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Summary. The purpose of this work is to define and study homogeneous fragmentation processes in continuous time, which are meant to describe the evolution of an object that breaks down randomly into pieces as time passes. Roughly, we show that the dynamic of such a fragmentation process is determined by some exchangeable measure on the set of partitions of $\mathbb{N}$, and results from the combination of two different phenomena: a continuous erosion and sudden dislocations. In particular, we determine the class of fragmentation measures which can arise in this setting, and investigate the evolution of the size of the fragment that contains a point pick at random at the initial time.

Key words. Fragmentation, exchangeable partitions, subordinator.

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1 Introduction

It is well-known that the analysis of random processes in continuous times may be much more subtle than that of their counterparts in discrete times. For instance, the law of a random walk, say in $\mathbb{R}^d$, is completely characterized by the distribution of its generic step, which is an arbitrary probability measure on $\mathbb{R}^d$. The continuous time analogue of a random walk is a Lévy process, that is a process with independent and homogeneous increments; its structure is only revealed by the combination of the celebrated Lévy-Khintchine formula and Lévy-Itô decomposition, which are two deep and difficult results of probability theory.

Similarly, it is easy to define random fragmentation chains in discrete times. Typically, consider at the initial time $t = 0$ a mass, say $m_0 \in]0, \infty[$. At time $t = 1$, this mass is broken, which gives rise to a random sequence of smaller masses, say, $m_{1,1} \geq m_{1,2} \geq \cdots$ with $m_{1,1} + m_{1,2} + \cdots \leq m_0$. Note that we do not impose equality to take into account the possibility that a part of the initial mass has been reduced to dust. It may be natural suppose the scaling invariance, that is that the law of the sequence of proportions $m_{1,1}/m_0, m_{1,2}/m_0, \ldots$ is the same whatever the initial mass m_0 is. The next steps consist of independent iterations, in the sense that at time $t = 2, 3, \ldots$, each fragment obtained at time $t-1$ is again randomly broken into pieces, independently of the other fragments and according to the same rule. Obviously, the law of such a fragmentation chain is completely determined by the distribution of the sequence of the ratios $m_{1,1}/m_0, m_{1,2}/m_0, \ldots$. The latter is called a fragmentation kernel by Pitman [13] to whom we also refer for some literature related to this topic. The purpose of this work is to investigate fragmentation in continuous times. Informally, we consider an object that breaks down into pieces as time passes in $[0, \infty)$, and suppose that the fragmentation process, which gives the masses of the fragments as a function of time, can be described as follows. The fragmentation at time t' is obtained from that at time $t < t'$ by breaking into pieces each component of the latter, independently of the other parts. We assume that the distribution of the random variable that gives the mass proportions of the fragments of this component only depends of the time increment $t' - t$. This means that the family of masses of the fragments resulting at time t' from a single object with mass $m > 0$ at time t, has the same distribution as that of $mF(t'-t)$, where $F(s)$ is the family of masses resulting from the fragmentation after a time duration s of an object with initial unit mass. Such a process will be referred to as a *homogeneous fragmentation*.

Let us give a simple example. Consider a continuous time Galton-Watson process, that is we have an ancestor who has a random number of children and dies after some independent exponential time. Suppose that initially the ancestor has some wealth that he spends at some fixed rate. When he dies, if he has no progeny, what remains of his belongings is lost (or is given to some institution). Otherwise a portion of the remaining wealth is used to pay the succession taxes, and the rest is split amongst his children, according to his will (the shares of the children may be different). Each child follows the same evolution, independently of the other children, except of course that he only has a portion of the initial fortune. And things go on, generation after generation. If we write $F(t)$ for the sequence of the fortunes of the individuals alive at time t, then the process $(F(t), t \geq 0)$ is a homogeneous fragmentation.

Our approach to investigate fragmentation processes is based on the fact that they can be encoded as random processes valued in the space of partitions of $\mathbb{N}$. This idea stems from the seminal work of Kingman [11], see also Evans and Pitman [10], Bolthausen and Sznitman [9], Pitman [12, 13], Bertoin and Le Gall [7] and others for further developments. Roughly, the connection can be understood as follows. One introduces a sequence of i.i.d. random points $U_1, \ldots$ which are picked according to the mass distribution of the object. We suppose that this sequence is independent of the fragmentation process, and we consider at each time t the random partition of the set of indices $\mathbb{N}$, specified by the rule that two indices, say i and j, belong to the same block of the partition at time t if and only if the points U_i and U_j belong to the same fragment of the object at time t. Plainly, this partition-valued process is *exchangeable*, in the sense that a permutation of indices does not affect its distribution. An application of the law of large numbers shows that the sequence of the masses of the fragments at time t can be recovered as the sequence of asymptotic frequencies of the blocks of the partition. Note also that considering the partition-valued process enables us to keep track of the 'genealogy' of the fragmentation.

Our basic result in this setting is a representation à la Lévy-Itô, i.e. based on some Poisson point process which describes the fragmentation on infinitesimal time intervals. This can also be viewed as an analogue of Pitman's description [13] of coalescent processes with multiple collisions (see also Schweinsberg [15]). More precisely, we shall construct a so-called characteristic measure, which specifies the law of the homogeneous fragmentation. We shall then show that the fragmentation at time t can be viewed as the result of possibly infinitely many partitions which appear according to some Poisson point process with intensity given by the characteristic measure. We determine all the possible characteristic measures by showing that they arise as the combination of two different mechanisms. The first describes the continuous erosion, whereas the second is related to the sudden dislocations. For instance, if we consider the example above describing inheritance for branching processes, the erosion corresponds to the fact that each individual spends his fortune at some rate, and the dislocation to the fact that his fortune is split when he dies.

As an application of the Lévy-Itô decomposition, we study the evolution of the mass of the fragment containing a point picked at random at the initial time, which can be viewed as the asymptotic frequency of the block containing some given integer, say 1, in the partition valued process. More precisely, we show that the latter is related to a certain subordinator whose distribution can be expressed in terms of the erosion and the dislocation measures.

In a forthcoming work (in progress), we shall show that certain non-homogeneous fragmentation processes, including those considered by Aldous and Pitman [4], Bertoin [6], Bolthausen and Sznitman [9] and Pitman [13], can be constructed from homogeneous fragmentation processes via some time-substitution which enables us to make the fragmentation rate depend on time and on the mass of the object.

2 Some notation

We write $\mathbb{N} = \{1, \dots\}$ for the set of positive integers and start by fixing the notation and introducing notions related to partitions.

Let $E \subseteq \mathbb{N}$ be some subset of integers. A partition $\Gamma = (B_1, \dots)$ of E is an infinite sequence of blocks $B_i \subseteq E$ such that

$$\bigcup_{i \in \mathbb{N}} B_i = E \quad \text{and} \quad B_i \cap B_j = \emptyset \text{ for } i \neq j\,.$$

It is convenient to agree that the blocks of a partition are always ranked according to the increasing order of their least element, i.e.

$$i \leq j \implies \min B_i \leq \min B_j\,,$$

with the convention $\min \emptyset = \infty$.

