Generalized scale functions and properties of refracted processes

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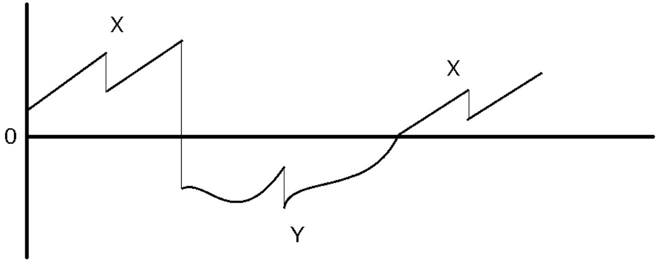
What is refracted processes?

X, Y: $\mathbb{R}$-valued standard processes with no positive jumps
Let U be a process behaves as X when U has positive values and Y when U has negative values. We call U a refracted process.



We generalize scale functions of spectrally negative Lévy processes and think a duality problem of refracted processes using the excursion theory.

Preceding study (scale functions)

X: spectrally negative Lévy process.
$\Psi(\lambda) := \log \mathbb{E}_0^X\left[e^{\lambda X_1}\right]$, $\lambda \geq 0$. $\Phi(\theta) := \sup\{\lambda \geq 0 : \Psi(\lambda) = \theta\}$, $\theta \geq 0$.

Let $W^{(q)}$ ($q \geq 0$) be a right-continuous function from $\mathbb{R}$ to $[0, \infty)$ satisfying

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\Psi(\beta) - q}, \quad \beta > \Phi(q).$$

(see e.g., Kyprianou(2014)).
$W^{(q)}$ is called the q-scale function of X.
For $q \geq 0$ and $b < x < a$,

$$\mathbb{E}_x^X\left[e^{-qT_a^+}; T_a^+ < T_b^-\right] = \frac{W^{(q)}(x-b)}{W^{(q)}(a-b)},$$

where $T_a^+ = \inf\{t > 0 : X_t \geq a\}$ and $T_b^- = \inf\{t > 0 : X_t \leq b\}$.

Preceding study (scale functions)

For $q \geq 0$, $b < x < a$ and non-negative measurable function f,

$$\mathbb{E}_x^X\left[\int_0^{T_a^+ \wedge T_b^-} e^{-qt} f(X_t) dt\right] = \int_b^a f(y)\left(\frac{W^{(q)}(x-b)W^{(q)}(a-y)}{W^{(q)}(a-b)} - W^{(q)}(x-y)\right) dy.$$

It is known that

$$W^{(q)}(x) = \frac{1}{n_0^X\left[e^{-qT_x^+}; T_x^+ < \infty\right]},$$

where n_0^X is an excursion measure away from 0 subject to the normalization

$$n_0^X\left[1 - e^{-qT_0}\right] = \frac{1}{\Phi'(q)}, \qquad q > 0$$

where $T_x = \inf\{t > 0 : X_t = x\}$ (see e.g., N.–Yano([7])).

Standard processes

X: $\mathbb{R}$-valued standard processes with no positive jumps satisfying following conditions:

- $(x, y) \to \mathbb{E}_x^X \left[e^{-T_y} \right] > 0$ is a $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$-measurable function.
- X has the Feller property.
- X has a reference measure m^X, i.e., for all $q \geq 0$ and $x \in \mathbb{R}$,

$$\mathbb{E}_x \left[\int_0^\infty e^{-qt} 1_{\{X_t \in \ \cdot \ \}} dt \right] \ll m_X(\ \cdot \ ).$$

Local times

By Blumenthal–Getoor(1968) pp.216, if $x \in \mathbb{R}$ is regular for itself, X has a continuous local time $L^{X,x}$ at x.
If $x \in \mathbb{R}$ is irregular for itself , define local time $L^{X,x}$ at x as

$$L_t^{X,x} = l_x \# \{0 \le s < t : X_s = x\}$$

where $l_x \in (0,\infty)$ is a constant.
By [3, Theorem 18.4], there exists local times $L^{X,x}$ satisfying

$$\int_0^t f(X_s) ds = \int_{\mathbb{R}} f(y) L_t^{X,y} m_X(dy), \quad \text{a.s.}$$

$$R_X^{(q)} f(x) := \mathbb{E}_x^X \left[\int_0^\infty e^{-qt} f(X_t) dt \right] = \int_{\mathbb{R}} f(y) \mathbb{E}_x^X \left[\int_0^\infty e^{-qt} dL_t^{X,y} \right] m_X(dy).$$

[3] D. Geman and J. Horowitz. Occupation densities. Ann. Probab. 8 (1980), no. 1, 1-67.

