Universités de Paris 6 & Paris 7 - CNRS (UMR 7599)

PRÉPUBLICATIONS DU LABORATOIRE

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MARS 2003

Prépublication n^o 808

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Additive martingales and probability tilting for homogeneous fragmentations

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Summary. Homogeneous fragmentations describe the evolution of a mass that breaks down into pieces as time passes. They can be thought of as continuous time analogs of branching random walks. Using Kingman's representation of exchangeable partitions of ℕ, we adapt to fragmentations the method of probability tilting of Lyons, Pemantle and Peres. Some applications to the asymptotic behavior of the fragmentation are derived.

Key words. Fragmentation, branching random walk, probability tilting, convergence of martingales.

A.M.S. Classification. 60 J 25, 60 G 09.

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1 Introduction

Homogeneous fragmentations form a family of random processes in continuous times which have been introduced in [3]. Roughly, these are particle systems meant to serve as models for a mass that breaks down randomly into pieces as time passes. More precisely, each particle is identified with its mass (i.e. it is specified by a positive real number), and the fragmentation property requires that different particles have independent evolutions. The homogeneity property means that the process started from a single particle with mass $x > 0$ has the same distribution as x times the process started from a single particle with unit mass.

This verbal description has obvious similarities with that of branching random walks. More precisely, let us write $Z^{(t)}$ for the random point measure which assigns a Dirac point mass at $\log x$ for every x varying over the set of particles at time t. Taking logarithms transforms the fragmentation and homogeneity properties into the branching property for random point measures. More precisely, for every $t, t' \geq 0$, $Z^{(t+t')}$ is obtained from $Z^{(t)}$ by replacing each atom $z = \log x$ of $Z^{(t)}$ by a family $\{z + y, y \in \mathcal{Y}\}$, where $\mathcal{Y}$ is distributed as the family of the atoms of $Z^{(t')}$ for $Z^{(0)} = \delta_0$, and to distinct atoms z of $Z^{(t)}$ correspond independent copies of $\mathcal{Y}$.

The literature on branching random walks is broadly concerned with discrete time processes, and homogeneous fragmentations can thus be viewed as their analogs for continuous times. This connection reminds us of course of superprocesses, which are continuous-time versions of branching Markov processes. Nonetheless, we stress that homogeneous fragmentations cannot be identified as a special class of superprocesses, because the branching mechanisms which govern the latter are not general enough to encompass the former. Informally, the reason is that in a homogeneous fragmentation, sudden dislocations may occur on a dense set of times and each may produce an infinite number of fragments which never die.

Homogeneous fragmentations may be seen as extensions of branching random walks in continuous time. These processes have been considered by Uchiyama [21], Biggins [8], Kyprianou [12]. Their main feature is that each particle has a lifetime exponentially distributed and at the instant of its death, scatters a random number of children-particles in space relative to its death point according to the point process.

The close connection between homogeneous fragmentations and branching random walks could suggest that one should investigate fragmentations using discrete time approximations by branching random walks. However, this approach would yield very difficult technical problems (see the forthcoming Section 2.5 for some details). The fundamental idea for circumventing these difficulties is due to Kingman [11], who pointed out that partitions of an object, say with a unit mass, can be fruitfully encoded by partitions of $\mathbb{N}$. In order to explain the coding, we introduce a sequence of i.i.d. random points $U_1, \ldots$ which are picked according to the mass distribution of the object. One then considers the random partition Γ of the set of indices $\mathbb{N}$, such that two indices, say i and j, belong to the same block of Γ if and only if the points U_i and U_j belong to the same fragment of the object. By the law of large numbers, we see that the masses of the fragments can be recovered as the asymptotic frequencies of the blocks of Γ. Roughly, the fundamental point in Kingman's coding is that it enables to discretize the state-space instead of time-space; moreover this discretization is exact and not merely an approximation. We refer to Pitman [19] for an important application of these ideas to a coalescent setting.

The purpose of this work is to present further applications of Kingman's idea to homogeneous fragmentations. Even though one cannot directly shift results from the theory of branching random walks to the homogeneous fragmentation setting, this theory provides highly valuable insights on potential results and methodology. Of course, the difficult task is to adapt arguments built for discrete time processes to the continuous time setting. For instance a most useful notion such as the first branching time has no analog for fragmentations since in general dislocations occur instantaneously. Specifically, we will be interested here in the method of probability tilting introduced by Lyons, Pemantle and Peres (see e.g. [16]) to investigate the convergence of so-called additive martingales for branching random walks. We shall show how this method can be adapted to homogeneous fragmentations, which yields some interesting limit theorems.

The rest of this paper is organized as follows. In Section 2, we present the necessary background on partitions of $\mathbb{N}$, the construction of fragmentations, and a few key results for homogeneous fragmentations proved in [3]. Section 3 is devoted to probability tilting techniques and their applications to additive martingales. In particular, we shall establish a version of Biggins' theorem for homogeneous fragmentations which extends that obtained in [4]. In Section 4, we use again tilted probabilities to derive information about the asymptotic behavior of the empirical measure associated to the fragmentation; these are related to results of Asmussen, Kaplan and Biggins in the branching random walk context (see [7] and references therein). Finally, Section 5 is devoted to the study of the convergence of the so-called derivative martingale.

2 Preliminaries

2.1 Partitions of $\mathbb{N}$, asymptotic frequencies and ranked partitions

A *partition* of $\mathbb{N} = \{1, \dots\}$ is a sequence $\pi = (\pi_1, \pi_2, \dots)$ of disjoint subsets, called *blocks*, such that $\bigcup \pi_i = \mathbb{N}$. The blocks π_i of a partition are enumerated in the increasing order of their least element, i.e. $\min \pi_i \leq \min \pi_j$ for $i \leq j$, with the convention that $\min \emptyset = \infty$. If π and π' are two partitions of $\mathbb{N}$, we say that π is finer than π' if every block of π is contained into some block of π'.

For every block $B \subseteq \mathbb{N}$, we denote by $\pi_{|B}$ the partition of B induced by π by an obvious restriction. For every integer k, the block $\{1, \dots, k\}$ is denoted by $[k]$. A partition π is entirely determined by the sequence of its restrictions $\left(\pi_{|[k]}, k \in \mathbb{N}\right)$, and conversely, if for every $k \in \mathbb{N}$, γ_k is a partition of $[k]$ such that the restriction of γ_{k+1} to $[k]$ coincides with γ_k (this will be referred to the compatibility property in the sequel), then there exists a unique partition $\pi \in \mathcal{P}$ such that $\pi_{|[k]} = \gamma_k$ for every $k \in \mathbb{N}$.

The space of partitions of $\mathbb{N}$ is denoted by $\mathcal{P}$ and endowed with the hyper-distance

$$\mathrm{dist}(\pi, \pi') := 1/\max\left\{k \in \mathbb{N} : \pi_{|[k]} = \pi'_{|[k]}\right\} ,$$

with the convention $\max \mathbb{N} := \infty$ and $1/\infty := 0$. This makes $\mathcal{P}$ compact. We also introduce the space of numerical sequences

$$\mathcal{S} := \left\{ s = (s_1, \dots) : s_1 \geq s_2 \geq \cdots \geq 0 \text{ and } \sum_1^\infty s_i \leq 1 \right\}$$

endowed with the uniform distance, which is also a compact set.

One says that a block $B \subseteq \mathbb{N}$ has an *asymptotic frequency* if the limit

$$|B| := \lim_{n \to \infty} n^{-1} \mathrm{Card}(B \cap [n])$$

exists. When every block of some partition $\pi \in \mathcal{P}$ has an asymptotic frequency, we write $|\pi| = (|\pi_1|, \dots)$, and then $|\pi|^{\downarrow} = (|\pi|_1^{\downarrow}, \dots)$ for the decreasing rearrangement[1] of the sequence $|\pi|$. In the case when some block of the partition π does not have an asymptotic frequency, we decide that $|\pi| = |\pi|^{\downarrow} = \partial$, where ∂ stands for some extra point added to $\mathcal{S}$. This defines a natural map $\pi \to |\pi|^{\downarrow}$ from $\mathcal{P}$ to $\mathcal{S} \cup \{\partial\}$ which is not continuous.