We write $\mathcal{P}(E)$ for the set of partitions of E and simply $\mathcal{P}$ for $\mathcal{P}(\mathbb{N})$. When E' is a subset of E and $\Gamma \in \mathcal{P}(E)$ a partition of E, we write $\Gamma_{E'}$ for the partition of E' obtained by restricting Γ to E', in the sense that two integers i and j in E' belong to the same block of $\Gamma_{E'}$ if and only if they belong to the same block of Γ. For the sake of simplicity, for every $n \in \mathbb{N}$ and $\Gamma \in \mathcal{P}$, we write $\Gamma_n = \Gamma_{\{1,\dots,n\}}$ for the partition restricted to the first n integers. We call $(E, \emptyset, \dots)$ the trivial partition of E and we write $\mathcal{P}_n^*$ for the subset of partitions $\Gamma \in \mathcal{P}$ such that the restriction Γ_n to $\{1, \dots, n\}$ is not trivial.

Note that if $n < n'$ are two positive integers and $\Gamma \in E$, then Γ_n coincides with the restriction to $\{1, \ldots, n\}$ of $\Gamma_{n'}$. This will be referred to as the *consistency property*. Conversely, if we are given for each $n \in \mathbb{N}$ a partition γ_n of $\{1, \ldots, n\}$ with the consistency property, then there is a unique partition $\Gamma \in \mathcal{P}$ such that $\Gamma_n = \gamma_n$ for every n. Given two partitions of $\mathbb{N}$, Γ and Γ', we set

$$\operatorname{dist}(\Gamma, \Gamma') \;=\; 2^{-n(\Gamma,\Gamma')} ,$$

where $n(\Gamma, \Gamma')$ is the largest integer n such that $\Gamma_n = \Gamma'_n$ (so $n(\Gamma, \Gamma') = \infty$ if and only if $\Gamma = \Gamma'$). It can be checked that $\operatorname{dist}(\Gamma, \Gamma')$ defines a metric on $\mathcal{P}$, and that $\mathcal{P}$ is a compact space.

Next, for every $C \subseteq \mathbb{N}$ and every partition $\Gamma \in \mathcal{P}$, we may define a partition $\Gamma \circ C$ of C as follows. We rank the elements of C in the increasing order, i.e. $C = \{c_1, \ldots\}$ where the (possibly finite, or even empty) sequence $c_1, \ldots$ is increasing. Then we denote by $\Gamma \circ C$ the partition of C defined by

$$\Gamma \circ C \;=\; (\{c_j : j \in B_i\}, i = 1, \ldots) ,$$

where $B_1, \ldots$ are the blocks of the partition Γ. Of course $\Gamma \circ \emptyset = (\emptyset, \ldots)$. Note that by definition, $\Gamma \circ C$ always has infinitely many blocks (even when C is finite, in which case only finitely many blocks of $\Gamma \circ C$ are not empty). We shall call $\Gamma \circ C$ the partition of C induced by Γ; note that in general $\Gamma \circ C$ does not coincide with the restricted partition Γ_C. In this vein, if Γ is a partition of E, Δ a partition of $\mathbb{N}$ and $k \in \mathbb{N}$ an integer, we write $B_1, \ldots$ for the blocks of Γ, and define the compound partition

$$\Delta \overset{k}{\circ} \Gamma \;\in\; \mathcal{P}(E)$$

as the unique partition whose blocks are given by the B_i's for $i \neq k$ and the blocks of $\Delta \circ B_k$, the partition of B_k induced by Δ.

It is often convenient to view a partition Γ as an equivalence relation, in the sense that $i \overset{\Gamma}{\sim} j$ if and only if i and j belong to the same block of the partition Γ. In this direction, we say that a partition Γ is finer than a partition Γ' if and only if

$$i \overset{\Gamma}{\sim} j \Longrightarrow i \overset{\Gamma'}{\sim} j \,.$$

Finally, call finite permutation any bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sigma(n) = n$ when n is large enough. The group of finite permutations acts on $\mathcal{P}$; more precisely for every finite permutation σ and every partition Γ, the relation

$$i {\sim} j \Longleftrightarrow \sigma(i) \overset{\Gamma}{\sim} \sigma(j) \,, \qquad i, j \in \mathbb{N}$$

is an equivalence relation which can be identified as a partition denoted by $\sigma(\Gamma)$.

3 Homogeneous fragmentation

3.1 Definitions and first properties

Consider a random process $(\Pi(t), t \geq 0)$ valued in $\mathcal{P}$, which is continuous in probability and starts a.s. from the trivial partition. The process $\Pi(\cdot)$ is called *exchangeable* provided that whenever σ is a finite permutation on $\mathbb{N}$, then the permuted process $(\sigma(\Pi(t)), t \geq 0)$ has the same the distribution as the original process $\Pi(\cdot)$. An exchangeable process $\Pi(\cdot)$ valued in $\mathcal{P}$ is called a homogeneous fragmentation if the following condition is fulfilled:

Definition 1 (Fragmentation property) *For every $t_0, t \geq 0$, conditionally on $\Pi(t_0) = (B_1, \ldots) \in \mathcal{P}$, the partition $\Pi(t_0 + t)$ is independent of the partitions $(\Pi(s), 0 \leq s \leq t_0)$ and has the same distribution as the partition obtained by the family of the blocks of the induced partitions $\Gamma_1 \circ B_1, \Gamma_2 \circ B_2, \ldots$, where $\Gamma_1, \ldots$ are independent copies of $\Pi(t)$.*

Plainly, for every $0 \leq t \leq t'$, the partition $\Pi(t')$ is finer than $\Pi(t)$. This enables us to choose a regular (i.e. càdlàg) version of the process $\Pi(\cdot)$. More precisely, if we define for every $t \geq 0$ the partition $\Pi(t+)$ corresponding to the equivalence relation

$$i \overset{\Pi(t+)}{\sim} j \iff i \overset{\Pi(t')}{\sim} j \text{ for some rational number } t' > t\,,$$

then the hypothesis of continuity in probability entails that $(\Pi(t+), t \geq 0)$ is a version of $\Pi(\cdot)$. This version has right-continuous paths with limits on the left a.s.; and we shall implicitly work with this version from now on. It can be easily deduced from the exchangeability and the fragmentation properties that if the fragmentation $\Pi(\cdot)$ is not identically trivial, then $\Pi(\infty-)$ is the discrete partition (i.e. the partition of $\mathbb{N}$ consisting of singletons) a.s.

A homogeneous fragmentation is a Markov process whose its the semigroup is given by a fragmentation kernel in the terminology used by Pitman [13]. In this direction, we denote by $\mathbb{P}$ the distribution of the homogeneous fragmentation $\Pi(\cdot)$ started from the trivial partition, and by $(\mathcal{F}_t)_{t \geq 0}$ its natural filtration, as usual completed up to $\mathbb{P}$-null sets. It is easy to deduce from Kolmogorov's zero-one law and the fragmentation property that this filtration is right-continuous (alternatively, this also follows from the Feller property stated in the forthcoming Proposition 1). Next, we have to introduce a probability measure that gives the law of the fragmentation process started from an arbitrary partition. Such probability measures are viewed on Skorohod space of càdlàg functions valued in the compact metric space $\mathcal{P}$. For any partition of $\mathbb{N}$, $\Gamma = (B_1, \ldots)$, we first introduce a sequence $\Pi^{(1)}(\cdot), \ldots$ of independent copies of $\Pi(\cdot)$, and we define $\Pi^{\Gamma}(t)$ as the partition of $\mathbb{N}$ whose blocks are those of $\Pi^{(1)}(t) \circ B_1, \ldots$. The law of the fragmentation process $\Pi^{\Gamma}(\cdot)$ is denoted by $\mathbb{P}_{\Gamma}$.

Next, we wish to reinforce the fragmentation (or Markov property), and this relies on the following result.