Excursion measures

[8] D. Revuz. Mesures associées aux fonctionnelles additives de Markov. I. (French) Trans. Amer. Math. Soc. 148 1970 501–531.

By [8], if X has a dual process, we can take $\{L^{X,x}\}_{x\in\mathbb{R}}$ such that $x \mapsto \mathbb{E}_x^X\left[\int_0^\infty e^{-qt} dL_t^{X,y}\right]$ is right-continuous and $y \mapsto \mathbb{E}_x^X\left[\int_0^\infty e^{-qt} dL_t^{X,y}\right]$ is left-continuous.

When X does not have dual processes, we define local times by Geman-Horowitz([3]).
When X has dual process, we define local times by Revuz([8]).

For $x \in \mathbb{R}$, define the excursion measure n_x^X away from x with the following normalization:

$$n_x^X\left[1 - e^{-qT_0}\right] = \frac{1}{\mathbb{E}_x^X\left[\int_{0-}^\infty e^{-qt} dL_t^{X,x}\right]}, \qquad q > 0.$$

Generalized scale functions

Definition 2.1

For $q \geq 0$ and $x, y \in \mathbb{R}$, we define q-scale function $W_X^{(q)}$ as

$$W_X^{(q)}(x,y) = \begin{cases} \frac{1}{n_y^X\left[e^{-qT_x^+}; T_x^+ < \infty\right]} & x \geq y, \\ 0 & x < y. \end{cases}$$

We fix $b < a \in \mathbb{R}$.

Theorem 2.1 (N.)

For $q \geq 0$ and $x \in (b, a)$, we have

$$\mathbb{E}_x^X\left[e^{-qT_a^+}; T_a^+ < T_b^-\right] = \frac{W_X^{(q)}(x,b)}{W_X^{(q)}(a,b)}.$$

Generalized scale functions

Since $\{L^{X,x}\}_{x\in\mathbb{R}}$ satisfies the occupation formula, we have

$$\mathbb{E}_x^X\left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} f(X_t)dt\right] = \int_{\mathbb{R}} f(y)\mathbb{E}_x^X\left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL_t^{X,y}\right] m_X(dy).$$

Theorem 2.2 (N.)

For $q \geq 0$ and $x, y \in (b, a)$, we have

$$\mathbb{E}_x^X\left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL_t^{X,y}\right] = \frac{W_X^{(q)}(x,b)}{W_X^{(q)}(a,b)} W_X^{(q)}(a,y) - W_X^{(q)}(x,y).$$

Generalized scale functions

We think about a property of generalized scale function when X has a dual process.

Definition 2.2 (see e.g., Chung–Walsh(2005))

Z and $\widehat{Z}$ are in duality relative to m_Z if

- *For $q > 0$, non-negative measurable functions f and g,*

$$\int_{\mathbb{R}} f(x) R_Z^{(q)} g(x) m_Z(dx) = \int_{\mathbb{R}} R_{\widehat{Z}}^{(q)} f(x) g(x) m_Z(dx).$$

- *For $q \geq 0$ and $x \in \mathbb{R}$, $R_Z^{(q)} 1_{(\cdot)}(x) \ll m_Z(\cdot)$ and $R_{\widehat{Z}}^{(q)} 1_{(\cdot)}(x) \ll m_Z(\cdot)$.*

Generalized scale functions

$\widehat{X}$: $\mathbb{R}$-valued standard processes with no negative jumps satisfying following conditions:

- $(x, y) \to \mathbb{E}_x^{\widehat{X}}\left[e^{-T_y}\right] > 0$ is a $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$-measurable function.
- $\widehat{X}$ has the Feller property.
- $\widehat{X}$ has a reference measure m^X.