[1] Ranking the asymptotic frequencies of the blocks of π in the decreasing order is just a simple procedure to forget the labels of these blocks. In other words, we want to consider the family of the asymptotic frequencies without keeping the additional information provided by the way blocks are labelled.

2.2 Nested partitions and discrete point measures

We call *nested partitions* a collection $\Pi = (\Pi(t), t \geq 0)$ of partitions of $\mathbb{N}$ such that $\Pi(t)$ is finer than $\Pi(t')$ when $t' \leq t$. There is a simple procedure for the construction of a large family of nested partitions which we now describe and will use throughout the rest of this work.

We call *discrete point measure* on $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$ any measure ω which can be expressed in the form

$$\omega = \sum_{(t,\pi,k)\in\mathcal{D}}^{\infty} \delta_{(t,\pi,k)}$$

where $\mathcal{D}$ is a subset of $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$ such that for every real number $t' \geq 0$ and integer $n \geq 1$

$$\text{Card}\left\{(t,\pi,k) \in \mathcal{D} : t \leq t', \pi_{|[n]} \neq \text{trivial}(n), k \leq n\right\} < \infty\,,$$

and $\text{trivial}(n) = ([n], \emptyset, \emptyset, \ldots)$ stands for the partition of $[n]$ which has a single non empty block (roughly, $\text{trivial}(n)$ plays the role of a neutral element in the space of partitions of $[n]$).

Starting from an arbitrary discrete point measure ω on $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$, we may construct nested partitions $\Pi = (\Pi(t), t \geq 0)$ as follows: Fix $n \in \mathbb{N}$; the assumption that the point measure ω is discrete enables us to construct a step-path $(\Pi(t,n), t \geq 0)$ with values in the space of partitions of $[n]$, which only jumps at times t at which the fiber $\{t\} \times \mathcal{P} \times \mathbb{N}$ carries an atom of ω, say (t,π,k), such that $\pi_{|[n]} \neq \text{trivial}(n)$ and $k \leq n$. In that case, $\Pi(t,n)$ is the partition obtained by replacing the k-th block of $\Pi(t-,n)$, viz. $\Pi_k(t-,n)$, by the restriction $\pi_{|\Pi_k(t-,n)}$ of π to this block, and leaving the other blocks unchanged. Now it is immediate from this construction that for each time $t \geq 0$, the sequence $(\Pi(t,n), n \in \mathbb{N})$ is compatible, and hence there exists a unique partition $\Pi(t)$ such that $\Pi(t)_{|[n]} = \Pi(t,n)$ for each $n \in \mathbb{N}$.

We denote the space of discrete point measures on $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$ by Ω, and the sigma-field generated by the restriction to $[0,t] \times \mathcal{P} \times \mathbb{N}$ by $\mathcal{G}(t)$. So $(\mathcal{G}(t))_{t\geq 0}$ is a filtration, and the nested partitions $(\Pi(t), t \geq 0)$ are $(\mathcal{G}(t))_{t\geq 0}$-adapted. We shall also need to consider the sigma-field $\mathcal{F}(t)$ generated by the decreasing rearrangement $|\Pi(r)|^{\downarrow}$ of the sequence of the asymptotic frequencies of the blocks of $\Pi(r)$ for $r \leq t$, and $(\mathcal{F}(t))_{t\geq 0}$ is a sub-filtration of $(\mathcal{G}(t))_{t\geq 0}$.

2.3 Homogeneous fragmentations

We call *dislocation measure* a measure ν on $\mathcal{S}$ with $\nu(\{(1,0,\ldots)\}) = 0$, which fulfils the requirement

$$\int_{\mathcal{S}} (1 - s_1)\nu(ds) < \infty\,. \tag{1}$$

According to Theorem 2 in [3], there exists a unique measure μ on $\mathcal{P}$ which is exchangeable (i.e. invariant by the action of finite permutations on $\mathcal{P}$), and such that ν is the image of μ by the map $\pi \to |\pi|^{\downarrow}$. Note that we assumed for simplicity that μ has no erosion component, which essentially induces no loss of generality; see Berestycki [1]. In order to avoid uninteresting discussions about sudden extinction, we shall further assume that $\nu(s_1 = 0) = 0$, which means that a mass cannot be suddenly reduced to dust, and that $\nu \neq 0$.

An important fact which stems from exchangeability, is that the distribution of the asymptotic frequency of the first block $|\pi_1|$ under the measure μ is that of a size-biased picked term from the ranked sequence s under ν. In other words, there is the identity

$$\int_{\mathcal{P}} f(|\pi_1|)\mu(d\pi) \;=\; \int_{\mathcal{S}} \sum_{i=1}^{\infty} s_i f(s_i)\nu(ds)$$

where $f : [0,1] \to \mathbb{R}_+$ denotes a generic measurable function with $f(0) = 0$.

Let $\mathbb{P}$ be the probability measure on Ω corresponding to the law of a Poisson point measure with intensity $dt \otimes \mu \otimes \#$, where $\#$ denotes the counting measure on $\mathbb{N}$. The assumption (1) on the dislocation measure ν ensures that ω is a discrete point measure $\mathbb{P}$-a.s. The nested partitions $(\Pi(t), t \geq 0)$ constructed in the preceding section from ω are exchangeable under $\mathbb{P}$, i.e. their distribution is invariant under the action of finite permutations on $\mathbb{N}$. See Section 3 in [3].

Furthermore, $\Pi = (\Pi(t), t \geq 0)$ is a Markov process. More precisely the Markov property is essentially a variation of the branching property; it will be referred to as the fragmentation property in the sequel. It can be stated as follows. For every $t, t' \geq 0$, the conditional distribution of $\Pi(t+t')$ given $\mathcal{G}(t)$ is the same as that of the random partition of $\mathbb{N}$ induced by the restrictions $\Pi^{(1)}(t')_{|B_1}, \Pi^{(2)}(t')_{|B_2}, \ldots$, where $\Pi^{(1)}, \ldots$ are independent copies of Π and $(B_1, \ldots) = \Pi(t)$ is the sequence of blocks of $\Pi(t)$. In the terminology of [3], we say that $\Pi = (\Pi(t), t \geq 0)$ is a *homogeneous fragmentation* under $\mathbb{P}$.

It is known that $\mathbb{P}$-a.s., $\Pi(t)$ has asymptotic frequencies for all $t \geq 0$; cf. Theorem 3(i) in [3]. The process of ranked asymptotic frequencies $|\Pi|^{\downarrow}$ is a Markov process with values in $\mathcal{S}$, which we call the *ranked fragmentation*; cf. [1]. When the dislocation measure ν is finite, there is a simple description of its evolution viewed as a particle system : Each particle, say with mass x, splits with rate ν, independently of the other particles in the system. This means that the splitting of the particle occurs after an exponential time with parameter $\nu(\mathcal{S})$ and produces a random sequence xs where s is distributed according to the probability measure $\nu(\cdot)/\nu(\mathcal{S})$. We refer to Berestycki [1] for a similar description in case when ν is infinite.

2.4 An important subordinator

The process $|\Pi_1(\cdot)|$ of the asymptotic frequencies of the first block and its logarithm,

$$\xi(t) \;:=\; -\log|\Pi_1(t)|\,, \qquad t \geq 0$$

will have a special role in this study.

