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Self-similar fragmentations

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Summary. We introduce a probabilistic model that is meant to describe an object that falls apart randomly as time passes and fulfills a certain scaling property. We show that the distribution of such a process is determined by its index of self-similarity $\alpha \in \mathbb{R}$, a rate of erosion $c \geq 0$, and a so-called Lévy measure that accounts for sudden dislocations. The key of the analysis is provided by a transformation of self-similar fragmentations which enables us to reduce the study to the homogeneous case $\alpha = 0$ which is treated in [6].

Key words. Fragmentation, self-similar, exchangeable partition.

Titre en Français. Fragmentations auto-similaires

Résumé en Français. On introduit un modèle probabiliste pour décrire l'évolution d'une masse qui se fragmente de façon aléatoire au cours du temps, tout en satisfaisant à une certaine propriété d'auto-similarité. On établit que la loi d'un tel processus est déterminée par son indice d'auto-similarité $\alpha \in \mathbb{R}$, un taux d'érosion $c \geq 0$, et une mesure de Lévy qui prend en compte les dislocations soudaines. La clef de l'analyse consiste en une transformation permettant de réduire l'étude à celle du cas homogène $\alpha = 0$ qui a déjà fait l'objet de [6].

Mots clefs. Fragmentation, auto-similaire, partition échangeable.

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1 Introduction

Informally, imagine an object with total unit mass that falls apart randomly as time passes. The state of the system at some given time consists in the sequence of the masses of the fragments. Suppose that its evolution is Markovian and obeys the following rule. There is a parameter $\alpha \in \mathbb{R}$, called the index, such that given that the system at time $t \geq 0$ consists in the ranked sequence of masses $m_1 \geq m_2 \geq \ldots \geq 0$, the system at time $t+r$ is obtained by dislocating every mass m_i independently of the other fragments to obtain a family of sub-masses, say $(m_{i,j}, j \in \mathbb{N})$, where the sequence of the ratios $(m_{i,j}/m_i, j \in \mathbb{N})$ has the same distribution as the sequence resulting from a single unit mass fragmented up to time $m_i^\alpha r$. Such a random process will be referred to as a *self-similar fragmentation* with index α.

Here is a simple example that is closely connected to Kingman's coalescent [13]. Consider a stick of length 1, which can be identified as the unit interval, and $U_1, \ldots$, a sequence of i.i.d. uniformly distributed variables. For $n = 1, \ldots$, cut the stick at the location U_n at the instant of the n-th jump of some Poisson process which is independent of the sequence $U_1, \ldots$. Then the process giving the lengths of the fragments of the stick as a function of time is easily seen to be a self-similar fragmentation with index $\alpha = 1$. Related examples based on binary splitting of intervals have been considered by Brennan and Durrett [7, 8] (in this direction, it may be interesting to recall that the case $\alpha = 2/3$ arises in a model for polymer degradation). More recently, Aldous and Pitman [3] have constructed a self-similar fragmentation with index 1/2 which has a central role in the study of the standard additive coalescent, by cutting randomly the continuum random tree along its skeleton (see also [5] for an alternative construction based on the Brownian excursion). We also refer to Aldous' survey [2] for more literature on fragmentation processes.

Roughly, the key result of this work is that the distribution of a self-similar fragmentation is characterized by its index $\alpha \in \mathbb{R}$, a coefficient $c \geq 0$ that measures the rate of erosion, and a so-called Lévy measure ν which accounts for the sudden dislocations. More precisely, introduce the natural state-space $\mathcal{S}^\downarrow$ for the ranked sequence of sub-masses resulting from the dislocation of a unit mass, i.e. $\mathcal{S}^\downarrow$ denotes the space of decreasing numerical sequences $s = (s_1, s_2, \ldots)$ with $\sum s_i \leq 1$. A Lévy measure ν on $\mathcal{S}^\downarrow$ is a measure that gives no mass to the sequence $(1, 0, \ldots)$ and fulfills the requirement

$$\int_{\mathcal{S}^\downarrow} (1 - s_1)\, \nu(ds) \, < \, \infty \, .$$

Conversely, given arbitrary numbers $\alpha \in \mathbb{R}$ and $c \geq 0$ and a measure ν on $\mathcal{S}^\downarrow$ that verifies the preceding integral condition, one can construct a self-similar fragmentation with index α, erosion rate c and Lévy measure ν.

Our approach relies on a recent related work [6] which focuses on the so-called *homogeneous* case $\alpha = 0$ where the fragmentation rate does not depend of the mass of the fragments. More precisely, the characterization alluded above has been established there for homogeneous fragmentations, and the first purpose of this work is to extend this to the self-similar case. This extension will be obtained by introducing a kind of random time-transformation that enables us to change the index in a self-similar fragmentation process, and hence to reduce the study to the homogeneous case.

This classical idea of transforming a Markov process into a simpler one by a suitable time-substitution raises important technical difficulties in the present setting. Specifically, it has been pointed out by Pitman [15] that in the homogeneous case $\alpha = 0$, it is much easier to analyze fragmentations as processes with values in the space of partitions of $\mathbb{N} = \{1, 2, \ldots\}$, and this is the key to the results in [6]. This trick is not so useful in the self-similar case $\alpha \neq 0$, because if one works in the space of partition of $\mathbb{N}$, the dynamics of the fragmentation depend on the so-called asymptotic frequencies of blocks (which correspond to the masses of the fragments), and the latter are *not* continuous functionals of partitions. As a consequence, it seems hopeless to prove the Feller property by this approach, and a fortiori, to develop techniques of random time substitutions.

We shall circumvent this difficulty by discussing two different aspects of fragmentation. We first consider fragmentation of the unit interval $]0,1[$ induced by a nested family of open sets, this framework being well-suited for establishing the Feller property in the self-similar case. Then we will turn our attention to a more general setting involving random exchangeable partitions of $\mathbb{N}$. We shall see that these two aspects are in fact essentially equivalent. This allows us to shift the time-substitution results established for interval-fragmentations to partition-valued fragmentations and to establish the desired characterization of self-similar fragmentations.

As an example of application, we consider the evolution as time passes of the size of the fragment that contains a tagged point picked randomly at the initial time, independently of the fragmentation process. We identify this process as a semi-stable Markov process in the terminology of Lamperti [14], and its distribution is made explicit in terms of the characteristics of the fragmentation. In some cases, such as for instance that considered by Aldous and Pitman [3], one can recover the characteristics of the fragmentation from the law of the mass of the tagged fragment.

2 Interval fragmentation

2.1 Definition

We write $\mathcal{V}$ for the space of open subsets $V \subseteq]0,1[$. Each $V \in \mathcal{V}$ is determined by the continuous function χ_V:

$$\chi_V(x) \;=\; \min\{|x-y| : y \in V^{\mathsf{c}}\}\;, \qquad x \in [0,1]\,,$$

where $V^{\mathsf{c}} = [0,1]\backslash V$. Define the distance between two open sets U and V as the uniform distance between the functions χ_U and χ_V, i.e.

$$\text{dist}(U,V) \;=\; ||\chi_U - \chi_V||_\infty \;=\; \max_{x\in[0,1]} |\chi_U(x) - \chi_V(x)|\,.$$

This coincides with the Hausdorff distance between the closed complementary sets U^{c} and V^{c}, and $\mathcal{V}$ a compact metric space. For instance, a sequence of open intervals, say $]a_n, b_n[$ for $n = 1, \ldots$, converges to $]a,b[$ where $0 \le a < b \le 1$ if and only if $\lim a_n = a$ and $\lim b_n = b$, and converges to $\emptyset$ if and only if $\lim(b_n - a_n) = 0$. We point out the following elementary lemma that will be useful for our future purpose.

Lemma 1 *For every $i \in \mathbb{N}$, let $(V_{n,i}, n \in \mathbb{N})$ be a sequence in $\mathcal{V}$ that converges to V_i. Assume that for each fixed $n \in \mathbb{N}$, the open sets $V_{n,1}, V_{n,2}, \ldots$ are disjoint and that the sequence $V_{n,i}$ converges in $\mathcal{V}$ as $i \to \infty$ to $\emptyset$, uniformly in $n \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \bigcup_{i\in\mathbb{N}} V_{n,i} = \bigcup_{i\in\mathbb{N}} V_i \qquad \text{in } \mathcal{V}.$$

Proof: It follows from the assumptions that the open sets $V_1, V_2, \ldots$ are disjoint and *a fortiori* converge to $\emptyset$ as $i \to \infty$. Set

$$W_n = \bigcup_{i\in\mathbb{N}} V_{n,i} \quad , \quad W = \bigcup_{i\in\mathbb{N}} V_i \, ,$$

and note that

$$\chi_{W_n} = \sum_{i=1}^{\infty} \chi_{V_{n,i}} \quad , \quad \chi_W = \sum_{i=1}^{\infty} \chi_{V_i} \, .$$

To complete the proof, note that by the triangular inequality, we have for every $k \in \mathbb{N}$

$$\begin{aligned} ||\chi_{W_n} - \chi_W||_\infty &\le \sum_{i=1}^{k} ||\chi_{V_{n,i}} - \chi_{V_i}||_\infty + ||\sum_{i=k+1}^{\infty} \chi_{V_{n,i}}||_\infty + ||\sum_{i=k+1}^{\infty} \chi_{V_i}||_\infty \\ &= \sum_{i=1}^{k} ||\chi_{V_{n,i}} - \chi_{V_i}||_\infty + \sup_{i\ge k+1} ||\chi_{V_{n,i}}||_\infty + \sup_{i\ge k+1} ||\chi_{V_i}||_\infty \, , \end{aligned}$$

where the identity stems from the fact that the open sets $V_{n,k+1}, \ldots$ (respectively $V_{k+1}, \ldots$) are disjoint). For every $\varepsilon > 0$, we can choose k large enough (independently of n) such that the last two terms in the sum in the right-hand side are both less than $\varepsilon/3$. The integer k being fixed, we can bound the first term by $\varepsilon/3$ whenever n is sufficiently large. □

Next, each open set $V \in \mathcal{V}$ can be expressed as the union of disjoint open intervals, and we call interval decomposition of V any infinite sequence $(I_i, i \in \mathbb{N})$ of disjoint open intervals such that $V = \bigcup I_i$. Of course, some of the I_i's may be empty, and any permutation of an interval decomposition is again an interval decomposition. We state the following simple connexion linking convergence in $\mathcal{V}$ and interval decompositions.