We define local times $\{L^{\widehat{X},x}\}_{x\in\mathbb{R}}$, excursion measures $\{n_x^{\widehat{X}}\}_{x\in\mathbb{R}}$ and scale functions $\{W_{-\widehat{X}}^{(q)}\}_{q\geq 0}$ by the same way as X's.

When we don't know if X and $\widehat{X}$ are in duality relative to m_X, we define local times by Geman–Horowitz([3]).
When X and $\widehat{X}$ are in duality relative to m_X, we define local times by Revuz([8]).

Theorem 2.3 (N.)

If X and $\widehat{X}$ are in duality relative to m_X, then we have

$$W_X^{(q)}(x,y) = W_{-\widehat{X}}^{(q)}(-y,-x), \qquad x,y \in \mathbb{R}.$$

The converse is also true.

Remark

When X is a spectrally negative Lévy process, the scale function defined by

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\Psi(\beta) - q}, \quad \beta > \Phi(q).$$

satisfies

$$W_X^{(q)}(x, y) = W^{(q)}(x - y), \quad x, y \in \mathbb{R}.$$

The theorem of duality is obvious since $-\widehat{X} \overset{d}{=} X$ and we have

$$W_X^{(q)}(x, y) = W^{(q)}(y - x) = W_{-\widehat{X}}^{(q)}(-y, -x).$$

Preceding study (refracted processes) [6]

[6] A. E. Kyprianou and R. L. Loeffen. Refracted Lévy processes. Ann. Inst. Henri Poincaré Probab. Stat. 46 (2010), no. 1, 24-44.

X, Y: spectrally negative Lévy processes satisfying

$$X_t + \delta t = Y_t, \quad t \geq 0, \qquad \text{a.s.}$$

Define a stochastic differential equation

$$U_t = U_0 + Y_t - \delta \int_0^t 1_{\{U_s > 0\}} ds, \qquad t \geq 0. \tag{1}$$

Theorem 3.1 ([6])

The stochastic differential equation (1) *has a unique strong solution.*

In this case, the difference between X and Y is only drift.

Preceding study (refracted processes) [6]

[6] A. E. Kyprianou and R. L. Loeffen. Refracted Lévy processes. Ann. Inst. Henri Poincaré Probab. Stat. 46 (2010), no. 1, 24-44.

For $q \geq 0$, they defined a function $W_U^{(q)} : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ satisfying for $b < x < a$ and non-negative measurable function f,

$$\mathbb{E}_x^U \left[e^{-qT_a^+} ; T_a^+ < T_b^- \right] = \frac{W_U^{(q)}(x, b)}{W_U^{(q)}(a, b)}, \tag{2}$$

$$\mathbb{E}_x^U \left[\int_0^{T_a^+ \wedge T_b^-} e^{-qt} f(U_t) dt \right] = \int_b^a \left(\frac{W_U^{(q)}(x, b)}{W_U^{(q)}(a, b)} W_U^{(q)}(a, y) - W_U^{(q)}(x, y) \right) f(y) dy. \tag{3}$$

Preceding study (refracted processes) [7]

[7] K. Noba and K. Yano. Generalized refracted Lévy process and its application to exit problem. arXiv:1608.05359; to appear in Stochastic Process. Appl.

X, Y: spectrally negative Lévy processes satisfying followings:

- X has no Gaussian part.
- X and Y may have different Lévy measures.

In order to define a refracted Lévy process U, we utilize the excursion theory.

In this study, we define $W_U^{(q)}$ satisfying (2) and (3).

Definition of refracted processes

We want to define refracted processes by standard processes X and Y.

Y: $\mathbb{R}$-valued standard processes with no positive jumps satisfying following conditions:

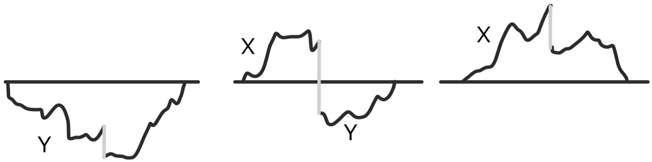
- $(x, y) \to \mathbb{E}_x^Y \left[e^{-T_y} \right] > 0$ is a $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$-measurable function.
- Y has the Feller property.
- Y has a reference measure m^Y.