Proposition 1 (Feller property) *The semigroup of a homogeneous fragmentation has the Feller property, that is for every continuous function $\Psi : \mathcal{P} \to \mathbb{R}$, the map*

$$\Gamma \to \mathbb{E}_{\Gamma}\left(\Psi(\Pi(t))\right), \qquad \Gamma \in \mathcal{P}$$

is continuous for every $t \geq 0$, and for every $\Gamma \in \mathcal{P}$,

$$\lim_{t \to 0+} \mathbb{E}_{\Gamma}\left(\Psi(\Pi(t))\right) = \Psi(\Gamma)\,.$$

Proof: For every $n \in \mathbb{N}$ and every partition $\pi \in \mathcal{P}$, write $\bar{\pi}_n$ for the unique partition of $\mathbb{N}$ whose restriction to $\{1, \ldots, n\}$ coincides with π_n and for which $\{n+1, \ldots\}$ is a block. In particular $\operatorname{dist}(\pi, \bar{\pi}_n) \leq 2^{-n}$.

On the one hand, since $\mathcal{P}$ is compact, every continuous function $\Psi : \mathcal{P} \to \mathbb{R}$ is uniformly continuous. For every $\varepsilon > 0$, there is an integer n_ε such that for every $\pi \in \mathcal{P}$,

$$n \geq n_\varepsilon \implies |\Psi(\pi) - \Psi(\bar{\pi}_n)| \leq \varepsilon .$$

On the other hand, the fragmentation property entails that whenever $\operatorname{dist}(\Gamma, \Gamma') \leq 2^{-n}$, the distribution of $\bar{\Pi}_n(t)$ is the same under $\mathbb{P}_\Gamma$ as under $\mathbb{P}_{\Gamma'}$. Hence we have

$$\mathbb{E}_\Gamma \left(\Psi(\bar{\Pi}_n(t)) \right) = \mathbb{E}_{\Gamma'} \left(\Psi(\bar{\Pi}_n(t)) \right) .$$

It follows now from the triangular inequality that

$$\left| \mathbb{E}_\Gamma \left(\Psi(\Pi(t)) \right) - \mathbb{E}_{\Gamma'} \left(\Psi(\Pi(t)) \right) \right| \leq 2\varepsilon ,$$

provided that $\operatorname{dist}(\Gamma, \Gamma') \leq 2^{-n_\varepsilon}$.

The map $\Gamma \to \mathbb{E}_\Gamma \left(\Psi(\Pi(t)) \right)$ is thus continuous, the continuity of $t \to \mathbb{E}_\Gamma \left(\Psi(\Pi(t)) \right)$ follows immediately from the assumption that the process $\Pi(\cdot)$ is continuous in probability under $\mathbb{P}$. $\square$

Feller processes enjoy the strong Markov property, and in our setting, this will be referred to the *strong fragmentation property.* Specifically, it holds that for every $(\mathcal{F}_t)$-stopping time T under the conditional probability measure $\mathbb{P}(\cdot \mid \Pi(T) = \Gamma)$, the shifted fragmentation $(\Pi(T+t), t \geq 0)$ is independent of $\mathcal{F}_T$ and has the law $\mathbb{P}_\Gamma$. Here, we implicitly used the fact that either $\Pi(\cdot)$ is identically trivial or $\Pi(\infty)$ is the discrete partition a.s., which enables us to define $\Pi(T+t)$ even in the case when $T = \infty$ with positive probability.

3.2 Lévy-Itô decomposition

Informally, the purpose of this section is to show that a homogeneous fragmentation process $\Pi(\cdot)$ can be decomposed into a Poisson point process of partitions, whose distribution is determined by a so-called the characteristic measure. This decomposition is a reminiscent of the celebrated Lévy-Itô decomposition of subordinators (see e.g. section I.2 in [5]), and is also closely related to Pitman's analysis of certain coalescent processes, cf. [13].

To start with, let us construct the characteristic measure. For each fixed integer $n \geq 2$, consider $\Pi_n(\cdot)$, the fragmentation restricted to $\{1, \ldots, n\}$. It should be plain that $\Pi_n(\cdot)$ is a (continuous time) strong Markov chain. The first instant T_n at which $\Pi_n(\cdot)$ jumps away from the trivial partition on $\{1, \ldots, n\}$ has an exponential distribution, say with parameter $q(n) \in]0, \infty[$. Denote the distribution of $\Pi_n(T(n))$ by $\rho(n)$. We stress that $q(n)$ and $\rho(n)$ determine the jump rates of $\Pi_n(\cdot)$ (by the strong fragmentation property), and thus characterize its distribution.

Next, consider the fragmentation $\Pi_{n+1}(\cdot)$ restricted to $\{1, \ldots, n+1\}$, so that $T(n+1)$ is its first jump time away from the trivial partition. Then either $\Pi_{n+1}(T(n+1))$ is the partition with non-void blocks $\{1, \ldots, n\}$ and $\{n\}$, or its restriction to $\{1, \ldots, n\}$ is a non-trivial partition.

More precisely, the first case occurs if and only if $T(n+1) < T(n)$, and the latter if and only if $T(n+1) = T(n)$. We thus see that the jump rate of $\Pi_{n+1}(\cdot)$ on the set of partitions of $\{1,\ldots,n+1\}$ whose restriction to $\{1,\ldots,n\}$ is non-trivial coincides with $q(n)$, and the conditional distribution of $\Pi_n(T(n+1))$, i.e. of the restriction of $\Pi_{n+1}(T(n+1))$ to $\{1,\ldots,n\}$, given that $\Pi_n(T(n+1))$ is non-trivial, is $\rho(n)$. This shows that $q(n)\rho(n)$ is the image measure of $q(n+1)\rho(n+1)$ by the restriction map $\mathcal{P}_{\{1,\ldots,n+1\}} \to \mathcal{P}_{\{1,\ldots,n\}}$, and restrained to the set of non-trivial partitions of $\{1,\ldots,n\}$.

Recall that $\mathcal{P}_n^*$ denotes the subset of partitions in $\mathcal{P}$ whose restriction to $\{1,\ldots,n\}$ is non-trivial. Invoking Kolmogorov's consistency theorem, we can construct a (unique) measure κ on $\mathcal{P}$ which gives no mass to the trivial partition and such that the image of κ restrained to $\mathcal{P}_n^*$ by the map $\mathcal{P}_n^* \to \mathcal{P}_{\{1,\ldots,n\}}$ is $q(n)\rho(n)$. As for each n, the probability measure $\rho(n)$ is invariant by the action of permutations of $\{1,\ldots,n\}$ (because the restricted fragmentation process $\Pi_n(\cdot)$ is exchangeable), κ is an exchangeable measure that assigns a finite mass to each $\mathcal{P}_n^*$.

This incites us to introduce the following definition

Definition 2 *We call a measure κ on $\mathcal{P}$ an exchangeable partition measure if:*

(i) κ exchangeable, i.e. invariant by the action of finite permutations,

(ii) κ assigns zero mass to the trivial partition,

(iii) $\kappa(\mathcal{P}_n^*) < \infty$ for every $n \in \mathbb{N}$.

We point out that an exchangeable partition measure is necessarily a sigma-finite measure. This is seen from the facts that each $\mathcal{P}_n^*$ is a compact subspace of $\mathcal{P}$ and that the complement of $\bigcup_{\mathbb{N}} \mathcal{P}_n^*$ reduces to the trivial partition. Note also that exchangeability implies that (iii) can be replaced by the apparently weaker requirement $\kappa\left(\mathcal{P}_2^*\right) < \infty$.

We are now able to record the preceding analysis of the jumps of the restricted fragmentation $\Pi_n(\cdot)$.