First, it is known (cf. Theorem 3(ii) in [4]) that under $\mathbb{P}$, $\xi = (\xi_t, t \geq 0)$ is a subordinator with Laplace exponent

$$\Phi(q) \;=\; \int_{\mathcal{S}} \left(1 - \sum_{i=1}^{\infty} s_i^{q+1}\right) \nu(ds)\,, \qquad q \geq \underline{p}\,,$$

where

$$\underline{p} \;:=\; \inf\left\{p \in \mathbb{R} : \int_{\mathcal{S}} \sum_{i=2}^{\infty} s_i^{p+1}\nu(ds) < \infty\right\}\,.$$

This means that $(\xi(t), t \geq 0)$ is a càdlàg process with independent and stationary increments (at least up to its lifetime), and the Laplace transform of its one-dimensional distribution is given by the identity

$$\mathbb{E}(\exp(-q\xi(t)) = \exp(-t\Phi(q)), \qquad q > \underline{p}.$$

In the case when ξ has a finite lifetime, the formula above has to be read with the convention $\mathrm{e}^{-\infty} = 0$; see Section 1.1 in [2] for details.

We also point out that the equation $\Phi(p) = (p+1)\Phi'(p)$ has a unique solution which will be denoted by $\bar{p} > 0$, i.e.

$$\Phi(\bar{p}) = (\bar{p}+1)\Phi'(\bar{p}).$$

This follows from an easy argument of convexity, cf. Lemma 1 in [4]. The quantities $\underline{p}$ and $\bar{p}$ will have a key role in this work.

Second, let us denote by $\mathcal{G}_1(t)$ the sigma-field generated by the restriction of the discrete point measure ω to the fiber $[0,t] \times \mathcal{P} \times \{1\}$. So $(\mathcal{G}_1(t))_{t\geq 0}$ is a sub-filtration of $(\mathcal{G}(t))_{t\geq 0}$, and the first block of $\Pi(t)$, $\Pi_1(t)$, and a fortiori its asymptotic frequency $\mathrm{e}^{-\xi_t}$, are $\mathcal{G}_1(t)$-measurable. Let $\mathcal{D}_1 \subseteq [0,\infty[$ be the random set of times $r \geq 0$ for which the discrete point measure has an atom on the fiber $\{r\} \times \mathcal{P} \times \{1\}$, and for every $r \in \mathcal{D}_1$, denote the second component of this atom by $\pi(r)$. The construction of the nested partitions from the discrete point measure yields the identity

$$\exp(-\xi_t) = |\Pi_1(t)| = \prod_{r \in \mathcal{D}_1 \cap [0,t]} |\pi_1(r)|, \tag{2}$$

for all $t \geq 0$, a.s. under $\mathbb{P}$; see e.g. the first remark at the end of Section 5 in [3]. More precisely, taking logarithm turns the identity (2) into the Lévy-Itô decomposition for subordinators.

Finally, the conditional distribution of $|\Pi_1(t)| = \mathrm{e}^{-\xi(t)}$ given $\mathcal{F}(t)$ (the sigma-field generated by the ranked asymptotic frequencies) is that of a size-biased sample from the ranked sequence $|\Pi(t)|^{\downarrow}$. In other words, we have

$$\mathbb{E}(f(\exp(-\xi(t))) = \mathbb{E}\left(\sum_{i=1}^{\infty} |\Pi_i(t)| f(|\Pi_i(t)|)\right) = \mathbb{E}\left(\sum_{j=1}^{\infty} |\Pi(t)|_j^{\downarrow} f(|\Pi(t)|_j^{\downarrow})\right)$$

where $f : [0,1] \to \mathbb{R}_+$ denotes a generic measurable function with $f(0) = 0$. More generally, exchangeability ensures that for every $t \geq 0$, the sequence $|\Pi(t)|$ of the asymptotic frequencies is a size-biased reordering of the ranked sequence $|\Pi(t)|^{\downarrow}$.

2.5 Connection with branching random walks

The fragmentation property entails that the empirical measure of the logarithms of the asymptotic frequencies of the blocks,

$$Z^{(t)} := \sum_{i=1}^{\infty} \delta_{\log |\Pi_i(t)|} = \sum_{j=1}^{\infty} \delta_{\log |\Pi(t)|_j^{\downarrow}}, \qquad t \geq 0$$

can be viewed as a continuous-time analog of a branching random walk. This observation suggests that we should be able to translate results from the literature on branching random walks to homogeneous fragmentations, at least when we restrict our attention to a discrete time skeleton. However, this is not straightforward as we shall now explain.

In this direction, let us identify some key quantities related to branching random walks in the fragmentation setting. First, the Laplace transform of the intensity of the point process $Z^{(1)}$ is given by

$$\begin{aligned} m(\theta) := \mathbb{E}\left(\int_{-\infty}^{\infty} \mathrm{e}^{\theta x} Z^{(1)}(dx)\right) &= \mathbb{E}\left(\sum_{i=1}^{\infty} |\Pi_i(1)|^{\theta}\right) \\ &= \mathbb{E}\left(\exp(-(\theta-1)\xi(1))\right) \\ &= \exp(-\Phi(\theta-1)), \end{aligned}$$

and this quantity is finite whenever $\theta > \underline{p}+1$. In particular, we have the identity

$$\theta m'(\theta)/m(\theta) + \log m(\theta) = \theta\Phi'(\theta-1) - \Phi(\theta-1).$$

Second, there is also the identification

$$W^{(n)}(\theta) := m(\theta)^{-n} \int_{-\infty}^{\infty} \mathrm{e}^{\theta x} Z^{(n)}(dx) = \exp(n\Phi(\theta-1)) \sum_{i=1}^{\infty} |\Pi_i(n)|^{\theta}.$$

Therefore, if for instance we were able to check directly that $\mathbb{E}\left(W^{(1)}(\theta)\log_+ W^{(1)}(\theta)\right) < \infty$, then we could apply Biggins' theorem [5] and derive information about the asymptotic behavior of $W^{(n)}(\theta)$. The problem is that this $L\log L$ condition is given in terms of the state of the fragmentation at time 1, and not in terms of its characteristic (i.e. the dislocation measure ν). In particular, this condition cannot be expressed in terms of the Laplace exponent Φ, because the function $L\log L$ is not linear.

3 Convergence of the additive martingales

In this section we investigate the convergence in $L^1(\mathbb{P})$ of some remarkable martingales which are naturally associated to the fragmentation. We shall first follow the lines of the method of Lyons [16] based on tilted probabilities. Then we shall reinforce the convergence, adapting the complex techniques of Biggins [8].

3.1 Additive martingales and tilted probability measures

There are two simple martingales connected to fragmentations for every parameter $p > \underline{p}$: First, a well-known fact for subordinators is that

$$\mathcal{E}(p,t) := \exp(-p\xi(t) + t\Phi(p)) = \mathrm{e}^{t\Phi(p)}|\Pi_1(t)|^p$$

is a positive $(\mathbb{P}, \mathcal{G}(t))$-martingale. Second, it follows readily from the fragmentation property that

$$M(p,t) := \exp(t\Phi(p)) \sum_{i=1}^{\infty} |\Pi_i(t)|^{p+1}$$

is also a $(\mathbb{P}, \mathcal{G}(t))$-martingale, called the *additive martingale*, which is adapted to the sub-filtration $\mathcal{F}(t)$. Observe that $M(p,t) = W^{(n)}(p+1)$ in the notation of Section 2.5, and note that $M(p,t)$ can be viewed as the projection of $\mathcal{E}(p,t)$ on the sub-filtration $\mathcal{F}(t)$.