Definition of refracted processes

$\psi : (0,\infty) \times (-\infty,0) \to (-\infty,0)$
We define an excursion measure n_0^U away from 0 that satisfy

$$\begin{aligned} n_0^U[F(U)] = c_0 n_0^Y \left[F(Y);\, T_0^- = 0\right] \\ + n_0^X \left[\mathbb{E}^{Y^0}_{\psi(X_{T_0^- -}, X_{T_0^-})}[F(\omega \circ Y)] \,\Big|_{\omega = k_{T_0^-} X};\, 0 < T_0^- < T_0\right] \\ + n_0^X \left[F(X);\, T_0^- = T_0\right] \end{aligned}$$

for all non-negative measurable functional F where $c_0 \in [0,\infty)$.
(if $\mathbb{P}_0^X[T_0 > 0] = 1$, we assume that $c_0 = 0$)

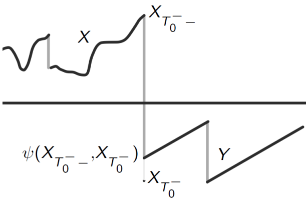


We define the lawt of stopped process $\mathbb{P}_x^{U^0}$ for $x \neq 0$ by the same way.

Definition of refracted processes

We assume that $\psi : (0,\infty) \times (-\infty,0) \to (-\infty,0)$ satisfies following condition:

$$n_0^X \left[1 - e^{-T_0^-} \mathbb{E}^Y_{\psi(X_{T_0^- -}, X_{T_0^-})} \left[e^{-T_0} \right] ; 0 < T_0^- < T_0 \right] < \infty.$$



$\Rightarrow$ we have $n_0^U \left[1 - e^{-T_0} \right] < \infty$.
We call ψ the landing function, because the landing point is changed from $X_{T_0^-}$ to $\psi(X_{T_0^- -}, X_{T_0^-})$.

Definition of refracted processes

[9] T. S. Salisbury. On the Itô excursion process. Probab. Theory Related Fields 73 (1986), no. 3, 319–350.

By the excursion theory, there is a right continuous strong Markov process U which has laws of stopped process $\mathbb{P}_x^{U^0}$, an excursion measure n_0^U and no stagnancy at 0 (see [9]).

Lemma 3.1 (N.)

U is a Feller process. So U is a standard process.

Duality problem

We suppose that X and Y have a dual process $\widehat{X}$ and $\widehat{Y}$, respectively. We make refracted processes U by X and Y, $\widehat{U}$ by $\widehat{X}$ and $\widehat{Y}$ and σ-finite measure m_U.
Think about the necessarily and sufficient condition that U and $\widehat{U}$ are in duality relative to m_U.

We suppose that X and Y are recurrent processes.
Let $\widehat{X}$ and $\widehat{Y}$ be dual processes of X and Y to m_X and m_Y, respectively.
Suppose that all excursions away from 0 of X and Y begin and end at 0.
U obtained from X, Y, ψ and $c_0 = 1$;
$\widehat{U}$ obtained from $\widehat{X}$, $\widehat{Y}$, ϕ and $c_0 = 1$.
Define $m_U = m_X|_{[0,\infty)} + m_Y|_{(-\infty,0)}$.

Duality problem

Theorem 4.1 (N.)

If we have, for non-negative measurable function h,

$$n_0^X\left[h(X_{T_0^- -}, \psi(X_{T_0^- -}, X_{T_0^-})); 0 < T_0^- < T_0\right] = n_0^Y\left[h(\phi(Y_{T_0^-}, Y_{T_0^- -}), Y_{T_0^-}); 0 < T_0^- < T_0\right], \tag{4}$$

then U and $\widehat{U}$ are in duality relative to m_U.
Conversely, if U and $\widehat{U}$ are in duality relative to m_U, we can normalize n_0^X and n_0^Y to satisfy (4).

Steps of the proof of Theorem 4.1

($\Rightarrow$)

1. $m_U = m_X|_{[0,\infty)} + m_Y|_{(-\infty,0)}$ is a reference measure of U and $\widehat{U}$.
2. We define $W_U^{(q)}$ and $W_{-\widehat{U}}^{(q)}$.
3. We prove $n_0^U[\ \cdot\ ] \overset{d}{=} n_0^{\widehat{U}}[\rho_0(\ \cdot\ )]$ by (4).
4. Using the identity above, we prove $W_U^{(q)}(x,y) = W_{-\widehat{U}}^{(q)}(-y,-x)$.