Lemma 1 *Given a homogeneous fragmentation $\Pi(\cdot)$, there exists a (unique) exchangeable partition measure κ on $\mathcal{P}$, called the characteristic measure of $\Pi(\cdot)$, such that for every $n \in \mathbb{N}$ and every non-trivial partition π_n of $\mathcal{P}_{\{1,\ldots,n\}}$, we have*

$$\kappa\left(\{\Gamma \in \mathcal{P} : \Gamma_n = \pi_n\}\right) = \frac{1}{\mathbb{E}(T(n))}\, \mathbb{P}(\Pi_n(T(n)) = \pi_n)\,,$$

where $T(n) = \inf\{t \geq 0 : \Pi(t) \in \mathcal{P}_n^*\}$. Moreover, κ determines the distribution of the fragmentation process $\Pi(\cdot)$.

Conversely, let us show how to construct a homogeneous fragmentation from a given exchangeable partition measure. To that end, consider a Poisson point process $((\Delta_t, k_t), t \geq 0)$ on $\mathcal{P} \times \mathbb{N}$ with intensity $\kappa \otimes \#$ where κ is some exchangeable partition measure on $\mathcal{P}$ and $\#$ stands for the counting measure on $\mathbb{N}$. This means that for every measurable set $A \subseteq \mathcal{P} \times \mathbb{N}$ with $\kappa \otimes \#(A) < \infty$, the counting process

$$N^A(t) \;=\; \mathrm{Card}\left(s \in [0,t] : (\Delta_s, k_s) \in A\right)\,, \qquad t \geq 0$$

is a Poisson process with intensity $\kappa \otimes \#(A)$, and to disjoint sets correspond independent counting processes. Note that the assumption that $\kappa(\mathcal{P}_n^*) < \infty$ ensures that for every integer j, the Poisson point process restricted to $\mathcal{P}^* \times \{1, \ldots, j\}$ is discrete.

Then, recall the notation $\Delta \overset{k}{\circ} \Gamma$ introduced in Section 2. By considering partitions restricted to $\{1, \ldots, n\}$, it can be checked that there a unique càdlàg partition-valued process $\Pi(\cdot)$ such that $\Pi(t) = \Pi(t-)$ when the Poisson point process has no point at t, and

$$\Pi(t) \;=\; \Delta_t \overset{k_t}{\circ} \Pi(t-) \tag{1}$$

otherwise. More precisely, fix n and observe the following. For every partition π of $\{1, \ldots, n\}$, $k \in \mathbb{N}$ and $\Delta \in \mathcal{P}$, if the k-th block of π has, say, j elements and if the restriction of Δ to $\{1, \ldots, j\}$ is trivial, then $\Delta \overset{k}{\circ} \pi = \pi$. Since the restriction of the Poisson point process to points (Δ_t, k_t) such that $\Delta_t \in \mathcal{P}_n^*$ and $k_t \leq n$ is discrete, we can construct a (unique) càdlàg process $\pi^{(n)}(\cdot)$ valued in the space of partitions of $\{1, \ldots, n\}$ and started from the trivial partition, such that $\pi^{(n)}(\cdot)$ only jumps at times $t \geq 0$ corresponding to points (Δ_t, k_t) of the Poisson point process, in which case

$$\pi^{(n)}(t) \;=\; \Delta_t \overset{k_t}{\circ} \pi^{(n)}(t-)\,.$$

We stress that t is not necessarily a jump time for $\pi^{(n)}(\cdot)$, and more precisely, the quantity above coincides with $\pi^{(n)}(t-)$ except for a discrete set of times.

Since the restriction of the trivial partition of $\{1, \ldots, n+1\}$ to $\{1, \ldots, n\}$ is again the trivial partition, and since for every partitions $\Delta, \Gamma \in \mathcal{P}$, one has obviously

$$\left(\Delta \overset{k}{\circ} \Gamma\right)_n \;=\; \Delta \overset{k}{\circ} \Gamma_n\,,$$

we see that the family $\left(\pi^{(n)}(t), n \in \mathbb{N}\right)$ fulfills the consistency property. We conclude that there exists a unique càdlàg process $\Pi(\cdot)$ valued in $\mathcal{P}$ such that the restriction $\Pi_n(t)$ of $\Pi(t)$ to $\{1, \ldots, n\}$ coincides with $\pi^{(n)}(t)$ for all n. This process $\Pi(\cdot)$ obviously fulfills (1).

The main result of this section is the following theorem which states the Lévy-Itô decomposition of a homogeneous fragmentation.

Theorem 1 *The partition-valued process $\Pi(\cdot)$ constructed above is a homogeneous fragmentation with characteristic measure κ.*

Proof: It should be clear from the construction and standard properties of Poisson point processes that the restricted processes $\Pi_n(\cdot)$ is continuous in probability and enjoys the fragmentation property. Hence so does $\Pi(\cdot)$.

Next, let us now check that the process $\Pi(\cdot)$ is exchangeable. In this direction, we only need to consider permutations σ of the following type. For some $n \in \mathbb{N}$, we have $\sigma(n) = n + 1$, $\sigma(n + 1) = n$ and $\sigma(i) = i$ for $i \neq n, n + 1$. For the sake of simplicity, we shall only treat the case $n = 1$ as the argument for general n's is similar, but with heavier notation.

Let T be the first instant at which $\Pi(t) \in \mathcal{P}_2^*$ (i.e. the integers 1 and 2 belong to the same block of $\Pi(t)$ for $t < T$ but not for $t \geq T$). By the construction of $\Pi(\cdot)$, we have

$$T \;=\; \inf\left\{t \geq 0 : \Delta_t \in \mathcal{P}_2^* \text{ and } k_t = 1\right\}\,.$$

Then, consider the point process $\left((\tilde{\Delta}_t, \tilde{k}_t), t \geq 0\right)$ defined as follows:

$$\tilde{\Delta}_t \;=\; \begin{cases} \Delta_t & \text{for } t \neq T \\ \sigma(\Delta_T) & \text{for } t = T\,, \end{cases} \qquad \tilde{k}_t \;=\; \begin{cases} k_t & \text{if } k_t \geq 3 \text{ or if } t \leq T \\ 2 & \text{if } k_t = 1 \text{ and } t > T \\ 1 & \text{if } k_t = 2 \text{ and } t > T\,. \end{cases}$$

The exchangeability of the partition measure κ ensures that $\tilde{\Delta}_T$ has the same law as Δ_T, and then it is easy to check from standard properties of Poisson point processes that $\left((\tilde{\Delta}_t, \tilde{k}_t), t \geq 0\right)$ has the same law as $((\Delta_t, k_t), t \geq 0)$. The process $\left((\tilde{\Delta}_t, \tilde{k}_t), t \geq 0\right)$ has been defined in such a way that if we set $\tilde{\Pi}(t) = \sigma(\Pi(t))$, then

$$\tilde{\Pi}(t) \;=\; \tilde{\Delta}_t \overset{\tilde{k}_t}{\circ} \tilde{\Pi}(t-)$$

at every instant t when the Poisson point process has a point, and $\tilde{\Pi}(t) = \tilde{\Pi}(t-)$ otherwise. We conclude that $\Pi(\cdot)$ and $\tilde{\Pi}(\cdot)$ have the same distribution and hence $\Pi(\cdot)$ is exchangeable.

Finally, it is immediate from the very construction of $\Pi(\cdot)$ that the characteristic measure of $\Pi(\cdot)$ is κ, which completes the proof. $\square$.