Lemma 2 *Let $(V_n, n \in \mathbb{N})$ be a sequence of open sets that converges to V in $\mathcal{V}$. Then there exists interval decompositions $(I_{n,i}, i \in \mathbb{N})$ and $(I_i, i \in \mathbb{N})$ of V_n and V, respectively, such that*

(i) $\lim_{n\to\infty} I_{n,i} = I_i$ *in $\mathcal{V}$ for each $i \in \mathbb{N}$,*

(ii) $\lim_{i\to\infty} I_{n,i} = \emptyset$ *in $\mathcal{V}$, uniformly in $n \in \mathbb{N}$.*

Proof: Let $(I_i, i \in \mathbb{N})$ be an interval decomposition of V such that the sequence of the lengths $|I_i|$ of these intervals is non-increasing. For every $i \in \mathbb{N}$ such that $I_i \neq \emptyset$, let m_i denote the mid-point of I_i. Set $d_n = \text{dist}(V_n, V)$ for $n \in \mathbb{N}$, so the sequence d_n converges to 0. For every $n \in \mathbb{N}$, set $i(n) = \max\{i \in \mathbb{N} : |I_i| > 4d_n\}$. For every $i \le i(n)$, we have $\chi_V(m_i) > 2d_n$, and hence $\chi_{V_n}(m_i) > d_n$. Thus $m_i \in V_n$, and we denote by $I_{n,i}$ the interval component of V_n that contains m_i. It is immediate that

$$|I_{n,i}| > 2d_n \quad \text{and} \quad \text{dist}(I_i, I_{n,i}) \le d_n \, , \qquad i \le i(n) \, . \tag{1}$$

Note that this entails that $I_{n,i} \cap I_{n,j} = \emptyset$ whenever $i \neq j \leq i(n)$, since otherwise we would have $I_{n,i} = I_{n,j}$ and by the triangular inequality $\mathrm{dist}(I_i, I_j) \leq 2d_n$, which is absurd (recall that $\chi_{I_i}(m_i) > 2d_n$ and $\chi_{I_j}(m_i) = 0$ since $m_i \notin I_j$).

Observe also that if J is an interval component of V_n with $|J| > 6d_n$, then $J = I_{n,i}$ for some $i \leq i(n)$. More precisely, consider the mid-point m of J. Since $\chi_{V_n}(m) > 3d_n$, we have $\chi_V(m) > 2d_n$, so $m \in V$ and the interval component I_i of V that contains m has length $|I_i| > 4d_n$, that is $i \leq i(n)$. The same argument as that used previously shows that $\mathrm{dist}(I_i, J) \leq d_n$, and by (1) and the triangular inequality, this forces $I_{n,i} = J$.

Next, we consider for every n the interval decomposition $(I_{n,i}, i \in \mathbb{N})$ of V_n obtained by adding to the finite sequence $(I_{n,i}, i \leq i(n))$ the infinite sequence $(I_{n,i}, i > i(n))$ of the remaining intervals components of V_n, where the latter are ranked according to the decreasing order of their lengths. We see from (1) that as $n \to \infty$, $I_{n,i}$ converges in $\mathcal{V}$ to I_i provided that $I_i \neq \emptyset$. The same holds in the case when $I_i = \emptyset$, because then $i(n) < i$ for every n and we have pointed out that this entails $|I_{n,i}| \leq 6d_n$. So all that we need is to verify the requirement (ii) of the statement.

Fix $\varepsilon > 0$, take $i > 3/\varepsilon$, and let n be an arbitrary integer. Note that $|I_i| < \varepsilon/3$ (because $|I_1| \geq |I_2| \geq \ldots$ and $|I_1| + \ldots \leq 1$), so $\mathrm{dist}(I_i, \emptyset) < \varepsilon/6$. First, if $d_n < \varepsilon/3$ and $i \leq i(n)$, then by (1) $\mathrm{dist}(I_i, I_{n,i}) < \varepsilon/3$, and it follows from the triangular inequality that $\mathrm{dist}(I_{n,i}, \emptyset) < \varepsilon/2$. Second, if $d_n < \varepsilon/3$ and $i > i(n)$, then we have already pointed out that $|I_{n,i}| \leq 6d_n$, and hence $\mathrm{dist}(I_{n,i}, \emptyset) \leq 3d_n < \varepsilon$. Third, if $d_n \geq \varepsilon/3$, then $i(n) \leq 3/4\varepsilon$ (because $|I_i| > 4\varepsilon/3$ when $i \leq i(n)$), and $i - i(n) > 9/4\varepsilon$. This implies that there are more than $9/4\varepsilon$ interval components of V_n with length at least $|I_{n,i}|$, so $|I_{n,i}| \leq 4\varepsilon/9$ and hence $\mathrm{dist}(I_{n,i}, \emptyset) < 2\varepsilon/9$. We have checked (ii) and the proof is complete. □

Finally, we introduce the space of numerical sequences

$$\mathcal{S}^{\downarrow} = \left\{ (s_1, \ldots) : s_1 \geq s_2 \geq \ldots \geq 0 \text{ and } \sum_{n=1}^{\infty} s_n \leq 1 \right\},$$

which is endowed with the topology of pointwise convergence. There is a natural map $s : \mathcal{V} \to \mathcal{S}^{\downarrow}$, where $s(V) = (s_1(V), \ldots)$ is given by the sequence of the lengths of the component intervals of V, ranked in the decreasing order, and it is readily checked from Lemma 2 that $V \to s(V)$ is continuous. We are now able to introduce the deterministic notion of fragmentation of $]0,1[$.

Definition 1 (Interval fragmentation) *A family $F = (F(t), t \geq 0)$ in $\mathcal{V}$ is called an interval fragmentation if it is nested , i.e. if $F(t) \subseteq F(r)$ whenever $0 \leq r \leq t$. The compound process $(s \circ F(t), t \geq 0)$ given by the decreasing sequence of the lengths of the interval components, is called the ranked fragmentation associated with F.*

It is easily checked that an interval fragmentation possesses a right-limit at any $t \geq 0$ and a left-limit at any $t' > 0$ which are given respectively by

$$F(t+) := \lim_{r \to t+} F(r) = \bigcup_{r > t} F(r) \quad , \quad F(t'-) := \lim_{r \to t'-} F(r) = \left(\bigcap_{r < t'} F(r) \right)^{\mathrm{i}} , \tag{2}$$

where A^{i} denotes the interior of A. Note that by the continuity of the map $s : \mathcal{V} \to \mathcal{S}^{\downarrow}$, we have

$$\lim_{r\to t+} s\circ F(r) \,=\, s\circ F(t+) \quad , \quad \lim_{r\to t'-} s\circ F(r) \,=\, s\circ F(t'-)\,.$$

In the converse direction, it is easily seen that an interval fragmentation F is continuous at t if and only if its associated ranked fragmentation $s\circ F$ is continuous at t.

Our next purpose is to define random self-similar fragmentations. To that end, our basic data consist in a family $(p_t(]0,1[), t\geq 0)$ of probability measures on $\mathcal{V}$, where $p_t(]0,1[)$ is meant to describe the distribution of the random open set resulting from the fragmentation at time t of the unit interval. We shall assume that the map $t \to p_t(]0,1[)$ is continuous and we construct a family of probability kernel on $\mathcal{V}$ as follows.

First, recall that we are given a real number α, the index of self-similarity. For every open interval $I \subseteq]0,1[$, we introduce the law $p_t^{(\alpha)}(I)$ of the fragmentation of I observed at time t by an obvious affine transformation. More precisely, $p_t^{(\alpha)}(\emptyset)$ is always the Dirac point mass at $\emptyset$. When $I =]a,b[$ is non-void, introduce the affine function $g_I :]0,1[\to I$ given by $g_I(x) = a + x(b-a)$. By a slight abuse of notation, we still denote by g_I the induced map on $\mathcal{V}$, so that $g_I(V)$ is an open subset of I. We then define the probability measure $p_t^{(\alpha)}(I)$ as the image of $p_r(]0,1[)$ by g_I, with $r = t|I|^{\alpha} = t(b-a)^{\alpha}$. Finally, if $V \in \mathcal{V}$ is an arbitrary open set with interval decomposition $(I_i, i \in \mathbb{N})$, we denote by $p_t^{(\alpha)}(V)$ the distribution of $\bigcup X_i$ where $X_1, \ldots$ are independent random variables distributed according to law $p_t^{(\alpha)}(I_1), \ldots$, respectively. We thus have defined for each $t \geq 0$ a kernel $p_t^{(\alpha)}$ of probability measures on $\mathcal{V}$.

