71], Vol. II, Ch. XVII.1, Thm. 2.

It follows that the limit X_1 in (CV_1) necessarily has infinitely divisible distribution.

Infinitely divisible distributions and Lévy processes

Lévy process: an $\mathbb{R}$-valued stochastic process $\mathbf{X} = (X_t)_{t \geq 0}$ is called a Lévy process if it has independent and stationary increments, namely, for all $0 \leq s \leq t < \infty$,

- $X_t - X_s$ is independent of $(X_u)_{u \leq s}$;
- $X_t - X_s$ is distributed as X_{t-s}.

Rem. A Lévy process has a càdlàg version.

We always suppose $X_0 = 0$.

Examples of Lévy processes include Brownian Motion, compound Poisson process, stable process.

Proposition

- If $X = (X_t)_{t \geq 0}$ is a Lévy process, then X_1 has infinitely divisible distribution.
- If μ is an infinitely divisible distribution, one can always find a Lévy process X such that X_1 has distribution μ.

Proof: See e.g. [Bertoin '96], Ch.1.1.

Convergence of random walks

Back to our initial problem:

$$(\mathrm{CV_S}) \qquad \left(\tfrac{1}{a_n} S^{(n)}_{\lfloor nt \rfloor} : t \geq 0\right) \xrightarrow[n\to\infty]{(d)} (X_t : t \geq 0)$$

Previous arguments imply that $X = (X_t)_{t\geq 0}$ is a Lévy process.

It turns out that to study the convergence in $(\mathrm{CV_S})$, it suffices to study the convergence for $t = 1$.

Theorem

The following statements are equivalent.

- $\left\{\frac{1}{a_n} S^{(n)}_n : n \geq 1\right\}$ converges in distribution to some random variable X_1.
- $\left\{(\frac{1}{a_n} S^{(n)}_{\lfloor nt \rfloor})_{t\geq 0} : n \geq 1\right\}$ converges in distribution to some process X in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$.

Moreover, if the above holds true, then X is necessarily a Lévy process.

Proof: See e.g. [Jacod & Shiryaev '03], Ch.VII.3, Cor.3.6

Restriction to spectrally positive case

From now on, we are going to focus on the spectrally positive (or skip free) case, which is most relative to random trees and graphs. Namely, we suppose that

- for all $n \geq 1$, the increments of the random walks satisfy $\Delta_i^{(n)} \geq -1$, a.s.
- The limit Lévy process $X = (X_t)$ does not have negative jumps.

We also suppose X is centred, namely, $\mathbb{E}|X| < \infty$ and $\mathbb{E}[X] = 0$.

Laplace exponent: under these assumptions, we can work with the Laplace transforms.

$$e^{\phi_n(\lambda)} := \mathbb{E}[e^{-\lambda S_n^{(n)}}] < \infty \quad \text{and} \quad e^{\phi(\lambda)} := \mathbb{E}[e^{-\lambda X_1}] < \infty, \text{ for all } \lambda \geq 0.$$

Theorem

We have

$$\frac{1}{a_n} S_n^{(n)} \xrightarrow{(d)} X_1 \quad \Longleftrightarrow \quad \forall \lambda \geq 0 : \quad \phi_n(\lambda/a_n) \to \phi(\lambda).$$

Proof: For $\Rightarrow$, see [Grimvall '74] Theorem 2.1. For $\Leftarrow$, see [B.-D.-W. '18+], Appendix A.

Rem. This is not the usual continuity theorem for Laplace transforms of measures on $\mathbb{R}_+$, as S_n and X_1 take values in $\mathbb{R}$.

Poisson approximation

We introduce a compensated compound Poisson process:

$$\forall\, t \geq 0: \quad \tilde{S}_t^{(n)} = -t + \sum_{i=1}^{N_t} \left(\Delta_i^{(n)} + 1\right), \quad \text{where } (N_t)_{t\geq 0} \text{ is a Poisson process of rate 1.}$$

Proposition

Suppose that $\{\frac{1}{a_n} S_n^{(n)} : n \geq 1\}$ converges. Then for each $t \geq 0$,

$$\frac{1}{a_n}\left(S_{\lfloor nt \rfloor}^{(n)} - \tilde{S}_{nt}^{(n)}\right) \xrightarrow{\mathbb{P}} 0.$$

Proof sketch: let $\varphi_n(\lambda) := \mathbb{E} e^{-\lambda \Delta_1^{(n)}}$. Then $\phi_n(\lambda) = n \log \varphi_n(\lambda)$. Since $\phi_n(\lambda / a_n)$ converges, necessarily $\varphi_n(\lambda) \to 1$. Note that $\tilde{S}_1^{(n)}$ has Laplace exponent $\tilde{\phi}_n(\lambda) = \lambda - 1 + e^{-\lambda} \varphi_n(\lambda)$. The statement follows by comparing ϕ_n and $\tilde{\phi}_n$.

Note that $\tilde{S}^{(n)}$ is itself a spectrally positive Lévy process. So the problem reduces to studying the convergence of Lévy processes.