4 Exchangeable partition measures

We have seen in the preceding section that the distribution of a homogeneous fragmentation is determined by its characteristic measure, which is an exchangeable partition measure, and that conversely, any exchangeable partition measure can be viewed as the characteristic measure of some homogeneous fragmentation. This incites us to investigate exchangeable partition measures. In this direction, we first give two fundamental examples.

First, for every $n \in \mathbb{N}$, we write ε_n for the partition of $\mathbb{N}$ that has exactly two non-void classes, $\{n\}$ and $\mathbb{N}\backslash\{n\}$. If δ_π stands for the Dirac point mass at $\pi \in \mathcal{P}$, then for every real number $c \geq 0$, the measure

$$\mu_c \;=\; c \sum_{n=1}^{\infty} \delta_{\varepsilon_n}$$

is an exchangeable partition measure. We shall refer to μ_c as an *erosion measure*, and to c as the rate of erosion.

To construct the second example, it is convenient to introduce the spaces

$$\begin{aligned} \mathcal{S}^{\downarrow} \;&=\; \left\{ s = (s_i, i \in \mathbb{N}) : s_1 \geq s_2 \geq \cdots \geq 0 \text{ and } \sum_{i=1}^{\infty} s_i \leq 1 \right\}, \\ \mathcal{S}^{*} \;&=\; \mathcal{S}^{\downarrow}\backslash\{(1,0,\ldots)\}\,. \end{aligned}$$

The spaces $\mathcal{S}^{\downarrow}$ and $\mathcal{S}^{*}$ are endowed with the topology of pointwise convergence. Each sequence $s \in \mathcal{S}^{\downarrow}$ can be viewed as a sub-probability measure on integers. Following Kingman [11], we associate to every $s \in \mathcal{S}^{\downarrow}$ an exchangeable random partition as follows. We first introduce a sequence $X_1, \ldots$ of i.i.d. integer-valued random variables with law

$$\mathbb{P}(X_i = n) \;=\; s_n \;\text{ for } n = 1, \ldots, \text{ and } \;\mathbb{P}(X_i = 0) = 1 - \sum_{n=1}^{\infty} s_n\,.$$

We consider the random partition of $\mathbb{N}$ induced by the equivalence relation defined for $i \neq j$ by

$$i \sim j \iff X_i = X_j > 0$$

(note that for $s = (1, 0, \ldots)$, this is just the trivial partition a.s.). We write μ_s for the distribution of this exchangeable random partition.

Next, we consider mixtures of μ_s's. To that end, it is easy to check that the map $s \to \mu_s$ is continuous. Then, call *Lévy measure* on $\mathcal{S}^*$ any sigma-finite measure ν on $\mathcal{S}^*$ such that

$$\int_{\mathcal{S}^*} (1 - s_1)\, \nu(ds) < \infty \tag{2}$$

(recall that $s_1 < 1$ stands for the first term of the sequence $s \in \mathcal{S}^*$).

Lemma 2 *Given a Lévy measure ν on $\mathcal{S}^*$, define a measure μ_ν on $\mathcal{P}$ by*

$$\mu_\nu(d\Gamma) = \int_{s \in \mathcal{S}^*} \mu_s(d\Gamma)\nu(ds)\,.$$

Then μ_ν is an exchangeable partition measure.

In the sequel we shall refer to μ_ν as the *dislocation measure* with Lévy measure ν.

Proof: Each μ_s is an exchangeable probability measure on $\mathcal{P}$. Exchangeability is preserved by mixing, so μ_ν is an exchangeable measure on $\mathcal{P}$. As for all $s \in \mathcal{S}^*$, the measures μ_s assign zero mass to the trivial partition, the same holds for the mixture μ_ν.

For every $n \in \mathbb{N}$ and $s \in \mathcal{S}^*$, the μ_s-probability of the event that the partition restricted to $\{1, \ldots, n\}$ is not trivial equals

$$\mu_s(\mathcal{P}_n^*) = 1 - \sum_{k=1}^{\infty} s_k^n \leq 1 - s_1^n \leq n(1 - s_1)\,.$$

Hence

$$\mu_\nu\left(\mathcal{P}_n^*\right) \leq n \int_{\mathcal{S}^*} (1 - s_1)\nu(ds)$$

and the right-hand side is finite by (2). □

The main result of this section, which is essentially a consequence of Kingman's representation of exchangeable random partitions [11], is that erosion measures and dislocation measures are essentially the most general type of exchangeable partition measure, in the sense that every exchangeable partition measure κ has a canonical decomposition $\kappa = \kappa_e + \kappa_d$ as the sum of an erosion measure κ_e and a dislocation measure κ_d. To give a precise statement in this direction, we say that a partition $\Gamma \in \mathcal{P}$ with blocks $B_1, \ldots$ has *asymptotic frequencies* if the following limit exists for every $i \in \mathbb{N}$:

$$\lim_{n \to \infty} n^{-1} \mathrm{Card}\,\{j \leq n : j \in B_i\} = \lambda_i \in [0, 1]\,.$$

We then write $\lambda_1^{\downarrow} \geq \lambda_2^{\downarrow} \geq \ldots$ for the decreasing rearrangement of the λ_i's and $\Lambda(\Gamma) = \left(\lambda_1^{\downarrow}, \ldots\right)$, so $\Lambda(\Gamma)$ is an element of $\mathcal{S}^{\downarrow}$. Observe from the strong law of large numbers that for every $s \in \mathcal{S}^{\downarrow}$, μ_s-almost every partition Γ has asymptotic frequencies with $\Lambda(\Gamma) = s$.

Theorem 2 *Let κ be an exchangeable partition measure. Then there exists a unique $c \geq 0$ and a unique Lévy measure ν on $\mathcal{S}^*$ such that $\kappa = \mu_c + \mu_\nu$. Specifically, the following holds:*

(i) *For κ-almost every $\Gamma \in \mathcal{P}$, Γ has asymptotic frequencies $\Lambda(\Gamma)$.*

(ii) *The restriction of κ to the subset of partitions Γ with $\Lambda(\Gamma) \neq (1, 0, \ldots)$ is a dislocation measure. More precisely, let $\Lambda(\kappa)$ be the image measure of κ by the mapping $\Gamma \to \Lambda(\Gamma)$. The restriction $\nu = \mathbf{1}_{\mathcal{S}^*}\Lambda(\kappa)$ of $\Lambda(\kappa)$ to $\mathcal{S}^*$ is a Lévy measure on $\mathcal{S}^*$ and*

$$\mathbf{1}_{\{\Lambda(\Gamma)\in\mathcal{S}^*\}}\kappa(d\Gamma) \;=\; \mu_\nu(d\Gamma)\,.$$

(iii) *The restriction of κ to the subset of partitions Γ with $\Lambda(\Gamma) = (1, 0, \ldots)$ is an erosion measure, i.e. there is a real number $c \geq 0$ such that*

$$\mathbf{1}_{\{\Lambda(\Gamma)=(1,0,\ldots)\}}\kappa(d\Gamma) \;=\; \mu_c(d\Gamma)\,.$$

Proof: (i) For every integer n, write κ_n for the restriction of κ to $\mathcal{P}_n^*$. Then κ_n is a finite measure on $\mathcal{P}$, and is invariant by the action of finite permutations that coincide with the identity on $\{1, \ldots, n\}$. Let $\vec{\kappa}_n$ be the image of κ_n by the n-shift on partitions, viz. the map $\Gamma \to \vec{\Gamma}$ defined by

$$i \overset{\vec{\Gamma}}{\sim} j \iff i + n \overset{\Gamma}{\sim} j + n\,, \qquad i, j \in \mathbb{N}.$$

Then $\vec{\kappa}_n$ is an exchangeable finite measure on $\mathcal{P}$, and by a result due to Kingman [11] (see Aldous [1] for a short proof), $\vec{\kappa}_n$-almost every partition has asymptotic frequencies, and more precisely,

$$\vec{\kappa}_n(d\Gamma) \;=\; \int_{\mathcal{S}^\downarrow} \mu_s(d\Gamma)\vec{\kappa}_n(\Lambda(\Gamma) \in ds) \tag{3}$$

is a regular disintegration of $\vec{\kappa}_n$. As shift does not affect asymptotic frequencies, κ_n-almost every partition has asymptotic frequencies. This establishes the first claim.