Definition 2 (Self-similar interval fragmentation*) A random interval fragmentation $F = (F(t), t\geq 0)$ is called self-similar with index $\alpha \in \mathbb{R}$ if F is a time-homogeneous Markov process which fulfills the following conditions:*

(i) *F is continuous in probability and starts from $F(0) =]0,1[$ a.s.*

(ii) *If $p_t(]0,1[)$ denotes the law of $F(t)$ for $t \geq 0$, then the transition semigroup of F is given by the kernels $\left(p_t^{(\alpha)}, t \geq 0\right)$ in the notation introduced above.*

Informally, (ii) means that disjoint intervals fall apart independently, which is a kind of branching property. In the sequel, this will be referred to as the (simple) fragmentation property. By (2), F possesses a càdlàg version given by $(F(t+), t\geq 0)$, where $F(t+) = \bigcup_{\varepsilon>0} F(t+\varepsilon)$ for every $t \geq 0$. We shall implicitly work with that version from now on, i.e. we assume that $F(t) = F(t+)$ in the sequel. We shall also suppose that the fragmentation is not trivial, i.e. $F \not\equiv]0,1[$ with positive probability; and it is then easy to see that $F(t)$ converges to $\emptyset$ a.s. Therefore we shall agree that $F(\infty) = \emptyset$; in this direction note that $\emptyset$ can be viewed as a cemetery point in the terminology of the theory of Markov processes.

2.2 First properties

Throughout the rest of this section, $F = (F(t), t \geq 0)$ will denote some (non-trivial, càdlàg) self-similar interval fragmentation, and $\mathbb{P}$ will stand for its distribution on Skorohod's space of paths with values in the compact metric space $\mathcal{V}$. For every open set $V \in \mathcal{V}$, we write $\mathbb{P}_V$ for the fragmentation process started from V, in particular, $\mathbb{P} = \mathbb{P}_{]0,1[}$.

The next two statements are devoted to the scaling and the Feller properties respectively, which are most useful tools in this work. Recall the notation $g_I : \mathcal{V} \to \mathcal{V}$ introduced above for a generic non-void interval I, and agree that $g_\emptyset(V) = \emptyset$ for every $V \in \mathcal{V}$. First, the scaling property is an immediate consequence of the very definition of a self-similar fragmentation.

Lemma 3 (Scaling property) *For every open interval $I \subseteq]0,1[$, the distribution of the process $(g_I \circ F(t|I|^\alpha), t \geq 0)$ under $\mathbb{P}$ is $\mathbb{P}_I$.*

Note that more generally, the combination of the scaling and the simple fragmentation properties entails that for every $V \in \mathcal{V}$, the law $\mathbb{P}_V$ of the fragmentation started at V can be constructed as follows. Introduce a sequence $F_1, F_2, \dots$ of independent copies of F (started from $]0,1[$), and pick an interval decomposition $(I_i, i \in \mathbb{N})$ of V. Next define for every $t \geq 0$ the random open set

$$X_t = \bigcup_{i \in \mathbb{N}} g_{I_i} \circ F_i\left(t|I_i|^\alpha\right) . \tag{3}$$

Then the distribution of the process $X = (X_t, t \geq 0)$ is $\mathbb{P}_V$.

Lemma 4 (Feller property) *The semigroup $\left(p_t^{(\alpha)}, t \geq 0\right)$ of F fulfills the Feller property, that is for each fixed $t \geq 0$ the map $V \to p_t^{(\alpha)}(V)$ is continuous on $\mathcal{V}$ and for each fixed $V \in \mathcal{V}$, $p_t^{(\alpha)}(V)$ converges to the Dirac point mass at V when $t \to 0$.*

Proof: Let $(V_n, n \in \mathbb{N})$ be a sequence in $\mathcal{V}$ converging to V, and pick interval decompositions $(I_{n,i}, i \in \mathbb{N})$ and $(I_i, i \in \mathbb{N})$ of V_n and of V respectively, that fulfill the conclusions (i) and (ii) of Lemma 2. For simplicity, we write $g_{n,i}$ and g_i for g_I when $I = I_{n,i}$ and $I = I_i$, respectively. Finally, fix $t \geq 0$ and set $t_{n,i} = t|I_{n,i}|^\alpha$ and $t_i = t|I_i|^\alpha$. Following (3), let $F_1, F_2, \dots$ be a sequence of independent copies of F, and introduce

$$Y_n = \bigcup_{i \in \mathbb{N}} g_{n,i} \circ F_i(t_{n,i}) \quad , \quad Y = \bigcup_{i \in \mathbb{N}} g_i \circ F_i(t_i) .$$

For every $i \in \mathbb{N}$, we have that $\lim_{n\to\infty} t_{n,i} = t_i$ and it is immediate to check that $\lim_{n\to\infty} g_{n,i} = g_i$ in the sense of uniform convergence of functions on the compact metric space $\mathcal{V}$. On the other hand, recall that F is continuous in probability. Provided that $t_i < \infty$, $F_i(t_{n,i})$ converges in probability to $F_i(t_i)$, and hence $g_{n,i} \circ F_i(t_{n,i})$ converges in probability to $g_i \circ F_i(t_i)$. The latter assertion also holds in the case $t_i = \infty$ because it only occurs when I_i is empty and $\alpha < 0$. As $I_{n,i}$ converges to $\emptyset$ as $i \to \infty$, uniformly in $n \in \mathbb{N}$, we have automatically that $g_{n,i} \circ F_i(t_{n,i})$ converges to $\emptyset$ as $i \to \infty$, uniformly in $n \in \mathbb{N}$.

Applying Lemma 1, we conclude that Y_n converges in probability to Y. On the other hand, we know from (3) that Y_n and Y have the law $p_t^{(\alpha)}(V_n)$ and $p_t^{(\alpha)}(V)$ respectively, and therefore the map $V \to p_t^{(\alpha)}(V)$ is continuous. This proves the first part of the statement; the argument for the second is similar (and easier). □

The Feller property ensures that the (simple) fragmentation property holds more generally for stopping times; this can be viewed as the strong fragmentation property. Our next purpose is to present a different extension of the simple fragmentation property which will play an important role in our analysis.

For every $x \in]0,1[$ and every $t \geq 0$, denote by $I_x(t)$ the interval component of $F(t)$ that contains x if $x \in F(t)$, and set $I_x(t) = \emptyset$ otherwise. Recall that the ultimate fragmentation $F(\infty)$ is empty a.s., so we also agree that $I_x(\infty) = \emptyset$. We write $\left(\mathcal{F}_t^{(x)}\right)_{t\geq 0}$ for the natural filtration (completed by null sets) generated the process $(I_x(t), t \geq 0)$. We are now able to introduce the notion of frost for an interval fragmentation, which bears roughly the same role as stopping times for Markov processes. It is also a close relative to the so-called stopping lines for branching processes; see Chauvin [10].

Definition 3 (Frost) *A random function $T :]0,1[\to [0,\infty]$ is called a frost for the interval fragmentation F if the following two conditions are satisfied:*

(i) *For every $x \in]0,1[$, $T(x)$ is an $\left(\mathcal{F}_t^{(x)}\right)$-stopping time.*

(ii) *For $x \in]0,1[$ and $y \in I_x(T(x))$, it holds that $T(x) = T(y)$.*

Of course, a deterministic constant function is a frost. To present a non-trivial example, we may consider $T(x) = \inf\{t \geq 0 : |I_x(t)| < \ell\}$, the first instant at which the length of the interval component of F containing x is less than some fixed $\ell \in]0,1[$.

When T is a frost, note from (ii) that for every $x, y \in]0,1[$, we have either $I_x(T(x)) = I_y(T(y))$ or $I_x(T(x)) \bigcap I_y(T(y)) = \emptyset$. This incites us to introduce the random open set

$$F(T) \;=\; \bigcup_{x\in]0,1[} I_x(T(x))\,,$$

which will be referred to as the fragmentation *frozen* at T. On the other hand, it is immediately seen that if T and T' are two frosts, then $T \wedge T'$ is again a frost, and moreover $F(T) \subseteq F(T \wedge T')$. Similarly, for every deterministic $t > 0$, $T + t$ is also a frost and $F(T + t) \subseteq F(T)$. These observations enable us to define the fragmentation *terminated* at T

$$F \circ \tau_T \;=\; (F(t \wedge T), t \geq 0)$$

and the *resumed* fragmentation

$$F \circ \theta_T \;=\; (F(T + t), t \geq 0)\ .$$

The notation τ and θ refer to the classical stop and shift operators in the canonical notation for Markov processes.

Theorem 1 (Extended fragmentation property) *Let T be a frost for F. For every open set $V \in \mathcal{V}$, under the conditional law $\mathbb{P}_V(\cdot \mid F(T) = V')$ of the fragmentation started from V and conditioned on the frozen fragmentation $F(T) = V'$, the fragmentation terminated at T, $F \circ \tau_T$, and the resumed fragmentation, $F \circ \theta_T$, are independent and the latter has the law $\mathbb{P}_{V'}$.*

Proof: We shall first check by induction Theorem 1 in the case when T only takes finitely many values. The statement is trivial when T is a deterministic constant, so let us assume that the extended fragmentation property has been proved for every frost taking at most n values, and consider a frost T taking values in $\{t_1, \cdots, t_{n+1}\}$ where $0 \leq t_1 < \cdots < t_{n+1} \leq \infty$. We may apply the extended fragmentation property to the frost $T \wedge t_n$, so conditionally on $F(T \wedge t_n) = V'$, $F \circ \tau_{T\wedge t_n}$ and $F \circ \theta_{T\wedge t_n}$ are independent and $F \circ \theta_{T\wedge t_n}$ has the law $\mathbb{P}_{V'}$.