[4] R. K. Getoor and M. J. Sharpe. Two results on dual excursions. Seminar on Stochastic Processes, 1981 (Evanston, Ill., 1981), pp. 31-52, Progr. Prob. Statist., 1, Birkhäuser, Boston, Mass., 1981.

We prove only when 0 is regular for itself for X.

Lemma 4.1 ([4, Lemma 4.16])

Let Z and $\widehat{Z}$ be standard processes. We assume that Z and $\widehat{Z}$ have the following conditions:

- *Z and $\widehat{Z}$ are in duality.*
- *x is recurrent.*
- *All excursions of Z away from x begin and end at x.*

Then we have

$$n^Z_x[\ \cdot\ ] \overset{d}{=} n^{\widehat{Z}}_x[\rho_x(\ \cdot\ )].$$

$(\Leftarrow)$

By the duality of U and $\widehat{U}$, we can normalize n^U_0 and $n^{\widehat{U}}_0$ to satisfy

$$n^U_0[\ \cdot\ ] = n^{\widehat{U}}_0[\ \rho_0(\ \cdot\ )\ ].$$

Example

Let $\alpha, \beta \in (1,2)$ with $\alpha > \beta$.
X: spectrally negative α-stable process
with Lévy measure $\Pi_X(dx) = c_X 1_{\{x<0\}} |x|^{-\alpha-1} dx$.
Y: spectrally negative β-stable process
with Lévy measure $\Pi_Y(dx) = c_Y 1_{\{x<0\}} |x|^{-\beta-1} dx$.
$m_X(dx) := \frac{\alpha-1}{c_X} dx$, $m_Y(dx) := \frac{\beta-1}{c_Y} dx$.
$\psi(x,y) := y(x-y)^{\frac{\alpha-1}{\beta-1}-1}$, $\phi(x,y) := y(y-x)^{\frac{\beta-1}{\alpha-1}-1}$.

Then, refracted processes U obtained from X and Y and $\widehat{U}$ obtained from $\widehat{X}$ and $\widehat{Y}$ are well defined and in duality relative to m_U.

Summary

1. We defined scale functions of standard processes with no positive jumps.
2. We defined refracted process U by n_0^X, $\mathbb{P}_x^X$, n_0^Y, $\mathbb{P}_x^Y$, ψ and the excursion theory.
3. We supposed that X and Y has a dual process $\widehat{X}$ and $\widehat{Y}$, respectively. We defined a refracted process U by X and Y, and $\widehat{U}$ by $\widehat{X}$ and $\widehat{Y}$. We saw the necessarily and sufficient condition that U and $\widehat{U}$ are in duality.

[1] R. M. Blumenthal and R. K. Getoor. Markov processes and potential theory. Pure and Applied Mathematics, Vol. 29 Academic Press, New York-London 1968 x+313 pp.

[2] K. L. Chung and J. Walsh. Markov processes, Brownian motion, and time symmetry. Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 249. Springer, New York, 2005.

[3] D. Geman and J. Horowitz. Occupation densities. Ann. Probab. 8 (1980), no. 1, 1-67.

[4] R. K. Getoor and M. J. Sharpe. Two results on dual excursions. Seminar on Stochastic Processes, 1981 (Evanston, Ill., 1981), pp. 31-52, Progr. Prob. Statist., 1, Birkhäuser, Boston, Mass., 1981.

[5] A. E. Kyprianou. Fluctuations of Lévy processes with applications. Introductory lectures. Second edition. Universitext. Springer, Heidelberg, 2014. xviii+455 pp.

[6] A. E. Kyprianou and R. L. Loeffen. Refracted Lévy processes. Ann. Inst. Henri Poincaré Probab. Stat. 46 (2010), no. 1, 24-44.

[7] K. Noba and K. Yano. Generalized refracted Lévy process and its application to exit problem. arXiv:1608.05359; to appear in Stochastic Process. Appl.

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