Following the genuine method of Lyons, Pemantle and Peres (see e.g. [16]), we introduce the *tilted probability measure* $\mathbb{P}^{(p)}$ on the space of discrete point measures Ω endowed with the filtration $(\mathcal{G}(t))_{t\geq 0}$ by

$$d\mathbb{P}^{(p)}_{|\mathcal{G}(t)} \;=\; \mathcal{E}(p,t)\, d\mathbb{P}_{|\mathcal{G}(t)}\,. \tag{3}$$

Observe that projections on the sub-filtration $\mathcal{F}(t)$ give the identity

$$d\mathbb{P}^{(p)}_{|\mathcal{F}(t)} \;=\; M(p,t)\, d\mathbb{P}_{|\mathcal{F}(t)}\,. \tag{4}$$

We stress that the tilting only affects the distribution of the discrete point measure on the fiber $\mathbb{R}_+ \times \mathcal{P} \times \{1\}$. In particular, the restriction to $\mathbb{R}_+ \times \mathcal{P} \times \{2,3,\ldots\}$ is independent of the restriction to the fiber $\mathbb{R}_+ \times \mathcal{P} \times \{1\}$, i.e. of $\{(r, \pi(r)), r \in \mathcal{D}_1\}$ in the notation of Section 2.4, and its distribution is same under $\mathbb{P}$ and $\mathbb{P}^{(p)}$. The fiber $\mathbb{R}_+ \times \mathcal{P} \times \{1\}$ can thus be viewed as a universal analog in the fragmentation setting of the so-called spine in the branching random walk framework. By the exponential tilting and the formula (2), we see that under $\mathbb{P}^{(p)}$, the family $\{(r, \pi(r)), r \in \mathcal{D}_1\}$ is that of the atoms of a Poisson random measure on $\mathbb{R}_+ \times \mathcal{P}$ with intensity $dr \otimes \mu^{(p)}$, where

$$\mu^{(p)}(d\pi) \;=\; |\pi_1|^p \mu(d\pi)\,.$$

We also point out that the tilted probability can be interpreted as follows :

First for $p > 0$, starting from a discrete point measure ω, we let $\omega^{(p)}$ denote a thinning of ω such that each atom (t, π, k) of ω is deleted for $k = 1$ with probability $1 - |\pi_1|^p$, independently of the other atoms. Then $\mathbb{P}^{(p)}$ is the image of $\mathbb{P}$ by the thinning transformation $\omega \to \omega^{(p)}$.

Second, for $\underline{p} < p < 0$, let ω' be a random Poisson point measure on the fiber $\mathbb{R}_+ \times \mathcal{P} \times \{1\}$ with intensity $dt \otimes (|\pi_1|^p - 1)\mu(d\pi)$, which is independent of ω. By superposition of independent Poisson measures, $\mathbb{P}^{(p)}$ can be identified as the law of $\omega + \omega'$ under $\mathbb{P}$. As a consequence, the original probability measure $\mathbb{P}$ can be recovered from $\mathbb{P}^{(p)}$ by thinning: $\mathbb{P}$ is the image of $\mathbb{P}^{(p)}$ by the map $\omega \to \omega^{(-p)}$.

Finally, we mention that by absolute continuity, the random partition $\Pi(t)$ obtained by evaluating the nested partitions at time t, possesses asymptotic frequencies a.s. under the tilted probability $\mathbb{P}^{(p)}$. In this direction, it is interesting to compare the evolution of the ranked-fragmentation $|\Pi(\cdot)|^{\downarrow}$ under $\mathbb{P}^{(p)}$ with the evolution under $\mathbb{P}$ which was described in section 2.3. For the sake of simplicity, we again suppose here that ν is finite. Because the first block $\Pi_1(\cdot)$ has a special role in the definition of the tilted probability $\mathbb{P}^{(p)}$, and cannot be recovered from the ranked sequence $|\Pi(\cdot)|^{\downarrow}$ alone, we need to introduce the following notion. Let us call "tagged" the unique particle (i.e. asymptotic frequency) at time t corresponding to $\Pi_1(t)$ and untagged the other ones. Under $\mathbb{P}^{(p)}$, the untagged particles follow the same evolution as under $\mathbb{P}$, i.e. they split according to ν, independently of the others, and only produce untagged particles. The tagged particle splits independently of the other particles, but with a different rate, namely

$$\nu^{(p)}(ds) \;:=\; \left(\sum_{i=1}^{\infty} s_i^{p+1}\right) \nu(ds)\,.$$

Indeed, $\nu^{(p)}$ is the image of the intensity measure $\mu^{(p)}$ by the map $\pi \to |\pi|^{\downarrow}$, and since under μ, $|\pi_1|$ can be viewed as a size-biased pick from the ranked sequence $|\pi|^{\downarrow}$, this yields the formula above. The "new" tagged particle is picked at random amongst the particles produced by the splitting of the "old" tagged particle as follows: Let x denote the mass of the old tagged particle and xs the ranked sequence of the masses of the particles produced after the splitting, where $s = (s_1, \ldots) \in \mathcal{S}$. Then the probability that the new tagged particle has mass xs_j equals $s_j^{p+1} / \sum_{i=1}^{\infty} s_i^{p+1}$.

3.2 Convergence of martingales via tilting

The following result extends Theorem 2 in [4], and can be viewed as an analog for fragmentations of Biggins' Theorem [5]. Recall that the quantities $\underline{p} \leq 0$ and $\bar{p} > 0$ have been defined at the beginning of Section 2.4.

Theorem 1 *The martingale $M(p,t)$ converges in $L^1(\mathbb{P})$ if $\underline{p} < p < \bar{p}$, and converges to 0 $\mathbb{P}$-a.s. if $p \geq \bar{p}$.*

Remark: It is easy to check that for $\underline{p} < p < \bar{p}$, the terminal value $M(p, \infty)$ of the uniformly integrable martingale $M(p,t)$, is strictly positive $\mathbb{P}$-a.s.; see e.g. [4]. Thus the probability measures $\mathbb{P}$ and $\mathbb{P}^{(p)}$ are equivalent on $\mathcal{F}(\infty)$.

Proof: We start with the easiest part. On the one hand, the function $p \to (p+1)\Phi'(p) - \Phi(p)$ has derivative $\Phi''(p) < 0$. It follows that

$$p > \overline{p} \iff (p+1)\Phi'(p) < \Phi(p) .$$

On the other hand, we have the lower bound

$$M(p,t) \geq e^{t\Phi(p)} |\Pi_1(t)|^{p+1} = \exp\{\Phi(p)t - (p+1)\xi_t\} .$$

It is well-known that under $\mathbb{P}^{(p)}$, ξ is a subordinator with Laplace exponent

$$\Phi^{(p)}(q) = \Phi(q+p) - \Phi(p) , \qquad q > \underline{p} - p .$$

In particular, under $\mathbb{P}^{(p)}$, the Lévy process $\Phi(p)t - (p+1)\xi_t$ has mean $\Phi(p) - (p+1)\Phi'(p)$, which is 0 if $p = \bar{p}$ and positive if $p > \bar{p}$. In both cases, we have

$$\limsup_{t\to\infty} (\Phi(p)t - (p+1)\xi_t) = \infty , \qquad \mathbb{P}^{(p)}\text{-a.s.}$$

This shows that $\limsup_{t\to\infty} M(p,t) = \infty$, $\mathbb{P}^{(p)}$-a.s. when $p \geq \bar{p}$, and hence the martingale $M(p,t)$ converges to 0, $\mathbb{P}$-a.s.

Assume henceforth that $\underline{p} < p < \overline{p}$; we have to check that $\liminf_{t\to\infty} M(p,t) < \infty$, $\mathbb{P}^{(p)}$-a.s. Observe first that now, the Lévy process $\Phi(p)t - (p+1)\xi_t$ drifts to $-\infty$ under $\mathbb{P}^{(p)}$, and hence we may focus on

$$M(p,t) - \exp\{\Phi(p)t\} |\Pi_1(t)|^{p+1} = \exp\{\Phi(p)t\} \sum_{i=2}^{\infty} |\Pi_i(t)|^{p+1} .$$

By construction of the fragmentation Π, each block $\Pi_i(t)$ for $i \geq 2$ got separated from 1 at some instant $r \in \mathcal{D}_1 \cap [0,t]$. More precisely, recall that at such an instant r, the block $\Pi_1(r-)$ splits into $\pi(r)_{|\Pi_1(r-)}$, and that the block after the split which contains 1 is $\Pi_1(r) = \pi_1(r) \cap \Pi_1(r-)$. Thus, there is then some index $j \geq 2$ such that $\Pi_i(t) \subseteq \pi_j(r) \cap \Pi_1(r-)$, where $\pi_j(r)$ stands for the j-th block of the partition $\pi(r)$. In other words, we may consider the partition of $\{2, \ldots\}$ whose blocks are of the type

$$B(r,j) \;=\; \{i \geq 2 : \Pi_i(t) \subseteq \pi_j(r) \cap \Pi_1(r-)\}\;,$$

and then $(\Pi_i(t) : i \in B(r,j))$ forms a partition of $\pi_j(r) \cap \Pi_1(r-)$ which we now analyze.