We next assign a mark $M(I)$ to each non-void interval component I of V' as follows: if $T(x) \leq t_n$ for some (and then all) $x \in I$, the mark $M(I)$ is 0 (stop); otherwise $M(I) = 1$ (continue). We stress that the random mark M is measurable with respect to the sigma-field generated by the stopped fragmentation $F \circ \tau_{T\wedge t_n}$, and hence the preceding extended fragmentation property can be reinforced as follows. Write $V_0 \subseteq V'$ for the open set obtained from the interval components of V' having mark 0 and $V_1 = V' \backslash V^{(0)}$ for that obtained from the intervals having mark 1, and denote by F_0 and F_1 the resumed fragmentation $F \circ \theta_{T\wedge t_n}$ restrained to V_0 and V_1, respectively. Then conditionally on $F(T \wedge t_n) = V'$ and M, $F \circ \tau_{T\wedge t_n}$, F_0 and F_1 are independent, F_0 has the law $\mathbb{P}_{V_0}$ and F_1 has the law $\mathbb{P}_{V_1}$. By an application the simple fragmentation property for F_1 at time $t_{n+1} - t_n$, we now easily conclude that the extended fragmentation property holds for T.

It is now straightforward to complete the proof for a general frost T. We may approximate T by a decreasing sequence $(T_n, n \in \mathbb{N})$ of frosts taking only finitely many values. For instance, one may consider

$$T_n(x) \;=\; \begin{cases} 2^{-n}[2^n T(x) + 1] & \text{if } T(x) \leq 2^n\,, \\ \infty & \text{otherwise.} \end{cases}$$

By a standard argument based on the right-continuity of the paths and the Feller property stated in Lemma 4, we see that the extended fragmentation property at T_n propagates to T. □

Recall that we are working with the right-continuous version of the fragmentation F. Turning our attention to left-continuity, we conclude this section with the following property.

Corollary 1 (Quasi-left-continuity) *Let $(T_n, n \in \mathbb{N})$ be an increasing sequence of frosts, and set $T = \lim_{n\to\infty} T_n$. Then T is a frost and $\lim_{n\to\infty} F(T_n) = F(T)$ a.s.*

Proof: That the increasing limit of a sequence of frosts is again a frost is immediate. The second assertion is established by the same argument based on the right-continuity of the path, the Feller property, and the extended fragmentation property as in the proof of the quasi-left-continuity for Feller processes. See for instance Proposition I.7 in [4]. □

2.3 Changing the index of self-similarity

The purpose of this section is to present a simple transformation based on the extended fragmentation property which allows us to change the index of self-similarity. Recall that $|I_x(t)|$ denotes the length of the interval component of $F(t)$ that contains x (with the convention that $|I_x(t)| = 0$ when $x \notin F(t)$), and introduce for an arbitrary $\beta \in \mathbb{R}$ and $t \geq 0$

$$T_t^{(\beta)}(x) \;=\; \inf\left\{u \geq 0 : \int_0^u |I_x(r)|^{-\beta} dr > t\right\}\,, \qquad x \in]0,1[\,.$$

It should be plain that each $T_t^{(\beta)}$ is a frost for F. This enables us to introduce the process of frozen fragmentations

$$F^{(\beta)}(t) \;:=\; F(T_t^{(\beta)})\,, \qquad t \geq 0.$$

Theorem 2 *The process of frozen fragmentations* $F^{(\beta)} = \left(F^{(\beta)}(t), t \geq 0\right)$ *is a self-similar interval fragmentation with index* $\alpha + \beta$.

Proof: As the function $t \to T_t^{(\beta)}$ is right-continuous and increasing, $F^{(\beta)}$ is a right-continuous interval fragmentation; and it is clear that $F^{(\beta)}(0) =]0,1[$, $\mathbb{P}$-a.s. Also for each $t > 0$, one has $T_r^{(\beta)}(x) < T_t^{(\beta)}(x)$ whenever $r < t$ and $T_r^{(\beta)}(x) < \infty$, and $\lim_{r \uparrow t} T_r^{(\beta)} = T_t^{(\beta)}$. It follows from the quasi-left-continuity property as stated in Corollary 1 that $F^{(\beta)}$ is continuous in probability.

Next, for every open set $V \subseteq]0,1[$, write $\mathbb{Q}_V$ for the distribution of $F^{(\beta)}$ under $\mathbb{P}_V$. On the one hand, for $u < t$, the frozen fragmentation $F^{(\beta)}(u) = F(T_u^{(\beta)})$ is measurable with respect to the fragmentation terminated at $T_t^{(\beta)}$, $F \circ \tau_{T_t^{(\beta)}}$. On the other hand, if we write $\tilde{F} = F \circ \theta_{T_t^{(\beta)}}$ for the resumed fragmentation, then we have in the obvious notation that

$$T_{t+r}^{(\beta)}(x) - T_t^{(\beta)}(x) \;=\; \tilde{T}_r^{(\beta)}(x) \qquad \text{whenever } T_t^{(\beta)}(x) < \infty \,.$$

Applying the extended fragmentation property at $T_t^{(\beta)}$, we now see that the conditional distribution of $\left(F^{(\beta)}(t+r), r \geq 0\right)$ under $\mathbb{P}_V$ given $\left(F^{(\beta)}(u), 0 \leq u \leq t\right)$ is $\mathbb{Q}_{V'}$ with $V' = F_t^{(\beta)}$. Hence $F^{(\beta)}$ is a Markov process, and more precisely, it enjoys the fragmentation property.

Finally, we have to check the self-similarity property, which relies on the scaling property. In this direction, let $I \subseteq]0,1[$ be an arbitrary non-void open interval, and recall the notation g_I introduced before Definition 2. Applying Lemma 3, we see that the distribution of $F_t^{(\beta)}$ under $\mathbb{P}_I$ is the same as that of $g_I \circ F\left(|I|^\alpha T'_t\right)$ under $\mathbb{P}$ with

$$T'_t(y) \;=\; \inf\left\{u \geq 0 : \int_0^u |J_y\left(|I|^\alpha r\right)|^{-\beta} dr > t\right\} \,,$$

where $J_y(\cdot)$ denotes the interval component of $g_I \circ F(\cdot)$ that contains $y \in]0,1[$. In other words, for $y = g_I(x)$, we have $J_y(\cdot) = g_I\left(I_x(\cdot)\right)$ and in particular $|J_y\left(|I|^\alpha r\right)| = |I||I_x\left(|I|^\alpha r\right)|$. It then follows from a few lines of elementary calculations that

$$|I|^\alpha T'_t(y) \;=\; T_{t|I|^{\alpha+\beta}}^{(\beta)}(x) \,,$$

and we conclude that the law of $F^{(\beta)}(t)$ under $\mathbb{P}_I$ is the same as that of $g_I \circ F^{(\beta)}\left(t|I|^{\alpha+\beta}\right)$ under $\mathbb{P}$. This shows that $F^{(\beta)}$ is a self-similar fragmentation with index $\alpha + \beta$, and the proof of Theorem 2 is now complete. □

We stress that F can be recovered from $F^{(\beta)}$, more precisely we have $F = (F^{(\beta)})^{(-\beta)}$ in the obvious notation.

3 Partition-valued fragmentation

Informally, focusing on interval fragmentation may appear as a rather restrictive point of view, and it could be more natural to consider fragmentation of abstract sets. In this direction, we first introduce some material on partitions of integers which are mostly lifted from Evans and Pitman [11].

3.1 Definition

An equivalence relation Γ on $\mathbb{N} = \{1, \dots\}$ can be identified as a *partition* of $\mathbb{N}$ into a sequence $(B_n, n \in \mathbb{N})$ of disjoint blocks. It is convenient to agree that the indexation of blocks obeys the following rule: B_n is the block of Γ that contains n provided that n is the smallest element in its block, otherwise $B_n = \emptyset$. The partition that has a unique non-void block, $B_1 = \mathbb{N}$, will be referred to as the *trivial partition*. The space of partitions of $\mathbb{N}$ is denoted by $\mathcal{P}$; recall there is a natural metric making $\mathcal{P}$ compact, which can be described as follows: for every $\Gamma, \Gamma' \in \mathcal{P}$, $\text{dist}(\Gamma, \Gamma') = 2^{-n}$ where n is the smallest integer such that the partitions induced by Γ and Γ' on $\{1, \dots, n\}$ differ. Next, for every $C \subseteq \mathbb{N}$ and every partition $\Gamma \in \mathcal{P}$, we may define a partition $\Gamma \circ C$ of C, called the partition of C induced by Γ, as follows. We rank the elements of C in the increasing order, i.e. $C = \{c_1, \dots\}$ with $c_1 < \dots$, and we denote by $\Gamma \circ C$ the partition of C defined by

$$\Gamma \circ C \;=\; \left(\{c_j : j \in B_i\}, i = 1, \dots\right) , \tag{4}$$

where $B_1, \dots$ are the blocks of Γ. Of course $\Gamma \circ \emptyset = (\emptyset, \dots)$.

A *$\mathcal{P}$-valued fragmentation* is a family of partitions $(\Pi(t), t \geq 0)$ such that for every $0 \leq r \leq t$, the partition $\Pi(t)$ is finer than $\Pi(r)$, in the sense that each block of $\Pi(t)$ is contained into some block of $\Pi(r)$. A random $\mathcal{P}$-valued fragmentation is called *exchangeable* if for every finite permutation σ of $\mathbb{N}$, the processes $(\sigma \circ \Pi(t), t \geq 0)$ and $(\Pi(t), t \geq 0)$ have the same distribution, where $\sigma \circ \Pi(t)$ is the random partition whose blocks are the images by σ of the blocks of $\Pi(t)$.