Lévy–Khintchine formula

Let $X = (X_t)_{t \geq 0}$ be a spectrally positive centred Lévy process. Then its Laplace exponent $\phi(\lambda) = \log \mathbb{E}[e^{-\lambda X_1}]$ takes the following form

$$\phi(\lambda) = \tfrac{1}{2}\beta\lambda^2 + \int (e^{-\lambda x} - 1 + \lambda x)\pi(dx) = \int \frac{e^{-\lambda x} - 1 + \lambda x}{x \wedge x^2}\nu(dx),$$

where the Gaussian coefficient $\beta \in \mathbb{R}_+$ and the Lévy measure π is supported on $(0, \infty)$ satisfying $\int (x \wedge x^2)\pi(dx) < \infty$,

$$\nu(dx) = \beta\delta_0(dx) + (x \wedge x^2)\pi(dx)$$

is a finite measure called the **characteristics** of X.

Theorem

Let $\{X^{(n)} : n \geq 1\}$ be a sequence of spectrally positive and centred Lévy processes with Laplace exponents ϕ_n and characteristics ν_n. Then

$$X^{(n)} \xrightarrow{(d)} X \text{ in } \mathbb{D} \iff \forall\, \lambda \geq 0 : \phi_n(\lambda) \to \phi(\lambda) \iff \nu_n \xrightarrow{w} \nu.$$

Rem. $\nu_n \xrightarrow{w} \nu \iff \nu_n(\mathbb{R}_+) \to \nu(\mathbb{R}_+)$ and $\forall\, f \in C_c((0,\infty)) : \int f\, d\nu_n \to \int f\, d\nu$.

$$\iff \beta_n + \int (x \wedge x^2)\pi_n(dx) \to \beta + \int (x \wedge x^2)\pi(dx)$$

and $\forall\, f \in C_c((0,\infty)) : \int f\, d\pi_n \to \int f\, d\pi$

Convergence of random walks: Main theorem

Applying the previous theorem to $(\frac{1}{a_n}\tilde{S}_{nt})_{t\geq 0}$, we find the following

Theorem

- $S_k^{(n)} = \sum_{i\leq k} \Delta_i^{(n)}$, $k \geq 0$, is a spectrally positive centred random walk.
- $X =$ a spect. positive centred Lévy process with Gaussian coeff. β and Lévy measure π.

$$(\mathrm{CV_S}) \qquad a_n \to \infty, \qquad \left(\frac{1}{a_n} S^{(n)}_{\lfloor nt \rfloor} : t \geq 0\right) \xrightarrow[n\to\infty]{(d)} (X_t : t \geq 0) \quad \text{in } \mathbb{D},$$

if and only if the following conditions hold:

$$(\mathrm{Cond_a}) \qquad n\,\mathbb{E}\left[\frac{\Delta_1^{(n)}}{a_n} \wedge \left(\frac{\Delta_1^{(n)}}{a_n}\right)^2\right] \to \beta + \int (x \wedge x^2)\pi(dx).$$

$$(\mathrm{Cond_b}) \qquad \forall\, f \in C_c((0,\infty)) : \quad n\,\mathbb{E}\left[f\left(\frac{\Delta_1^{(n)}}{a_n}\right)\right] \to \int f(x)\pi(dx).$$

Convergence of random walks: more general form

The above arguments can be easily extended to the case

- $\mathbb{E}[\Delta_1^{(n)}] = \alpha_n \to 0$.
- Consider the rescaled random walk $(\frac{1}{a_n} S^{(n)}_{\lfloor b_n t \rfloor})_{t \geq 0}$

Theorem

- $S_k^{(n)} = \sum_{i \leq k} \Delta_i^{(n)}$, $k \geq 0$, is a spectrally positive random walk.
- X = a spect. positive Lévy process with drift α, Gaussian coeff. β and Lévy measure π.

$$(\mathrm{CV_S}) \qquad a_n \to \infty, \; b_n \to \infty, \qquad \left(\frac{1}{a_n} S^{(n)}_{\lfloor b_n t \rfloor} : t \geq 0\right) \xrightarrow[n\to\infty]{(d)} (X_t : t \geq 0) \quad \text{in } \mathbb{D},$$

if and only if the following conditions hold:

$$(\mathrm{Cond_a}) \qquad b_n \, \mathbb{E}\left[\frac{\Delta_1^{(n)}}{a_n} \wedge \left(\frac{\Delta_1^{(n)}}{a_n}\right)^2\right] \to \beta + \int (x \wedge x^2) \pi(dx).$$

$$(\mathrm{Cond_b}) \qquad \forall f \in C_c((0,\infty)) : \quad b_n \, \mathbb{E}\left[f\left(\frac{\Delta_1^{(n)}}{a_n}\right)\right] \to \int f(x) \pi(dx).$$

$$(\mathrm{Cond_c}) \qquad \frac{b_n}{a_n} \, \mathbb{E}[\Delta_1^{(n)}] \to \alpha.$$

Summary

Scaling limits of random walks:

$$(\mathrm{CV_S}) \quad a_n \to \infty, \ \frac{b_n}{a_n} \to \infty, \quad \left(\frac{1}{a_n} S^{(n)}_{\lfloor b_n t \rfloor} : t \geq 0\right) \xrightarrow[n\to\infty]{(d)} (X_t : t \geq 0) \quad \text{in } \mathbb{D},$$

- Characterisation of the limit
- Poisson approximation of random walks
- Necessary and sufficient conditions for the convergence

¡Muchas Gracias!