Standard properties of Poisson random measures and the very construction of Π entail that for every $r \in [0,t]$ and $j \geq 2$, conditionally on $r \in \mathcal{D}_1$, $\Pi_1(r-)$ and $\pi_j(r)$, the partition $(\Pi_i(t) : i \in B(r,j))$ can be given in the form $\tilde{\Pi}(t-r)_{|\pi_j(r)\cap\Pi_1(r-)}$ where $\tilde{\Pi}$ is a homogeneous fragmentation distributed as Π under $\mathbb{P}$ and is independent of the sigma-field $\mathcal{G}_1(t)$. Because $|\Pi_1(t-r)|$ is distributed as a size-biased pick from the ranked sequence $|\Pi(t-r)|^{\downarrow}$, there is the identity

$$\mathbb{E}\left(\sum_{i=1}^{\infty} |\Pi_i(t-r)|^{p+1}\right) \;=\; \mathbb{E}\left(|\Pi_1(t-r)|^p\right) \;=\; \exp(-(t-r)\Phi(p))\,,$$

this entails that

$$\begin{aligned}
&\mathbb{E}^{(p)}\left(\exp\{\Phi(p)t\}\sum_{i=2}^{\infty}|\Pi_i(t)|^{p+1} \mid \mathcal{G}_1(t)\right)\\
=\;& \sum_{r\in\mathcal{D}_1\cap[0,t]}\sum_{j=2}^{\infty}\exp\{\Phi(p)r\}\,|\pi_j(r)\cap\Pi_1(r-)|^{p+1}\\
=\;& \sum_{r\in\mathcal{D}_1\cap[0,t]}\exp\{\Phi(p)r-(p+1)\xi(r-)\}\left(\sum_{j=2}^{\infty}|\pi_j(r)|^{p+1}\right)\,,
\end{aligned}$$

where the last identity stems from the easy fact that $|\pi_j(r) \cap \Pi_1(r-)| = |\pi_j(r)||\Pi_1(r-)|$ for all $r \in \mathcal{D}_1$, $\mathbb{P}$-a.s. and hence also $\mathbb{P}^{(p)}$-a.s.

As pointed out by Lyons [16], by the conditional Fatou's theorem, all that we need is to check that the limit of above quantity as $t \to \infty$ is finite $\mathbb{P}^{(p)}$-a.s. This is established in the next lemma, hence completing the proof of Theorem 1. ■

Lemma 2 *For every $p \in]\underline{p}, \bar{p}[$, it holds that*

$$\sum_{t\in\mathcal{D}_1}\exp\{\Phi(p)t-(p+1)\xi(t-)\}\left(\sum_{j=2}^{\infty}|\pi_j(t)|^{p+1}\right) \;<\; \infty \qquad \mathbb{P}^{(p)}\text{-}a.s.$$

Proof: We start by recalling that under $\mathbb{P}^{(p)}$, $t\Phi(p) - (p+1)\xi(t)$ is a Lévy process with a strictly negative mean, so by the law of large numbers, there exists some constant $\varepsilon > 0$ such that $\mathbb{P}^{(p)}$-a.s.

$$\exp(t\Phi(p)-(p+1)\xi(t)) \;=\; o(\mathrm{e}^{-2\varepsilon t})\,, \qquad t\to\infty\,. \tag{5}$$

We denote by $\Sigma(t)=\sum_{j=2}^{\infty}|\pi_j(t)|^{p+1}$ and first claim that the set of times t when $\Sigma(t)\geq \mathrm{e}^{\varepsilon t}$ is finite $\mathbb{P}^{(p)}$-a.s. Indeed, recall that under $\mathbb{P}^{(p)}$, $\{(t,\pi(t)), t\in\mathcal{D}_1\}$ is the family of the atoms of a Poisson random measure with intensity $dt\otimes|\pi_1|^p\mu(d\pi)$. So we simply need to check the finiteness of

$$\begin{aligned}\int_0^{\infty}dt\int_{\mathcal{P}}\mu(d\pi)|\pi_1|^p\mathbf{1}_{\{\sum_{j=1}^{\infty}|\pi_j|^{p+1}\geq \mathrm{e}^{\varepsilon t}\}} &= \varepsilon^{-1}\int_{\mathcal{P}}\mu(d\pi)|\pi_1|^p\log^+\left(\sum_{j=1}^{\infty}|\pi_j|^{p+1}\right)\\ &= \varepsilon^{-1}\int_{\mathcal{S}}\nu(ds)\left(\sum_{j=1}^{\infty}s_j^{p+1}\right)\log^+\left(\sum_{j=1}^{\infty}s_j^{p+1}\right),\end{aligned}$$

where the second identity derives from easy calculations based on the fact that $|\pi_1|$ is distributed under μ as a size-biased pick from the ranked sequence s under ν. That the latter quantity is indeed finite follows readily from the assumption $p>\underline{p}$.

Now we only need to check the finiteness of

$$\sum_{t\in\mathcal{D}_1}\exp\{\Phi(p)t-(p+1)\xi(t-)\}\,\Sigma(t)\mathbf{1}_{\{\Sigma(t)<\mathrm{e}^{\varepsilon t}\}}.$$

To that end, we compute the $(\mathcal{G}_1(t-))$-predictable compensator of the sum above and find

$$\exp\{\Phi(p)t-(p+1)\xi(t-)\}\,C(\varepsilon,t)dt\,, \tag{6}$$

where

$$\begin{aligned}C(\varepsilon,t) &= \int_{\mathcal{P}}\mu(d\pi)|\pi_1|^p\left(\sum_{j=2}^{\infty}|\pi_j|^{p+1}\right)\mathbf{1}_{\{\sum_{j=2}^{\infty}|\pi_j|^{p+1}<\mathrm{e}^{\varepsilon t}\}}\\ &= \int_{\mathcal{S}}\nu(ds)\left(\sum_{k=1}^{\infty}s_k^{p+1}\left(\sum_{j=1}^{\infty}s_j^{p+1}-s_k^{p+1}\right)\mathbf{1}_{\{\sum_{j=1}^{\infty}s_j^{p+1}-s_k^{p+1}<\mathrm{e}^{\varepsilon t}\}}\right).\end{aligned}$$

This quantity can be bounded from above by

$$\begin{aligned}&\int_{\mathcal{S}}\nu(ds)\left(\left(\sum_{k=1}^{\infty}s_k^{p+1}\right)^2-\sum_{k=1}^{\infty}s_k^{2(p+1)}\right)\mathbf{1}_{\{\sum_{j=1}^{\infty}s_j^{p+1}\leq 1+\mathrm{e}^{\varepsilon t}\}}\\ \leq\;&\int_{\mathcal{S}}\nu(ds)\left(\left(\left(\sum_{k=1}^{\infty}s_k^{p+1}\right)^2-1\right)+\left(1-\sum_{k=1}^{\infty}s_k^{2(p+1)}\right)\right)\mathbf{1}_{\{\sum_{j=1}^{\infty}s_j^{p+1}\leq 1+\mathrm{e}^{\varepsilon t}\}}\\ \leq\;&(2+\mathrm{e}^{\varepsilon t})\left|\int_{\mathcal{S}}\nu(ds)\left(\sum_{k=1}^{\infty}s_k^{p+1}-1\right)\right|+\int_{\mathcal{S}}\nu(ds)\left(1-\sum_{k=1}^{\infty}s_k^{2(p+1)}\right)\mathbf{1}_{\{\sum_{j=1}^{\infty}s_j^{p+1}\leq 1+\mathrm{e}^{\varepsilon t}\}};\end{aligned}$$

and we conclude that

$$C(\varepsilon,t)=O(\mathrm{e}^{\varepsilon t})\qquad\text{as } t\to\infty.$$

Finally, the compensated sum

$$\sum_{r\in\mathcal{D}_1\cap[0,t]}\exp\{\Phi(p)r-(p+1)\xi(r-)\}\,\Sigma(r)\mathbf{1}_{\{\Sigma(r)\leq \mathrm{e}^{\varepsilon r}\}}-\int_0^t C(\varepsilon,r)\exp\{\Phi(p)r-(p+1)\xi(r)\}\,dr$$

is a $\mathbb{P}^{(p)}$-martingale with bounded variation and only positive jumps. It follows from (5) that under $\mathbb{P}^{(p)}$, first, the integral above converges a.s. when $t \to \infty$, and second,

$$\lim_{r\to\infty} \exp\left\{\Phi(p)r - (p+1)\xi(r-)\right\} \Sigma(r) \mathbf{1}_{\{r\in\mathcal{D}_1, \Sigma(r)\leq e^{\varepsilon r}\}} \;=\; 0\,.$$