By a fundamental result of Kingman [13] (see also Aldous [1] for a simpler proof), for each $t \geq 0$, the blocks of $\Pi(t)$ have asymptotic frequencies a.s., in the sense that the limits

$$\lim_{n \to \infty} \frac{1}{n} \text{Card} \{k \leq n : k \in B_i(t)\} \;:=\; \lambda_i(t)$$

exist with probability one for $i = 1, \dots$. We write $\lambda^{\downarrow}(t) = \left(\lambda_1^{\downarrow}(t), \dots\right)$ for the decreasing rearrangement of the $\lambda_i(t)$'s. The $\mathcal{S}^{\downarrow}$-valued process $\lambda^{\downarrow} = \left(\lambda^{\downarrow}(t), t \geq 0\right)$ will be referred to as the ranked fragmentation corresponding to Π. We stress that $\lambda^{\downarrow}(t)$ is *not* a continuous functional of the exchangeable partition $\Pi(t)$.

We call an exchangeable $\mathcal{P}$-valued fragmentation Π *nice* if it fulfills the (apparently) stronger assumption that with probability one, $\Pi(t)$ has asymptotic frequencies for all $t \geq 0$ simultaneously. Evans and Pitman [11] have pointed out that this requirement is always fulfilled whenever Π has proper frequencies, in the sense that $\sum_{i=1}^{\infty} \lambda_i(t) = 1$ a.s. for every $t > 0$. Similarly, it has been observed in Section 5 of [5] that so-called homogeneous $\mathcal{P}$-valued fragmentations are always nice, and we are not aware of any exchangeable $\mathcal{P}$-valued fragmentation which is not nice.

Finally, we define self-similar $\mathcal{P}$-valued fragmentation. Recall the notation (4).

Definition 4 (Self-similar $\mathcal{P}$-valued fragmentation*) A nice exchangeable $\mathcal{P}$-valued fragmentation $\Pi = (\Pi(t), t \geq 0)$ is called self-similar with index $\alpha \in \mathbb{R}$ if Π is a time-homogeneous Markov process which fulfills the following conditions:*

(i) Π *starts a.s. from the trivial partition.*

(ii) *The ranked fragmentation* $\lambda^{\downarrow}$ *associated to* Π *is continuous in probability.*

(iii) *For every* $t, r \geq 0$*, the conditional distribution of* $\Pi(t+r)$ *given* $\Pi(t) = (B_1, \ldots)$ *is the law of the random partition whose blocks are those of the partitions* $\Pi^{(i)}(r_i) \circ B_i$ *for* $i = 1, \ldots,$ *where* $\Pi^{(1)}, \ldots$ *is a sequence of independent copies of* Π *and* $r_i = r\lambda_i(t)^{\alpha}$ *(recall that* $\lambda_i(t)$ *denotes the asymptotic frequency of the block* B_i*).*

3.2 Connection with interval fragmentation

Here is a prototype of an exchangeable $\mathcal{P}$-valued fragmentation. Let E be an abstract space endowed with a sigma-field $\mathcal{E}$ and a probability measure ρ. Consider for each $t \geq 0$ a sequence $(E_n(t), n \in \mathbb{N})$ of disjoint measurable sets such that for every $0 \leq s \leq t$ and every $i \in \mathbb{N}$ there is some $j \in \mathbb{N}$ such that $E_i(t) \subseteq E_j(s)$. So informally we may think of E as an object that falls apart as time runs, and of the family $(E_n(t), n \in \mathbb{N})$ as the sequence of fragments at time t. Next, pick a sequence $U_1, \ldots$ of random points in E such that each U_i has the law ρ, and $U_1, \ldots$ are independent. For each $t \geq 0$, consider $\Pi(t)$, the random partition of $\mathbb{N}$ such that two distinct integers i and j belong to the same block of $\Pi(t)$ if and only if the points U_i and U_j both belong to $E_n(t)$ for some $n \in \mathbb{N}$. It should be plain that Π is an exchangeable $\mathcal{P}$-valued process. Moreover, it follows from the strong law of large numbers that for each $t \geq 0$,

$$\lim_{n\to\infty} \frac{1}{n} \text{Card}\,\{i \leq n : U_i \in E_k(t)\} \;=\; \rho(E_k(t))\,, \text{a.s.}$$

so the ranked fragmentation $\lambda^{\downarrow}$ is the process that describes the ranked sequence of masses of the fragments in the dislocation process of the space E.

We may of course apply the construction above in the special case when $E =]0,1[$, ρ is the Lebesgue measure and for each $t \geq 0$, $(E_n(t) = I_n(t), n \in \mathbb{N})$ is an interval decomposition of $F(t)$, where $F = (F(t), t \geq 0)$ is some interval fragmentation. In that case, we write $\Pi(t) = \Pi_F(t)$ and refer to $(\Pi_F(t), t \geq 0)$ as the $\mathcal{P}$-valued fragmentation associated with the interval fragmentation F. (To be completely rigorous, we should rather call this a version as this process also depends on the uniform random variables $U_1, \ldots$; but since we are only interested in the law of such $\mathcal{P}$-valued fragmentation, we will not indicate the dependency on the U_i's). Note that the Glivenko-Cantelli theorem enables us to assert that Π_F is nice.

The following lemma is essentially straightforward.

Lemma 5 *If* $F = (F(t), t \geq 0)$ *is a self-similar interval fragmentation with index* α*, then the associated* $\mathcal{P}$*-valued fragmentation* $(\Pi_F(t), t \geq 0)$ *is self-similar with index* α*, and has the same ranked fragmentation as* F*, i.e.* $s \circ F(t) = \lambda^{\downarrow}(t)$ *a.s. for each* $t \geq 0$*.*

Proof: We have already observed that Π_F is a nice exchangeable $\mathcal{P}$-valued fragmentation. As $F(0) =]0,1[$ a.s., the partition $\Pi_F(0)$ is trivial a.s. Moreover, F is continuous in probability, and this entails that the corresponding ranked fragmentation $s \circ F$ is also continuous in probability. By the strong law of large numbers, $s \circ F$ coincides with the ranked fragmentation $\lambda^{\downarrow}$ of the $\mathcal{P}$-valued process Π_F, so (i) and (ii) of Definition 4 have been checked.

Next, fix $t > 0$ and consider an interval decomposition $(I_n(t), n \in \mathbb{N})$ of $F(t)$ (for instance we may rank the interval components of $F(t)$ in the decreasing order of their lengths and from the left to the right in the case of intervals with the same length); it is convenient to set $I_0(t) = F(t)^c$. Introduce for $n = 0, 1, \dots$

$$\beta_n = \{k \in \mathbb{N} : U_k \in I_n(t)\} ,$$

so β_0 is the set of indices corresponding to singletons in the partition $\Pi_F(t)$, and the blocks of $\Pi_F(t)$ which are neither empty nor reduced to singletons coincide with the β_n's for $n = 1, \dots$ and $I_n(t) \neq \emptyset$. Whenever $I_i(t)$ is not empty, we index the elements of β_i according to the increasing order, $\beta_{i,1} < \beta_{i,2} < \dots$, and set for simplicity $U_{i,j} = U_{\beta_{i,j}}$. It is easily seen that conditionally on the $I_n(t)$'s and β_n's, the families of variables $(U_{i,1}, \dots)$ for $i = 1, \dots$ are independent, and more precisely, provided that $I_i(t)$ is not empty, $U_{i,1}, U_{i,2} \dots$ is a sequence of i.i.d. variables that are uniformly distributed on $I_i(t)$. As for $r \leq t$, the partition $\Pi(r)$ can be recovered from $F(r)$, the sequence of blocks $(\beta_n, n \in \mathbb{N})$, and the variables $(U_j, j \in \beta_0)$, the preceding observations easily entail that Π_F is a Markov process, and the self-similarity property derives from that for F. □

In the converse direction [1], we first show that given a nice exchangeable $\mathcal{P}$-valued fragmentation Π, we can construct an interval fragmentation $(F_\Pi(t), t \geq 0)$ having the same ranked fragmentation as Π. For every $t \geq 0$ and $k \in \mathbb{N}$, let $B_k(t)$ denote the block of the partition $\Pi(t)$ that contains k provided that k is the least element of its block, and $B_k(t) = \emptyset$ otherwise. Let $\lambda_k(t)$ be the asymptotic frequency of $B_k(t)$, and define the instant when the k-th block appears,

$$t_k = \inf\{t \geq 0 : B_k(t) \neq \emptyset\} .$$

Next, for $k \geq 2$, call $j \in \mathbb{N}$ the father of k if k was an element of the j-th block immediately before B_k emerges, that is if $k \in B_j(t_k-)$. Define by induction the notion of ancestor of $k \geq 1$, so that k is an ancestor of k, and the father of an ancestor of k is again an ancestor of k. Call $k' \geq 2$ a twin brother of k if $t_k = t_{k'}$ and k and k' have the same father. Finally, define for every $k \geq 2$ the predecessor $p(k)$ of k as the largest twin brother k' of k such that $k' < k$ whenever such k' exists, and otherwise define $p(k)$ as the father of k. Plainly, $p(k) < k$ for all $k \geq 2$.