(ii) Note that if we write $\{i \not\sim j\}$ for the event that i and j do not belong to the same block, then by (3), for every $s \in \mathcal{S}^\downarrow$

$$\begin{aligned}
\kappa_n(n+1 \not\sim n+2 \mid \Lambda(\Gamma) = s) \;&=\; \mu_s(1 \not\sim 2) \\
&=\; 1 - \sum_{k=1}^{\infty} s_k^2 \\
&\geq\; 1 - s_1 \left(\sum_{k=1}^{\infty} s_k\right) \\
&\geq\; 1 - s_1\,.
\end{aligned}$$

Hence, if we denote by $\nu_n = \mathbf{1}_{\mathcal{S}^*}\Lambda(\kappa_n)$ the restriction to $\mathcal{S}^*$ of the image measure of κ_n by Λ, then

$$\kappa_n(n+1 \not\sim n+2) \;\geq\; \int_{\mathcal{S}^*} (1 - s_1)\nu_n(ds)\,.$$

On the one hand, the finite measure ν_n increases as $n \uparrow \infty$ to the measure ν defined in the statement, so

$$\lim_{n\to\infty} \int_{\mathcal{S}^*} (1 - s_1)\nu_n(ds) \;=\; \int_{\mathcal{S}^*} (1 - s_1)\nu(ds)\,.$$

On the other hand,

$$\kappa_n(n+1 \not\sim n+2) \,\leq\, \kappa(n+1 \not\sim n+2) \,=\, \kappa(1 \not\sim 2) \,=\, \kappa\left(\mathcal{P}_2^*\right) \,<\, \infty\,.$$

We thus see that ν is a Lévy measure on $\mathcal{S}^*$.

Finally, fix $k \in \mathbb{N}$ and pick a non-trivial partition π of $\{1,\ldots,k\}$. We have by monotone convergence

$$\kappa(\Gamma_k = \pi, \Lambda(\Gamma) \in \mathcal{S}^*) \,=\, \lim_{n\to\infty} \kappa(\Gamma_k = \pi, \Gamma_{\{k+1,\ldots,k+n\}} \text{ is not trivial}, \Lambda(\Gamma) \in \mathcal{S}^*)\,.$$

In the notation introduced in (i), we see from an obvious permutation that

$$\kappa(\Gamma_k = \pi, \Gamma_{\{k+1,\ldots,k+n\}} \text{ is not trivial}, \Lambda(\Gamma) \in \mathcal{S}^*) \,=\, \vec{\kappa}_n(\Gamma_k = \pi, \Lambda(\Gamma) \in \mathcal{S}^*)\,.$$

Applying (3) and then letting n tend to ∞, we conclude that

$$\kappa(\Gamma_k = \pi, \Lambda(\Gamma) \in \mathcal{S}^*) \,=\, \int_{\mathcal{S}^*} \mu_s(\Gamma_k = \pi)\nu(ds)\,.$$

This establishes (ii) as k is arbitrary and the restriction $\Lambda(\Gamma) \in \mathcal{S}^*$ excludes the trivial partition.

(iii) Consider $\tilde{\kappa}$, the restriction of κ to the event $\{1 \not\sim 2, \Lambda(\Gamma) = (1, 0\ldots)\}$, which has finite mass. Its image by the 2-shift as defined in (i) an exchangeable finite measure on $\mathcal{P}$ for which almost every partition has asymptotic frequencies $(1,0,\ldots)$, and hence it must be proportional to the Dirac mass at the trivial partition. Let us denote by π the partition with non-void blocks $\{1\}$ and $\{2,\ldots\}$, π' the partition with non-void blocks $\{2\}$ and $\{1,3,\ldots\}$, and π'' the partition with non-void blocks $\{1\}$, $\{2\}$ and $\{3,\ldots\}$. We thus have that

$$\tilde{\kappa} \,=\, c\delta_\pi + c'\delta_{\pi'} + c''\delta_{\pi''}$$

where $c, c', c'' \geq 0$ are some real numbers and δ stands for the Dirac point mass. By exchangeability, we see that the assumption $\kappa(1 \not\sim 2) < \infty$ forces $c'' = 0$. It is also immediate that $c = c'$, and the fact that κ restricted to the event $\{\Lambda = (1,0,\ldots)\}$ coincide with μ_c is now plain again by exchangeability. $\square$

5 Asymptotic frequencies of blocks

In this section, we consider a homogeneous fragmentation $\Pi(\cdot)$ with characteristic measure κ. Recall from Theorem 2 that $\kappa = \mu_c + \mu_\nu$ is the canonical decomposition of the exchangeable partition measure κ as the sum of an erosion measure with erosion rate $c \geq 0$, and a dislocation measure with Lévy measure ν. Our purpose is to investigate the process that gives the asymptotic frequency of the block of the partition $\Pi(t)$ containing a given point, say 1, as time passes.

For every $t \geq 0$, we write $B_1(t)$ for the first block of the random partition $\Pi(t)$. We consider its upper and lower asymptotic frequencies,

$$\begin{aligned}
\bar{\lambda}_1(t) &= \limsup_{n\to\infty} n^{-1}\mathrm{Card}\left(B_1(t) \cap \{1,\ldots,n\}\right)\,, \\
\underline{\lambda}_1(t) &= \liminf_{n\to\infty} n^{-1}\mathrm{Card}\left(B_1(t) \cap \{1,\ldots,n\}\right)\,.
\end{aligned}$$

When these two quantities coincide, their common value is denoted by $\lambda_1(t)$ and we say that $B_1(t)$ has an asymptotic frequency.

Recall that $(\mathcal{F}_t)_{t\geq 0}$ denotes the natural filtration of the homogeneous fragmentation $\Pi(\cdot)$. Call $(\mathcal{F}_t)$-subordinator a right-continuous process $(\tau_t, t \geq 0)$ with values in $[0,\infty]$ such that $\tau_0 = 0$ a.s. and given $\tau_t < \infty$, the increment $\tau_{t+t'} - \tau_t$ is independent of $\mathcal{F}_t$ and has the same law as $\tau_{t'}$. The distribution of a subordinator is specified by its Laplace exponent Φ that is given by the identity

$$\mathbb{E}\left(\exp\{-q\tau_t\}\right) = \exp\{-t\Phi(q)\}, \qquad t, q > 0,$$

and the Lévy-Khintchine formula

$$\Phi(q) = \mathtt{k} + \mathtt{d}q + \int_{]0,\infty[} \left(1 - \mathrm{e}^{-qx}\right) L(dx)$$

where $\mathtt{k} \geq 0$ is the so-called killing rate, $\mathtt{d} \geq 0$ the drift coefficient and L a measure on $]0,\infty[$ with $\int (1 \wedge x) L(dx) < \infty$, called the Lévy measure of τ. See Section I.1 in [5] for material on this area. We are now able to state the main result of this section.