This entails (see e.g. the corollary on page 484 in [20]) that

$$\sum_{r\in\mathcal{D}_1} \exp\left\{\Phi(p)r - (p+1)\xi(r-)\right\} \Sigma(r) \mathbf{1}_{\{\Sigma(r)\leq e^{\varepsilon r}\}} \;<\; \infty \qquad \mathbb{P}^{(p)}\text{-a.s.}$$

This completes the proof of our statement. ■

3.3 Uniformity via complex analysis

In this section, we adapt arguments developed by Biggins [8] (see also Uchiyama [21]) for branching random walks to derive a uniform reinforcement of Theorem 1. The fact that we are dealing with processes in continuous time induces new difficulties which are solved by stochastic calculus, using the Poissonian structure of homogeneous fragmentations, as in the proof of Theorem 2 in [4].

For the sake of simplicity, we shall assume throughout this section that no mass is lost during sudden dislocations, i.e.

$$\nu\left(\sum_{i=1}^{\infty} s_i < 1\right) \;=\; 0\,.$$

Proposition 3 *Let $K \subseteq]\underline{p}, \bar{p}[$ be a compact set. Then the martingale $M(p,\cdot)$ converges a.s. and in $L^1(\mathbb{P})$, uniformly for $p \in K$.*

We shall work with complex numbers $\lambda = p + i\eta$; the Laplace exponent Φ is defined by analytic continuation on $\{\lambda : p > \underline{p}\}$. For $1 < q \leq 2$, we introduce the quantities

$$c(\lambda, q) = \int_{\mathcal{S}} \mid 1 - \sum_{i=1}^{\infty} s_i^{\lambda+1} \mid^q \;\nu(ds) \;\;,\;\; d(\lambda,q) := \Phi(q\Re\lambda) - q\Re\Phi(\lambda)\,.$$

and the sets

$$\mathcal{O}_q^1 := \mathrm{int}\{\lambda : c(p,q) < \infty\} \;\;,\;\; \mathcal{O}_q^2 := \mathrm{int}\{\lambda : d(\lambda,q) > 0\}\;,\; \mathcal{O} := \cup_{1<q\leq 2}\left(\mathcal{O}_q^1 \cap \mathcal{O}_q^2\right)\,.$$

Lemma 4 *For every compact set $K \subset]\underline{p}, \bar{p}[$, there exists $\epsilon_0 < 1$ such that $\{\lambda : p \in K \;,\; |\eta| \leq \epsilon_0\} \subset \mathcal{O}$.*

Proof: To start with, we recall from the proof of Theorem 2 in [4] that if $\underline{p} < p < \bar{p}$, then we may find $q > 1$ sufficiently close to 1 such that $c(p,q) < \infty$. We now check $c(\lambda,q) < \infty$ in this situation.

First, suppose $p = \Re\lambda \neq 0$. The map $x \mapsto |(x^{\lambda+1} - x)/(x^{\Re\lambda+1} - x)|$ is bounded on $]0,1[$ by some constant $a_\lambda > 0$. Thus

$$\begin{aligned} c(\lambda, q) &= \int_{\mathcal{S}} \mid \sum_{i=1}^{\infty}(s_i^{\lambda+1} - s_i) \mid^q \ \nu(ds) \\ &\leq \int_{\mathcal{S}} \left(\sum_{i=1}^{\infty} \mid s_i^{\lambda+1} - s_i \mid\right)^q \ \nu(ds) \\ &\leq a_\lambda^q \int_{\mathcal{S}} \left(\sum_{i=1}^{\infty} \mid s_i^{\Re\lambda+1} - s_i \mid\right)^q \ \nu(ds) = a_\lambda^q \ c(p,q)\,, \end{aligned}$$

and thus $c(\lambda, q) < \infty$.

For $\Re\lambda = p = 0$, we observe that

$$\mid 1 - \sum_{i=1}^{\infty} s_i^{\lambda+1} \mid^q \leq 2^{q-1} \left(\mid 1 - s_1^{\lambda+1} \mid^q + \left(\sum_{i=2}^{\infty} s_i\right)^q \right)$$

The function $x \mapsto (1 - x^{\lambda+1})/(1-x)$ is bounded on $]0,1[$ by some constant a'_λ, and since $\sum_{i=2}^{\infty} s_i = 1 - s_1$, we conclude that

$$c(\lambda, q) \leq 2^{q-1} \left(1 + a'^q_\lambda\right) \int_{\mathcal{S}} (1 - s_1)\nu(ds) \ < \infty\,.$$

Next, we point out as (cf. again the proof of Theorem 2 in [4]) that if $c(p_0, q) < \infty$ then there is $h_0 > 0$ such that $c(p,q) < \infty$ for every $p \in [p_0 - h_0, p_0 + h_0]$. We conclude that $\{\lambda : p \in [p_0 - h_0, p_0 + h_0]\} \subset \mathcal{O}^1_{q_0}$.

We now turn our attention to the function d. If $p_0 \in]\underline{p}, \bar{p}[$, then we know again from the proof of Theorem 2 in [4] that there exists $q_0 \in]1,2]$ such that $d(p_0, q_0) > 0$. The function

$$(p, \eta) \mapsto d(p + i\eta, q_0) = \Phi(q_0 p - 1) - q_0 \Re\Phi(p + i\eta - 1)$$

is defined and continuous on some neighborhood of p_0. Thus there exists some rectangle $[p_0 - h, p_0 + h] \times [-\epsilon, +\epsilon]$ on which this function remains positive.

Putting the pieces together, for every $p_0 \in]\underline{p}, \bar{p}[$, we have been able to construct some rectangle containing p_0 and included into $\mathcal{O}^1_q \cap \mathcal{O}^2_q$. We complete the proof of the lemma by a compacity argument. ■

We are now able to establish Proposition 3.

Proof: By compacity, it suffices to check that for every $x \in \mathcal{O}$ we can construct a disk $D_x(r)$ on which the stated property holds true.

We first pick γ and r such that $D_x(3r) \subset \mathcal{O}^1_\gamma \cap \mathcal{O}^2_\gamma$. Since $M(\lambda, T) - M(\lambda, t)$ is an analytic function of the variable λ on $D_x(3r)$, an application of Cauchy's formula yields

$$\sup_{\lambda \in D_x(r)} \mid M(\lambda, T) - M(\lambda, t) \mid \leq \pi^{-1} \int_0^{2\pi} \mid M(x + 2re^{i\theta}, T) - M(x + 2re^{i\theta}, t) \mid d\theta\,.$$

Fix $t > 0$, take the supremum over $T \geq t$ and then the expectation; we obtain

$$\begin{aligned} & \mathbb{E}\left(\sup_{T\geq t} \sup_{\lambda \in D_x(r)} \mid M(\lambda, T) - M(\lambda, t) \mid \right) \\ \leq \ & \pi^{-1} \int_0^{2\pi} \mathbb{E}\left(\sup_{T\geq t} \mid M(x+2re^{i\theta}, T) - M(x+2re^{i\theta}, t) \mid \right) d\theta \,. \end{aligned} \tag{7}$$

For $q > 1$, we have one the one hand

$$\begin{aligned} & \mathbb{E}\left(\sup_{T\geq t} \mid M(x+2re^{i\theta}, T) - M(x+2re^{i\theta}, t) \mid \right) \\ \leq \ & \left(\mathbb{E}\left(\sup_{T\geq t} \mid M(x+2re^{i\theta}, T) - M(x+2re^{i\theta}, t) \mid^q \right) \right)^{1/q} \end{aligned} \tag{8}$$

and on the other hand, by an inequality of Burkholder-Davis-Gundy type involving the q-variation of pure jumps martingales (see Lépingle [14]), we have for $\lambda \in \mathbb{C}$ and $0 < t_0 < t_1$,

$$\mathbb{E}\left(\sup_{t_0 < t \leq t_1} \mid M(\lambda, t) - M(\lambda, t_0) \mid^q \right) \leq k_q \, \mathbb{E}\left(\sum_{t_0 < s \leq t_1} \mid M(\lambda, s) - M(\lambda, s-) \mid^q \right) , \tag{9}$$

where k_q denotes some constant depending only on q.