We then introduce for every $t \geq 0$ and $k \in \mathbb{N}$ the open interval

$$I_k(t) =]x_k, x_k + \lambda_k(t)[\subseteq]0, 1[,$$

where $x_1 = 0$ and for $k \geq 2$

$$x_k = x_{p(k)} + \lambda_{p(k)}(t_k) .$$

The following properties are clear from this very construction. First $I_k(t) = \emptyset$ if $t < t_k$ and $I_k(t') \subseteq I_k(t)$ if $t_k \leq t < t'$. Second, if $k' \neq k$ is either the father of k or one of its twin brothers, then $I_k(t_k) \cap I_{k'}(t_k) = \emptyset$. Third, if j is the father of $k \geq 2$, then $I_k(t) \subseteq I_j(t_k-)$.

Combining these elementary observations, we now see that we have $I_i(t) \cap I_j(t) = \emptyset$ whenever $i \neq j$ (consider the largest common ancestor of i and j and the last instant when i and j are in the same block), so the sequence of intervals $(I_i(t), i \in \mathbb{N})$ can be viewed as interval decomposition of an open set in $]0, 1[$ which we denote by $F_\Pi(t)$. It is also easy to check that the family $(F_\Pi(t), t \geq 0)$ is nested. Indeed, let $0 \leq r < t$. We already known that if $r \geq t_k$, then $I_k(t) \subseteq I_k(r)$, and if $t < t_k$, then $I_k(t)$ is empty. So suppose that $r < t_k \leq t$ and consider

[1] We stress that the notation Π_F and F_Π is *not* meant to suggest that one could be viewed as the inverse of the other.

the largest ancestor i of k with $t_i \le r$. It is immediate that $I_k(t) \subseteq I_i(r)$. We conclude that $(F_\Pi(t), t \ge 0)$ is an interval fragmentation. Finally, we have by construction that the length $\lambda_k(t)$ of $I_k(t)$ coincides with the asymptotic frequency of the block $B_k(t)$.

We now state the following counterpart of lemma 5

Lemma 6 *Let Π be a self-similar $\mathcal{P}$-valued fragmentation with index α. Then the following assertions hold:*

(i) *The interval fragmentation $F_\Pi = (F_\Pi(t), t \ge 0)$ constructed above is also self-similar with index α.*

(ii) *The $\mathcal{P}$-valued fragmentation Π_{F_Π} associated to F_Π (cf. Lemma 5) has the same distribution as Π.*

Proof: (i) The statement is intuitively obvious, however making the intuition rigorous is somewhat heavy. For every $t \ge 0$, the sequence of intervals $(I_k(t), k \in \mathbb{N})$ are constructed from the family of partitions $(\Pi(u), 0 \le u \le t)$. Choose an integer k such that $I_k(t) \neq \emptyset$ and recall that the k-th block $B_k(t)$ of the partition $\Pi(t)$ has asymptotic frequency $|I_k(t)|$. Recall also from the fragmentation property (iii) in Definition 4 that the partition $\Pi(t+r)$ restricted to $B_k(t)$ can be expressed in the form $\tilde{\Pi}(r|I_k(t)|^\alpha) \circ B_k(t)$, where $\tilde{\Pi}$ is independent of $(\Pi(u), 0 \le u \le t)$ and has the same distribution as Π. We shall now see that this entails that the interval fragmentation F_Π is self-similar with index α.

Write for simplicity $g = g_{I_k(t)}$ for the affine function that maps $]0,1[$ to $I_k(t)$, and let $k_1 = k < k_2 < \ldots$ be the ordered sequence of the elements of the block $B_k(t)$. We claim that the family $(I_{k_i}(t+r), i \in \mathbb{N})$ of intervals that result at time $t+r$ from $I_k(t)$ can be expressed in the form $\left(g(\tilde{I}_i), i \in \mathbb{N}\right)$, where the family $\left(\tilde{I}_i, i \in \mathbb{N}\right)$ is independent of $(\Pi(u), 0 \le u \le t)$ and has the same law as $(I_i(r|I_k(t)|^\alpha), i \in \mathbb{N})$. More precisely, denote by $\tilde{I}_i(u) =]\tilde{x}_i, \tilde{x}_i + \tilde{\lambda}_i(u)[$ for $i = 1, \ldots$, the family of intervals obtained from the $\mathcal{P}$-valued fragmentation $\tilde{\Pi}$ at time u. By construction, the instant t_{k_i} at which emerges the k_i-th block in the fragmentation Π can be expressed as

$$t_{k_i} = t_k + |I_k(t)|^\alpha \tilde{t}_i \, ,$$

where $\tilde{t}_i$ is the instant at which emerges the i-th block in the fragmentation $\tilde{\Pi}$. Also, the asymptotic frequency of $B_{k_i}(t+u)$ is clearly given by

$$\lambda_{k_i}(t+u) = \lambda_k(t) \tilde{\lambda}_i \left(u|I_k(t)|^\alpha\right) \, ,$$

It follows readily that $x_{k_i} = g(\tilde{x}_i)$ and hence

$$I_{k_i}(t+r) = g\left(I_i\left(r|I_k(t)|^\alpha\right)\right) \, .$$

This establishes our claim; more generally, a variation of this argument that now fully exploits the fragmentation property of the $\mathcal{P}$-valued process Π shows that disjoint intervals in the interval-fragmentation F_Π fall apart independently. Putting the pieces together, this completes the proof of (i).

(ii) For simplicity, write Π' for Π_{F_Π}. We know from Lemma 5 that Π' a self-similar $\mathcal{P}$-valued fragmentation with index α, which has the same ranked fragmentation as Π. According to

Kingman [13], two exchangeable partitions with the same ranked asymptotic frequencies have the same distribution, so the one-dimensional distributions of Π and Π' are the same. Because Π and Π' both are self-similar, their semigroups are the same, and we conclude that they have the same law. □

3.3 Characteristics of self-similar fragmentations

We are now able to tackle the problem that motivated this work, that is the characterization of self-similar $\mathcal{P}$-valued fragmentations. In this direction, we start by recalling the results obtained in [6] in the homogeneous case $\alpha = 0$.

First, recall that $\mathcal{S}^{\downarrow}$ denotes the natural state-space for ranked fragmentations, i.e. the space of decreasing numerical sequences $s = (s_1, \ldots)$ with $\sum_{i=1}^{\infty} s_i \leq 1$. Following Kingman [13], we can associate to each $s \in \mathcal{S}^{\downarrow}$ a unique exchangeable probability measure μ_s on $\mathcal{P}$ such that the ranked sequence of the asymptotic frequencies of the blocks of the generic partition is $\lambda^{\downarrow} = s$, μ_s-a.s. Finally, call a measure ν on the space $\mathcal{S}^{\downarrow}$ a Lévy measure if ν has no atom at $(1, 0, \ldots)$ and verifies the integral condition

$$\int_{\mathcal{S}^{\downarrow}} (1 - s_1)\nu(ds) \; < \; \infty \,, \tag{5}$$

where $s = (s_1, s_2, \ldots)$ denotes a generic sequence in $\mathcal{S}^{\downarrow}$. The mixture

$$\mu_\nu \; = \; \int_{\mathcal{S}^{\downarrow}} \mu_s \nu(ds)$$

is a sigma-finite measure on $\mathcal{P}$, called the dislocation measure corresponding to the Lévy measure ν. Next, for every integer k, denote by δ_k the measure on $\mathcal{P}$ given by the Dirac point mass at the partition that has only two non-void blocks, $\{k\}$ and $\mathbb{N}\backslash\{k\}$. For every $c \geq 0$, call

$$\mu_c \; = \; c \sum_{k=1}^{\infty} \delta_k$$

the erosion measure with rate c.

Given an erosion measure μ_c and a dislocation measure μ_ν, one can construct a homogeneous $\mathcal{P}$-valued fragmentation as follows. First, one considers $((\Delta_t, k_t), t \geq 0)$, a Poisson point process with values in $\mathcal{P} \times \mathbb{N}$ with characteristic measure $M := (\mu_c + \mu_\nu) \otimes \#$, where $\#$ stands for the counting measure on $\mathbb{N}$. This means that for every measurable set $A \subseteq \mathcal{P} \times \mathbb{N}$ with $M(A) < \infty$, the counting process

$$N^A(t) \; = \; \mathrm{Card}\left(u \in [0, t] : (\Delta_u, k_u) \in A\right) \,, \qquad t \geq 0$$

is a Poisson process with intensity $M(A)$, and to disjoint sets correspond independent counting processes. One can then construct a unique $\mathcal{P}$-valued process $\Pi_{c,\nu} = (\Pi(t), t \geq 0)$ started from the trivial partition and with càdlàg sample paths such that $\Pi_{c,\nu}$ only jumps at times t when a point (Δ_t, k_t) occurs in the Poisson point process, and in that case, $\Pi(t)$ is the partition obtained from $\Pi(t-)$ as follows. In the notation (4), consider the partition $\Delta_t \circ B_{k_t}(t-)$ of the

k_t-th block [2] of $\Pi(t-)$ induced by Δ_t. The blocks of the partition $\Pi(t)$ are formed by the blocks of $\Delta_t \circ B_{k_t}(t-)$ and the blocks $B_i(t-)$ of $\Pi(t-)$ for $i \neq k_t$. Then $\Pi_{c,\nu}$ is a homogeneous $\mathcal{P}$-valued fragmentation. Conversely, any homogeneous $\mathcal{P}$-valued fragmentation Π has the same law as $\Pi_{c,\nu}$ for some unique $c \geq 0$ and Lévy measure ν, see [6] for details.