Theorem 3 (i) *With probability one, for all $t \geq 0$, the block $B_1(t)$ has an asymptotic frequency, $\lambda_1(t)$.*
(ii) *The process $(-\log(\lambda_1(t)), t \geq 0)$ is an $(\mathcal{F}_t)$-subordinator. Its drift coefficient coincides with the erosion rate c of κ, its killing rate is given by*

$$\mathtt{k} = c + \int_{\mathcal{S}^*} \left(1 - \sum_{j=1}^{\infty} s_j\right) \nu(ds),$$

and its Lévy measure by

$$L(dx) = \mathrm{e}^{-x} \sum_{j=1}^{\infty} \nu(-\log s_j \in dx), \qquad x \in]0,\infty[.$$

The proof of Theorem 3 is broken into several lemmas. To start with, we first observe that for each fixed $t \geq 0$, $\Pi(t)$ is an exchangeable partition, and thus $B_1(t)$ has an asymptotic frequency a.s. (cf. [11]) , i.e. $\bar{\lambda}_1(t) = \underline{\lambda}_1(t)$ and $\lambda_1(t)$ is a well-defined random variable.

Our analysis relies on the following elementary formula for the moments of the frequency.

Lemma 3 *We have for every $t \geq 0$ and $k \in \mathbb{N}$*

$$\mathbb{E}\left((\lambda_1(t))^k\right) = \exp\left\{-t\left(c(k+1) + \int_{\mathcal{S}^*} \left(1 - \sum_{n=1}^{\infty} s_n^{k+1}\right) \nu(ds)\right)\right\}.$$

Proof: Let us first present the argument for $k = 1$. Observe that the expectation of the asymptotic frequency is

$$\mathbb{E}\left(\lambda_1(t)\right) = \mathbb{P}\left(2 \in B_1(t)\right).$$

Indeed, this identity is seen using dominated convergence and the facts that

$$\lambda_1(t) \;=\; \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}_{\{j\in B_1(t)\}}$$

and that (by exchangeability)

$$\mathbb{P}\left(j \in B_1(t)\right) \;=\; \mathbb{P}\left(2 \in B_1(t)\right)\,, \qquad \text{for every } j \geq 2\,.$$

On the one hand, recall the construction of Theorem 1 of $\Pi(\cdot)$ in terms of the Poisson point process $((\Delta_t, k_t), t \geq 0)$. Write $\left(\Delta_t^{(1)}, t \geq 0\right)$ for the point process derived from the former by retaining only the points for which the second coordinate (i.e. the integer one) equals 1. Then $\left(\Delta_t^{(1)}, t \geq 0\right)$ is a Poisson point process on $\mathcal{P}$ with intensity κ. It should be clear that for all $t \geq 0$, the blocks $B_1(t)$ can be constructed using only the restrained point process, and more precisely, that

$$2 \in B_1(t) \iff 1 \overset{\Pi(t)}{\sim} 2 \iff 1 \overset{\Delta_s^{(1)}}{\sim} 2 \text{ for all } s \in [0,t]\,.$$

We deduce from a basic result on Poisson point processes that

$$\mathbb{P}\left(2 \in B_1(t)\right) \;=\; \exp\left\{-t\kappa(1 \not\sim 2)\right\}\,.$$

On the other hand, it follows from Theorem 2 that

$$\kappa(1 \not\sim 2) \;=\; \mu_c(1 \not\sim 2) + \mu_\nu(1 \not\sim 2) \;=\; 2c + \int_{\mathcal{S}^*} \left(1 - \sum_{n=1}^{\infty} s_n^2\right) \nu(ds)\,,$$

which completes the proof for $k = 1$.

The argument for $k \geq 2$ is similar. One first checks that the k-th moment of $\lambda_1(t)$ coincides with the probability that the restricted partition $\Pi_{k+1}(t)$ is trivial. Then one identifies this quantity with $\exp\left\{-t\kappa(\mathcal{P}^*_{k+1})\right\}$ (recall that $\mathcal{P}^*_{k+1}$ denotes the space of partitions in $\mathcal{P}$ such that their restriction to $\{1, \ldots, k+1\}$ is non-trivial); and one calculates the latter using Theorem 2. $\square$

Each random variable $\lambda_1(t)$ takes values in $[0,1]$, and since $\lambda_1(t) \geq \lambda_1(t')$ a.s. whenever $0 \leq t \leq t'$, this enables us to define a right-continuous increasing process $(\tau_t, t \geq 0)$ with values in $[0,\infty]$ by

$$\tau_t \;=\; \lim_{\mathbb{Q}\ni t'\downarrow t} \log\left(1/\lambda_1(t')\right)\,, \qquad t \geq 0\,.$$

The notation indicates that we consider the decreasing limit as the rational number t' decreases to t.

Lemma 4 *The process τ is an $(\mathcal{F}_t)$-subordinator.*

Proof: Lemma 3 implies that $\mathbb{E}(\lambda_1(t))$ tends to 1 when $t \to 0+$, and thus $\lambda_1(t)$ converges in probability to 1. An argument of monotonicity entails that with probability one

$$\lim_{t\to 0+} \bar{\lambda}_1(t) \;=\; \lim_{t\to 0+} \underline{\lambda}_1(t) \;=\; 1 \tag{4}$$

and a fortiori that $\lambda_1(t)$ converges a.s. to 1 when t decreases to $0+$ along rational numbers. In other words, $\tau_0 = 0$ a.s.

On the other hand, the fragmentation property tells us that for every $t' \geq 0$, the block $B_1(t+t')$ of the partition $\Pi(t+t')$ containing 1 can be viewed as the first block of the partition $\Gamma \circ B_1(t)$ of $B_1(t)$ induced by Γ, where Γ is a random exchangeable partition independent of $\mathcal{F}_t$ and distributed as $\Pi(t')$. This implies that the asymptotic frequency $\lambda_1(t+t')$ of $B_1(t+t')$ is given by the product frequencies $\lambda_1(t)\lambda_1'(t')$ where $\lambda_1'(t')$ is the asymptotic frequency of the first block of the partition Γ. It follows readily that the process τ has stationary and independent increments in the filtration $(\mathcal{F}_t)_{t\geq 0}$. □

We next show that asymptotic frequencies exist simultaneously for all times, and are given in terms of τ.

Lemma 5 *The probability that for all $t \geq 0$ the block $B_1(t)$ has an asymptotic frequency which is given by $\lambda_1(t) = \exp\{-\tau_t\}$, equals one.*

Proof: Using the fact that for $t' < t < t''$, the partition $\Pi(t')$ is coarser than $\Pi(t)$ and $\Pi(t'')$ finer than $\Pi(t)$, we see that

$$\exp\{-\tau_t\} \leq \underline{\lambda}_1(t) \quad \text{and} \quad \bar{\lambda}_1(t) \leq \exp\{-\tau_{t-}\} .$$

Hence, whenever τ is continuous at time t, $B_1(t)$ has an asymptotic frequency which is given by $\exp\{-\tau_t\}$.