The mean of the q-variation in the right-hand side of (9) can be evaluated by an application of the compensation formula for Poisson point processes applied to the Poissonian construction of the fragmentation (cf. the proof of Theorem 2 in [4]). One gets

$$\begin{aligned} \mathbb{E}\left(\sum_{t_0 \leq s \leq t_1} \mid M(\lambda, s) - M(\lambda, s-) \mid^q \right) &= c(\lambda, q) \int_{t_0}^{t_1} \exp(sq\Re\Phi(\lambda)) \mathbb{E}(M(q\Re\lambda, s)) \ ds \\ &= c(\lambda, q) \int_{t_0}^{t_1} \exp\left(sq\Re\Phi(\lambda) - s\Phi(q\Re\lambda)\right) \ ds \\ &= c(\lambda, q) \left(\mathrm{e}^{-t_0 d(\lambda, q)} - \mathrm{e}^{-t_1 d(\lambda, q)} \right) / d(\lambda, q) \,, \end{aligned} \tag{10}$$

provided that $d(\lambda, q) \neq 0$. Finally, combining (7), (8), (9) and (10) we obtain for each fixed t

$$\mathbb{E}\left(\sup_{T\geq t} \sup_{\lambda \in D_x(r)} \mid M(\lambda, T) - M(\lambda, t) \mid \right) \leq K_q \sup_{\lambda \in C_x(2r)} \left(\frac{c(\lambda, q)}{d(\lambda, q)} \mathrm{e}^{-td(\lambda, q)} \right)^{1/q} \tag{11}$$

Since $C_x(2r) \subset \mathcal{O}_q^1 \cap \mathcal{O}_q^2$ is a compact set on which $c(., q)$ and $d(., q)$ are continuous, the right-hand side of (11) converges to 0 as $t \to \infty$, which completes the proof. ■

4 Applications to the empirical measure

In this section, we use the tilted probability measures to investigate the asymptotic behavior as time tends to infinity of an empirical measure associated to the fragmentation. More precisely, we are interested in the random measure

$$Z^{(t)}(dy) := \sum_{i=1}^{\infty} \delta_{\log |\Pi_i(t)|}(dy) \,, \qquad y \in \mathbb{R} \,,$$

where δ stands for the Dirac point mass. The following result can be viewed as a weak analog of a result of Biggins [6]. We mention that a stronger version (i.e. strengthening the convergence in probability below into a.s. convergence uniformly on compact sets of $]\underline{p}, \bar{p}[$) can be obtained by an adaptation of the method of Biggins [8] just as in Section 3.2. Even though the approach used here does not yield the sharpest result, it may still be interesting as it makes full use of the tilted probability and is rather elementary.

For the sake of simplicity, we shall implicitly assume in this section that the fragmentation is non-lattice, in the sense that the lattice $r\mathbb{Z}$ does not carry the measure $Z^{(t)}$ (or equivalently, the subordinator ξ does not live on $r\mathbb{Z}$) for any $r > 0$.

Theorem 5 *Fix $p \in]\underline{p}, \bar{p}[$, let $M(p,\infty)$ be the terminal value of the uniformly integrable martingale $M(p,\cdot)$. The following limits hold in probability under $\mathbb{P}$:*
(i) *If $f : \mathbb{R} \to \mathbb{R}$ is a continuous bounded function, then*

$$\lim_{t\to\infty} \mathrm{e}^{-t((p+1)\Phi'(p)-\Phi(p))} \int_{\mathbb{R}} f\left(\frac{t\Phi'(p)+y}{\sqrt{t|\Phi''(p)|}}\right) Z^{(t)}(dy) \;=\; \frac{M(p,\infty)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\mathrm{e}^{-y^2/2}dy\,.$$

(ii) *If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with compact support, then*

$$\lim_{t\to\infty} \sqrt{t}\mathrm{e}^{-t((p+1)\Phi'(p)-\Phi(p))} \int_{\mathbb{R}} f(t\Phi'(p)+y)Z^{(t)}(dy) \;=\; \frac{M(p,\infty)}{\sqrt{2\pi|\Phi''(p)|}} \int_{-\infty}^{\infty} f(y)\mathrm{e}^{-(p+1)y}dy\,.$$

Roughly, the first and second parts of Theorem 5 can be derived from the central limit theorem and the local central limit theorem, respectively, for the process $\xi = (\xi(t), t \geq 0)$ under the tilted probability $\mathbb{P}^{(p)}$. For the sake of brevity, will shall only give a complete argument for the second part[2], the first being easier. The proof amounts to establish the following asymptotics.

Lemma 6 *Let $g : \mathbb{R} \to \mathbb{R}_+$ be a continuous bounded function with compact support and $p \in]\underline{p}, \bar{p}[$. Set*

$$A(t) \;=\; \sqrt{t}\mathrm{e}^{t\Phi(p)} \sum_{i=1}^{\infty} |\Pi_i(t)|^{p+1} g(t\Phi'(p) + \log|\Pi_i(t)|)$$

and

$$I \;=\; \frac{1}{\sqrt{2\pi|\Phi''(p)|}} \int_{-\infty}^{\infty} g(y)dy\,.$$

Then we have

$$\lim_{t\to\infty} \mathbb{E}^{(p)}(A(t)/M(p,t)) \;=\; I \tag{12}$$

and

$$\limsup_{t\to\infty} \mathbb{E}^{(p)}\left((A(t)/M(p,t))^2\right) \;\leq\; I^2\,. \tag{13}$$

[2] A version of ii) for indicator functions of bounded intervals would give a sharp large deviation statement, extending Corollary 2 of [4].

Let us take Lemma 6 for granted and explain how Theorem 5 follows. By first and second moments calculus, Lemma 6 shows that $A(t)/M(p,t)$ converges to I in $L^2(\mathbb{P}^{(p)})$. The variables $A(t)$ and $M(p,t)$ are measurable with respect to the sigma-field $\mathcal{F}(\infty)$, and we know that $M(p,t)$ converges a.s. as $t \to \infty$ to $M(p,\infty)$. Moreover, the probability measures $\mathbb{P}$ and $\mathbb{P}^{(p)}$ are equivalent on $\mathcal{F}(\infty)$ since $p \in]\underline{p}, \bar{p}[$, and we conclude that $A(t)$ converges in probability under $\mathbb{P}$ towards $M(p,\infty)I$. Rewriting this in terms of the random measure $Z^{(t)}$ for $f(y) = \mathrm{e}^{(p+1)y}g(y)$ yields Theorem 5.