It might be useful to further explain this construction. A point (Δ_t, k_t) in the Poisson point process affects the fragmentation if and only if the k_t-th block of $\Pi(t-)$ is neither empty nor reduced to a singleton, which we shall assume in the sequel. Points in the Poisson point process can be of two types. First, the partition Δ_t may have trivial asymptotic frequencies, which occurs if and only if Δ_t has exactly two non-void blocks, say $\{j\}$ and $\mathbb{N}\backslash\{j\}$. The effect of the occurrence of such a point is that at time t, the k_t-block of $\Pi(t-)$ splits into two, more precisely its j-th element becomes a singleton (and the other blocks are unchanged). This alone does not affect the ranked fragmentation, in the sense that the asymptotic frequencies of $\Pi(t-)$ and $\Pi(t)$ are the same; however the accumulation of such points (note that μ_c has an infinite total mass when $c > 0$) in the Poisson point process induces a continuous erosion for the blocks of Π. Second, Δ_t may have non-trivial asymptotic frequencies, say $s \in \mathcal{S}^{\downarrow}\backslash\{(1, 0 \ldots)\}$. When such point (Δ_t, k_t) occurs, the k_t-block of $\Pi(t-)$ is dislocated into smaller blocks, more precisely the ranked sequence of the asymptotic frequencies of these blocks is $\lambda_{k_t}(t-)s$, where $\lambda_{k_t}(t-)$ is the asymptotic frequency of the k_t-th block of $\Pi(t-)$.

Recall from Theorem 2 that one can change the index in a self-similar interval fragmentation by a suitable time-substitution. It is therefore natural to look for a similar result for $\mathcal{P}$-valued self-similar fragmentations, in order to reduce their construction to the construction described above in the homogeneous case. In this direction, for every $i \in \mathbb{N}$ and $r \geq 0$, denote by $\ell_i(r)$ the asymptotic frequency of block of $\Pi(r)$ that contains i (so that $\ell_i(r) = \lambda_j(r)$ where j is the least element of the block that contains i at time r). Then introduce for an arbitrary $\beta \in \mathbb{R}$

$$T_i^{(\beta)}(t) \;=\; \inf\left\{u \geq 0 : \int_0^u \ell_i(r)^{-\beta}\, dr > t\right\}, \qquad t \geq 0\,,$$

and consider the random partition $\Pi^{(\beta)}(t)$ of $\mathbb{N}$ such that $i, j \in \mathbb{N}$ are in the same block of $\Pi^{(\beta)}(t)$ if and only if there are in the same block of $\Pi\left(T_i^{(\beta)}(t)\right)$ (or equivalently in the same block of $\Pi\left(T_j^{(\beta)}(t)\right)$). We are now able to state the main result of this work.

Theorem 3 (i) *If Π is a self-similar $\mathcal{P}$-valued fragmentation with index α, then the process $\Pi^{(\beta)} = \left(\Pi^{(\beta)}(t), t \geq 0\right)$ is a self-similar $\mathcal{P}$-valued fragmentation with index $\alpha + \beta$. Moreover Π can be recovered from $\Pi^{(\beta)}$, more precisely $\Pi = (\Pi^{(\beta)})^{(-\beta)}$ in the obvious notation.*

(ii) *As a consequence, the law of a self-similar $\mathcal{P}$-valued fragmentation is determined by its index $\alpha \in \mathbb{R}$, and by the erosion coefficient $c \geq 0$ and the Lévy measure ν on $\mathcal{S}^{\downarrow}$ of the homogeneous $\mathcal{P}$-valued fragmentation $\Pi^{(-\alpha)}$. We call (α, c, ν) the characteristics of Π.*

Proof: (i) Denote by $F = F_\Pi$ the interval fragmentation associated with Π and $\tilde{\Pi} = \Pi_F$ the $\mathcal{P}$-valued fragmentation associated to F, so that $\tilde{\Pi}$ and Π have the same law by Lemma 6(ii). Next, consider the interval fragmentation $F^{(\beta)}$ constructed from F as in Theorem 2. A

[2] In [6], we used a different convention to enumerate the blocks of a partition; however it is easy to check that these two conventions yield two homogeneous fragmentations with the same distribution.

(short) moment of reflection shows that the $\mathcal{P}$-valued fragmentation $\Pi_{F^{(\beta)}}$ associated to $F^{(\beta)}$ coincides with $\tilde{\Pi}^{(\beta)}$ in the obvious notation, and thus has the same distribution as $\Pi^{(\beta)}$. We know from Lemma 6(i) that F is self-similar with index α, we deduce from Theorem 2 that $F^{(\beta)}$ is self-similar with index $\alpha+\beta$, and conclude by Lemma 5 that $\Pi^{(\beta)}$ is self-similar with index $\alpha+\beta$. Finally the identity $\Pi = (\Pi^{(\beta)})^{(-\beta)}$ is immediate.

(ii) follows from (i) and the characterization of homogeneous fragmentations recalled at the beginning of this section. □

For instance, recall from the Introduction the example obtained by cutting the interval $]0,1[$ at i.i.d. points picked according to the uniform distribution, that arrive at the jump times of a Poisson process, say with parameter 1. One can check that this fragmentation is self-similar with index $\alpha = 1$ and its erosion rate is $c = 0$. Moreover it is *binary*, in the sense with probability one, when a fragment with mass m splits, it gives rises to exactly two fragments with masses say m_1 and m_2 and such that $m_1 + m_2 = m$. It follows that the Lévy measure ν is carried by the subset of $\mathcal{S}^{\downarrow}$ consisting of decreasing sequences $(s_1, s_2, \ldots)$ such that $s_1 + s_2 = 1$ and $s_2 > 0$, and therefore is completely by the obvious identity $\nu(s_1 \in dx) = 2dx$ for $x \in [1/2, 1[$.

We conclude this section by noting that the following construction of a self-similar $\mathcal{P}$-valued fragmentation Π with characteristics (α, c, ν) is implicit in Theorem 3: one first constructs a homogeneous $\mathcal{P}$-valued fragmentation $\tilde{\Pi}$ with erosion rate c and Lévy measure ν as in [6], and then one takes $\Pi = \tilde{\Pi}^{(\alpha)}$. In particular, this yields an interesting probabilistic interpretation for the Lévy measure ν in terms of the evolution of the first block $B_1(\cdot)$. More precisely, suppose for simplicity that the erosion coefficient is $c = 0$, and consider the point process $\Sigma = (\Sigma_t, t \geq 0)$ with values in $\mathcal{S}^{\downarrow}\backslash\{(1,0\ldots)\}$ defined as follows. If the asymptotic frequency $\lambda_1(\cdot)$ of the first block $B_1(\cdot)$ is continuous at time t, then $\Sigma_t = (1,0\ldots)$. Otherwise, the ranked sequence of the asymptotic frequencies of the blocks resulting at time t from the dislocation of the block $B_1(t-)$ can be expressed in the form $\lambda_1(t-)s$ for some $s \in \mathcal{S}^{\downarrow}\backslash\{(1,0\ldots)\}$, and we set $\Sigma_t = s$. We claim that the intensity of the point process Σ (see Jacod [12]) is given by

$$\mathbf{1}_{\{\lambda_1(t-)>0\}}\lambda_1(t-)^{\alpha}\nu(ds)dt\,, \qquad s \in \mathcal{S}^{\downarrow}\backslash\{(1,0\ldots)\} \text{ and } t \geq 0\,. \tag{6}$$

To see this, consider first the homogeneous case $\alpha = 0$, and recall the construction of the fragmentation from a Poisson point process $((\Delta_t, k_t), t \geq 0)$. Then introduce the $\mathcal{S}^{\downarrow}$-valued Poisson point process $D = (D_t, t \geq 0)$ where the points D_t occur at instants t when $k_t = 1$ and are then given by the ranked asymptotic frequencies of the blocks of the partition Δ_t. On the one hand, by construction, the characteristic measure of Φ coincides with the Lévy measure ν. On the other hand, a moment of reflection shows that $\Sigma_t = D_t$ provided that $\lambda_1(t-) > 0$. This establishes (6) in the homogenenous case. The self-similar case $\alpha \neq 0$ now follows from Theorem 3.

4 Mass of a tagged fragment

In this section, we consider a self-similar fragmentation with characteristics (α, c, ν), and at the initial time, we tag a point picked at random according to the mass distribution. Our purpose is to describe the evolution as time passes of mass $\lambda(\cdot)$ of the tagged fragment, i.e. that contains the tagged point. Equivalently, we may identify $\lambda(\cdot) = \lambda_1(\cdot)$ as the process of the asymptotic frequencies of the first block $B_1(\cdot)$ in a $\mathcal{P}$-valued self-similar fragmentation. In the case of an interval fragmentation, this simply means that we introduce a random variable U uniformly distributed on $]0,1[$ which is independent of the fragmentation process, and aim at studying the process

$$\lambda(t) := |I_U(t)|, \qquad t \geq 0,$$

where $|I_x(t)|$ denotes the length of the interval component of $F(t)$ that contains x.

On the one hand, it follows from Theorem 2 that if we define

$$\lambda^{(-\alpha)}(t) := \lambda\left(T^{(-\alpha)}(t)\right), \qquad t \geq 0, \tag{7}$$

where

$$T^{(-\alpha)}(t) = \inf\left\{u \geq 0 : \int_0^u \lambda(r)^\alpha dr > t\right\}, \qquad t \geq 0,$$

then the process $\lambda^{(-\alpha)} = \left(\lambda^{(-\alpha)}(t), t \geq 0\right)$ can be viewed as the process of the mass of the tagged fragment in a homogeneous fragmentation with characteristics $(0, c, \nu)$.