We next turn our attention to discontinuity points of τ. In this direction, fix an arbitrary integer $k \in \mathbb{N}$ and a real number $\varepsilon > 0$, and consider the instant $T(k,\varepsilon)$ when τ makes its k-th jump of size at least ε. Then $T(k,\varepsilon)$ is an $(\mathcal{F}_t)$-stopping time (recall that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous), so the strong fragmentation property entails that conditionally on $\{T(k,\varepsilon) < \infty\}$, for every $t > 0$

$$\begin{aligned}
\underline{\lambda}_1(T(k,\varepsilon))\underline{\lambda}_1'(t) \quad &\leq \quad \underline{\lambda}_1(T(k,\varepsilon)+t) \leq \underline{\lambda}_1(T(k,\varepsilon))\bar{\lambda}_1'(t) \\
\bar{\lambda}_1(T(k,\varepsilon))\underline{\lambda}_1'(t) \quad &\leq \quad \bar{\lambda}_1(T(k,\varepsilon)+t) \leq \bar{\lambda}_1(T(k,\varepsilon))\bar{\lambda}_1'(t) ,
\end{aligned}$$

where $\bar{\lambda}_1'(t)$ and $\underline{\lambda}_1'(t)$ denote the upper and lower asymptotic frequencies at time t of the first block of some homogeneous fragmentation $\Pi'(\cdot)$ which is has the same law as $\Pi(\cdot)$. Recall from (4) that $\bar{\lambda}_1'(t)$ and $\underline{\lambda}_1'(t)$ both converge to 1 a.s. when $t \to 0+$, we deduce that

$$\lim_{t\to 0+} \underline{\lambda}_1(T(k,\varepsilon)+t) = \underline{\lambda}_1(T(k,\varepsilon)) \quad \text{and} \quad \lim_{t\to 0+} \bar{\lambda}_1(T(k,\varepsilon)+t) = \bar{\lambda}_1(T(k,\varepsilon))$$

a.s. on $\{T(k,\varepsilon) < \infty\}$. Since the family of positive t's for which τ is continuous at time $T(k,\varepsilon)+t$ has 0 as an accumulation point, and since τ is right-continuous, we conclude that almost surely on $\{T(k,\varepsilon) < \infty\}$

$$\underline{\lambda}_1(T(k,\varepsilon)) = \bar{\lambda}_1(T(k,\varepsilon)) = \exp\left\{-\tau_{T(k,\varepsilon)}\right\} .$$

This identity is valid simultaneously for all integers $k \in \mathbb{N}$ and all $\varepsilon \in \{1/n, n \in \mathbb{N}\}$, so for all the jump times of τ. □

By Lemmas 4 and 5, all that is needed to complete the proof of Theorem 3 is the identification of the killing rate, drift coefficient and Lévy measure. To that end, recall from Lemma 3 that for every integer $q = k$, we have

$$\begin{aligned}\mathbb{E}\left(\exp\{-q\tau_t\}\right) &= \mathbb{E}\left((\lambda_1(t))^q\right)\\ &= \exp\left\{-t\left(c(q+1)+\int_{\mathcal{S}^*}\left(1-\sum_{n=1}^{\infty}s_n^{q+1}\right)\nu(ds)\right)\right\}.\end{aligned}$$

In order to interpret the last quantity as a Lévy-Khintchine formula, we introduce the real numbers $\mathtt{d} = c$ and

$$\mathtt{k} = c + \int_{\mathcal{S}^*}\left(1-\sum_{j=1}^{\infty}s_j\right)\nu(ds),$$

and the measure

$$L(dx) = \mathrm{e}^{-x}\sum_{j=1}^{\infty}\nu(-\log s_j \in dx), \qquad x \in]0,\infty[.$$

It is readily checked that $\int_{]0,\infty[}(1\wedge x)L(dx) < \infty$ and that we can now re-write the preceding identity as

$$\mathbb{E}\left(\exp\{-q\tau_t\}\right) = \exp\{-t\Phi(q)\}, \qquad q \in \mathbb{N} \tag{5}$$

where

$$\Phi(q) = \mathtt{k} + \mathtt{d}q + \int_{]0,\infty[}\left(1-\mathrm{e}^{-qx}\right)L(dx), \qquad q \in]0,\infty[. \tag{6}$$

Recall from the Stone-Weierstrass theorem that linear combinations of maps $x \to \mathrm{e}^{-qx}, q = 0, 1, \ldots$ are everywhere dense in the space of continuous functions on $[0,\infty[$ having a finite limit at ∞. Thus (5) characterizes the law of τ_t, and since, by the Lévy-Khintchine formula, (6) defines the Laplace exponent of a subordinator, we conclude that holds more generally for all real numbers $q \geq 0$. The proof of Theorem 3 is complete.

Remarks. • The calculation of the killing rate $\mathtt{k}$ and the Lévy measure L can also be understood as follows (the argument is only heuristic). Recall the notation $\Delta^{(1)}$ in the proof of Lemma 3 related to the point process of partitions of the first block $B_1(\cdot)$. Consider an instant t a point $\Delta_t^{(1)}$ appears in the Poisson point process. It is intuitively clear that t is an instant at which the asymptotic frequency $\lambda_1(\cdot)$ decreases, and more precisely that $\lambda_1(t) = \gamma_1\lambda_1(t-)$ where γ_1 is the frequency of the first block of the partition $\Delta_t^{(1)}$ and $\lambda_1(t-)$ that of $B_1(t-)$, the first block of the partition $\Pi(t-)$. Hence the subordinator τ jumps at time t, and the size of its jump is $\log(1/\gamma_1)$. We see that the rate of jumps of τ of size x corresponds to the rate of partitions for which the first block has frequency e^{-x} in the Poisson point process $\left(\Delta_t^{(1)}, t \geq 0\right)$. For $x = \infty$, this gives the killing rate. These quantities can be computed from Theorem 2.

• The expression

$$\mathtt{k} = c + \int_{\mathcal{S}^*}\left(1-\sum_{j=1}^{\infty}s_j\right)\nu(ds)$$

for the killing rate of τ reflects two distinct phenomena. The term c is due to the continuous erosion, it has to be viewed as the rate at which a unique point is separated (individually) from its block. The second term in the sum can be understood as follows. When $s \in \mathcal{S}^*$ is such that $1-\sum_{i=1}^{\infty} s_i := s_0 > 0$, the family of points which are isolated in an exchangeable random partition with law μ_s has almost surely asymptotic frequency s_0. Hence $\int_{\mathcal{S}^*} s_0 \nu(ds)$ is the rate at which a given point becomes isolated as the result of a sudden dislocation of its block in which infinitely many singletons appear.

• It is interesting to observe that the Lévy measure L of τ restricted to the interval $]0, \log 2[$ only depends on the 'distribution' of the first term s_1 of the sequence $s \in \mathcal{S}^*$ under the measure ν. More precisely one has $\nu(s_j > 1/2) = 0$ for $j = 2, \ldots$, and hence

$$L(dx) \;=\; \mathrm{e}^{-x} \nu(-\log s_1 \in dx)\,, \qquad x \in]0, \log 2[\,.$$

In particular, it is easily checked that the so-called upper index of the subordinator τ,

$$\beta \;=\; \inf\left\{\eta > 0 : \int_{(0,1)} x^{\eta} L(dx) < \infty\right\}$$

can be identified as

$$\beta \;=\; \inf\left\{\eta > 0 : \int_{\mathcal{S}^*} (1-s_1)^{\eta} \nu(ds) < \infty\right\}\,.$$

Applying a classical result due to Blumenthal and Getoor [8], this yields the following information about the small time behavior of the asymptotic frequency $\lambda_1(t) = \mathrm{e}^{-\tau_t}$. With probability one, it holds that

$$\limsup_{t \to 0+} \left(1 - \lambda_1(t)\right) t^{-\eta} \;=\; \begin{cases} 0 & \text{for } \eta < \beta \\ \infty & \text{for } \eta > \beta\,. \end{cases}$$

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