On the other hand, we recall from Section 5 in [6] that in the homogeneous case, if we set

$$\xi_t = -\log \lambda^{(-\alpha)}(t), \qquad t \geq 0,$$

then the process $\xi = (\xi_t, t \geq 0)$ is a subordinator, that is an increasing Lévy process, and its law can be specified in terms of the erosion rate c and the Lévy measure ν. More precisely, its drift coefficient coincides with the erosion coefficient c, its killing rate is

$$\mathtt{k} = c + \int_{\mathcal{S}^\downarrow} \left(1 - \sum_{j=1}^{\infty} s_j\right) \nu(ds),$$

and its Lévy measure

$$L(dx) = \mathrm{e}^{-x} \sum_{j=1}^{\infty} \nu(-\log s_j \in dx), \qquad x \in]0, \infty[. \tag{8}$$

Equivalently, the Laplace exponent Φ of ξ, which is determined by the identity

$$\mathbb{E}\left((\exp(-q\xi_t)\right) = \exp(-t\Phi(q)), \qquad q \geq 0,$$

is given by the Lévy-Khintchine formula

$$\Phi(q) = c(q+1) + \int_{\mathcal{S}^\downarrow} \left(1 - \sum_{n=1}^{\infty} s_n^{q+1}\right) \nu(ds). \tag{9}$$

Putting the pieces together, we obtain at the following description of the process of the mass of a tagged fragment.

Corollary 2 *Let Π be a self-similar fragmentation with characteristics (α, c, ν), and let $\xi = (\xi_t, t \geq 0)$ be a subordinator with Laplace exponent Φ given by (9). Introduce the time-change*

$$\rho(t) = \inf\left\{u : \int_0^u \exp(\alpha \xi_r) dr > t\right\}, \qquad t \geq 0,$$

and set $Z_t = \exp(-\xi_{\rho(t)})$ (with the convention that $Z_t = 0$ if $\rho(t) = \infty$). Then the processes $(Z_t, t \geq 0)$ and $(\lambda(t), t \geq 0)$ have the same law.

The representation of Corollary 2 can be viewed as a special case of the construction by Lamperti [14] of so-called semi-stable Markov processes (more precisely, Lamperti has considered the same transformation in the more general case where ξ is a Lévy process, not necessarily a subordinator).

It is interesting to point out that the first instant when the mass of the marked fragment vanished (which can be thought as the time when this fragment is reduced to dust),

$$\zeta \;=\; \inf\{t \geq 0 : \lambda(t) = 0\}\ ,$$

has the same distribution as the so-called exponential functional $\int_0^\infty \exp(\alpha\xi_r)dr$, which has been studied by Carmona et al. [9]. In particular Proposition 3.3 there shows that for $\alpha < 0$, the integral moments of ζ determine its distribution and are given in terms of the Laplace exponent Φ by the formula

$$\mathbb{E}\left(\zeta^k\right) \;=\; \frac{k!}{\Phi(-\alpha)\cdots\Phi(-k\alpha)}\,, \qquad k \in \mathbb{N}\,. \tag{10}$$

To conclude this work, let us discuss two related examples. First, let us consider the fragmentation introduced by Aldous and Pitman [3] in the study of the standard additive coalescent. This is a self-similar fragmentation with index 1/2, and it has been proved in Theorem 6 of [3] that the mass $\lambda(t)$ of the tagged fragment at time t fulfills the following identity in distribution:

$$(\lambda(t), t \geq 0) \;\stackrel{d}{=}\; (1/(1+\sigma(t)), t \geq 0)\ ,$$

where $\sigma(\cdot) = \inf\{u \geq 0 : W_u > \cdot\}$ is the first passage process of a standard Brownian motion $(W_u, u \geq 0)$. Combining this with Corollary 2, we obtain that the subordinator ξ can be taken in the form

$$\xi_t \;=\; \log\left(1 + \sigma(\gamma_t)\right)\ ,$$

with

$$\gamma_t \;=\; \inf\left\{u \geq 0 : \int_0^u \frac{dr}{\sqrt{1+\sigma(r)}} > t\right\}\ .$$

Using the well-known fact that $\sigma(\cdot)$ is a stable subordinator with index 1/2, and more precisely with no drift, no killing, and Lévy measure $(2\pi x^3)^{-1/2}dx$ on $]0,\infty[$, it is easy to deduce that the subordinator ξ has no drift, no killing rate and Lévy measure

$$L_{\mathrm{AP}}(dx) \;=\; \frac{\mathrm{e}^x}{\sqrt{2\pi(\mathrm{e}^x - 1)^3}}\,dx\,, \qquad x > 0\,. \tag{11}$$

Equivalently, the Laplace exponent Φ_{AP} of ξ is given by

$$\begin{aligned}
\Phi_{\mathrm{AP}}(q) \;&=\; \int_0^\infty \left(1 - \mathrm{e}^{-qx}\right) \frac{\mathrm{e}^x}{\sqrt{2\pi(\mathrm{e}^x - 1)^3}}\,dx \\
&=\; q\sqrt{\frac{2}{\pi}} \int_0^\infty \mathrm{e}^{-qx}(\mathrm{e}^x - 1)^{-1/2}dx \qquad \text{(integration by parts)} \\
&=\; q\sqrt{\frac{2}{\pi}} \int_0^1 t^{q-1/2}(1-t)^{-1/2}dt \qquad (t = \mathrm{e}^{-x})\,,
\end{aligned}$$

so finally

$$\Phi_{\rm AP}(q) \;=\; q\sqrt{\frac{2}{\pi}}\,{\rm B}(q+1/2,1/2)\,. \tag{12}$$

Comparing (11) with (8) readily yields the following formula for the distribution of the first term s_1 of the generic sequence $s = (s_1, \ldots)$ under the Lévy measure $\nu_{\rm AP}$ of the Aldous-Pitman fragmentation:

$$\nu_{\rm AP}(s_1 \in dx) \;=\; \left(2\pi x^3(1-x)^3\right)^{-1/2}\,dx\,, \qquad x \in [1/2, 1[\tag{13}$$

(note that all the other terms $s_2, s_3, \ldots$ must be less than 1/2). Identity (13) is essentially a variation of formula (39) in section 4.1 of [3]. On the other hand, it is seen from the construction of the Aldous-Pitman fragmentation based on the continuum random tree (cf. [3]) that this fragmentation is binary, i.e. the Lévy measure $\nu_{\rm AP}$ is carried by the subset of sequences $(s_1, s_2, \ldots)$ with $s_1 > s_2 > 0$, $s_1 + s_2 = 1$, $s_3 = s_4 = \ldots = 0$. In particular $\nu_{\rm AP}(s_1 < 1/2) = 0$ and (13) completely determines the Lévy measure $\nu_{\rm AP}$. On the other hand, we already know that the index of self-similarity is $\alpha = 1/2$, and it is clear that the erosion coefficient is $c = 0$ (because the drift coefficient of ξ is zero), so we have specified the characteristics of the Aldous-Pitman fragmentation.

Our second example is based on the Brownian excursion with unit duration, $e = (e(r), 0 \leq r \leq 1)$, and is a close relative to the alternative construction of the Aldous-Pitman fragmentation in [5]. Specifically, let us consider the interval fragmentation

$$F(t) \;=\; \{r \in]0,1[: e(r) > t\}\;, \qquad t \geq 0.$$

That $F = (F(t), t \geq 0)$ is a nested family of open sets is trivial, and it follows from standard arguments of excursion theory (for details, see [5]) that F is self-similar with index $\alpha = -1/2$. In this framework, we see that the instant ζ when the tagged fragment vanishes is simply $\zeta = e(U)$, where U is the tagged point. Since U is uniformly distributed on $[0,1]$ and independent of the excursion, it is well-known that $2e(U)$ follows the Rayleigh distribution, i.e.

$$\mathbb{P}\,(2\zeta \in dr) \;=\; \mathbb{P}\,(2e(U) \in dr) \;=\; r\exp(-r^2/2)dr\,, \qquad r \geq 0\,,$$

and the integral moments of ζ are thus given by

$$\mathbb{E}\left(\zeta^k\right) \;=\; 2^{-k/2}\Gamma(1+k/2)\,, \qquad k \in \mathbb{N}\,.$$

Using the identity (10), we deduce that the Laplace exponent Φ_e of the subordinator ξ (cf. Corollary 2) is given by

$$\Phi_e(k) \;=\; 2^{3/2}k\frac{\Gamma(k+1/2)}{\Gamma(k+1)} \;=\; 2k\sqrt{\frac{2}{\pi}}\frac{\Gamma(k+1/2)\Gamma(1/2)}{\Gamma(k+1)} \;=\; 2k\sqrt{\frac{2}{\pi}}{\rm B}(k+1/2,1/2)\,.$$

Comparing with the formula (12), we arrive at the striking identity

$$\Phi_e \;=\; 2\Phi_{\rm AP}\,.$$

This enables us to determine the characteristics of the present fragmentation. More precisely, as Φ_e has zero drift, the erosion coefficient is zero, and we have already observed that the index is $\alpha = -1/2$. On the other hand, it follows from the fact that the values of the local minima of the Brownian path are all distinct a.s. that the present fragmentation is binary, and hence its Lévy measure ν_e is again determined by Φ_e. More precisely, using the identity (13), we see that

$$\nu_e(s_1 \in dx) \;=\; 2\left(2\pi x^3(1-x)^3\right)^{-1/2} dx\,, \qquad x \in [1/2, 1[\,,$$

and this completely determines ν_e.

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