We now tackle the proof of Lemma 6. The first moment convergence is easy :

Proof of (12): By the absolute continuity on the sigma field $\mathcal{F}(t)$, we have

$$\mathbb{E}^{(p)}\left(A(t)/M(p,t)\right) = \mathbb{E}(A(t)).$$

Next, using the fact that under $\mathbb{P}$, $\mathrm{e}^{-\xi(t)} = |\Pi_1(t)|$ is distributed as a size-biased pick from the sequence $|\Pi(t)|$, we have

$$\begin{aligned} \mathbb{E}(A(t)) &= \sqrt{t}\mathrm{e}^{t\Phi(p)}\mathbb{E}\left(\exp(-p\xi(t))g(t\Phi'(p) - \xi(t))\right) \\ &= \sqrt{t}\mathbb{E}^{(p)}\left(g(t\Phi'(p) - \xi(t))\right), \end{aligned}$$

where the second identity follows from the very definition of the tilted probability measure. Since under $\mathbb{P}^{(p)}$ the process ξ is a subordinator with Laplace exponent $\Phi^{(p)}(q) = \Phi(p+q) - \Phi(p)$, its mean and variance at time 1 are given respectively by $\Phi'(p)$ and $-\Phi''(p)$, and the local central limit theorem completes the proof of (12). ∎

Next, we turn our attention to the second moment. In this direction, it is convenient to observe the fragmentation at time $\sqrt{t}$, kill the block $\Pi_1(\sqrt{t})$ and let evolve all the other blocks $\Pi_2(\sqrt{t}), \ldots$ up to time t. More precisely, we introduce

$$\hat{A}(t) := \sqrt{t}\mathrm{e}^{t\Phi(p)} \sum_{j \in \mathbb{N}\setminus J(t)} |\Pi_j(t)|^{p+1} g(t\Phi'(p) + \log|\Pi_j(t)|),$$

where $J(t)$ denotes the set of indices j such that $\Pi_j(t) \subseteq \Pi_1(\sqrt{t})$. We defined analogously $\hat{M}(p,t)$. As a first step, we observe that $A(t)$ and $M(p,t)$ are close to $\hat{A}(t)$ and $\hat{M}(p,t)$ respectively, in the L^1 sense.

Lemma 7 *For every $k > 0$, we have as $t \to \infty$*

$$\mathbb{E}\left(|A(t) - \hat{A}(t)|\right) = o(t^{-k}) \quad \textit{and} \quad \mathbb{E}\left(|M(p,t) - \hat{M}(p,t)|\right) = o(t^{-k}).$$

As a consequence, we have also

$$\mathbb{E}^{(p)}\left(\left|\frac{A(t)}{M(p,t)} - \frac{\hat{A}(t)}{\hat{M}(p,t)}\right|\right) = o(t^{-k}).$$

Proof: By the very definition of $\hat{M}(p,t)$, we have

$$M(p,t) - \hat{M}(p,t) = \mathrm{e}^{t\Phi(p)} \sum_{j \in J(t)} |\Pi_j(t)|^{p+1}.$$

The sum over $J(t)$ in the right hand side corresponds to blocks at time t which are issued from the block $\Pi_1(\sqrt{t})$. Thus, applying the fragmentation property at time $\sqrt{t}$ and the martingale property of $M(p,\cdot)$ yields

$$\mathbb{E}\left(\sum_{j\in J(t)} |\Pi_j(t)|^{p+1} \mid \mathcal{G}(\sqrt{t})\right) = \exp(-\Phi(p)(t-\sqrt{t}))|\Pi_1(\sqrt{t})|^{p+1}.$$

It follows that

$$\mathbb{E}\left(|M(p,t)-\hat{M}(p,t)|\right) = \exp\left(-\sqrt{t}(\Phi(p+1)-\Phi(p))\right) = o(t^{-k}).$$

As the function g is bounded, same calculations apply for $|A(t)-\hat{A}(t)|$ (the extra factor $\sqrt{t}$ can be neglected).

Next, we have

$$\begin{aligned}\mathbb{E}^{(p)}\left(\left|\frac{A(t)}{M(p,t)}-\frac{\hat{A}(t)}{\hat{M}(p,t)}\right|\right) &= \mathbb{E}\left(\left|A(t)-\hat{A}(t)\frac{M(p,t)}{\hat{M}(p,t)}\right|\right)\\ &\le \mathbb{E}\left(|A(t)-\hat{A}(t)|\right)+\mathbb{E}\left(\left|\frac{\hat{A}(t)}{\hat{M}(p,t)}\right||M(p,t)-\hat{M}(p,t)|\right).\end{aligned}$$

Since g is bounded, we have $\hat{A}(t)/\hat{M}(p,t)=O(\sqrt{t})$, and this completes the proof of our statement. ■

We are now able to finish the proof of Lemma 6.

Proof of (13): The same calculation as for (12) shows that

$$\begin{aligned}\mathbb{E}^{(p)}\left((A(t)/M(p,t))^2\right) &= \sqrt{t}\mathbb{E}\left(\frac{A(t)}{M(p,t)}\exp(-p\xi(t)+t\Phi(p))g(t\Phi'(p)-\xi(t))\right)\\ &= \sqrt{t}\mathbb{E}^{(p)}\left(\frac{A(t)}{M(p,t)}g(t\Phi'(p)-\xi(t))\right)\\ &= \sqrt{t}\mathbb{E}^{(p)}\left(\frac{\hat{A}(t)}{\hat{M}(p,t)}g(t\Phi'(p)-\xi(t))\right)+o(1),\end{aligned}$$

where the last identity stems from Lemma 7. By an application of the fragmentation property at time $\sqrt{t}$, we may rewrite the preceding quantity in the form

$$\sqrt{t}\mathbb{E}^{(p)}\left(\frac{\hat{A}(t)}{\hat{M}(p,t)}\mathbb{E}^{(p)}_{\xi(\sqrt{t})}\left(g(t\Phi'(p)-\xi(t-\sqrt{t}))\right)\right)+o(1),$$

where the notation $\mathbb{E}^{(p)}_y$ refers to the expectation for the subordinator ξ started from y, under the tilted probability measure. Now fix $\eta>0$ and recall that we assumed that $g\ge 0$. By the local central limit theorem, we have for every sufficiently large t the uniform upper-bound

$$\sqrt{t}\mathbb{E}^{(p)}_z\left(g(t\Phi'(p)-\xi(t-\sqrt{t})\right) \le \eta+I, \qquad \forall z\in\mathbb{R},$$

which yields

$$\mathbb{E}^{(p)}\left((A(t)/M(p,t))^2\right) \leq \mathbb{E}^{(p)}\left(\hat{A}(t)/\hat{M}(p,t)\right)(\eta + I) + o(1)\,.$$

Because η can be chosen arbitrarily small, another application of Lemma 7 gives

$$\limsup_{t\to\infty} \mathbb{E}^{(p)}\left((A(t)/M(p,t))^2\right) \leq I \limsup_{t\to\infty} \mathbb{E}^{(p)}\left(A(t)/M(p,t)\right) = I^2\,,$$

where the last equality is given by (12). ■

5 The derivative martingale

We end up this work by considering the so-called derivative martingale that we now introduce. Recall that $\bar{p} > 0$ is the critical value for the convergence in $L^1(\mathbb{P})$ of the additive martingales. The process

$$\mathcal{E}'(t) := (t\Phi'(\bar{p}) - \xi(t)) \exp(-\bar{p}\xi(t) + t\Phi(\bar{p}))\,, \qquad t \geq 0$$

is clearly a $(\mathbb{P}, \mathcal{G}(t))$-martingale; its projection on the sub-filtration $(\mathcal{F}(t))_{t\geq 0}$ is a $(\mathbb{P}, \mathcal{F}(t))$ martingale, called the *derivative martingale* and given by

$$M'(t) = \sum_{i=1}^{\infty} (t\Phi'(\bar{p}) + \log(|\Pi_i(t)|)) \exp(t\Phi(\bar{p})) |\Pi_i(t)|^{\bar{p}+1}\,.$$

We stress that the derivative martingale is not always positive, which contrasts with the case of additive martingales. The idea of considering the derivative martingale at the critical value goes back to Neveu [18] for the branching Brownian motion. For the branching random walk, it has been considered by Kyprianou [13], Liu [15] with the help of a functional equation and by Biggins and Kyprianou with the measure change method in [9].

Proposition 8 (i) *The martingale M' converges $\mathbb{P}$-a.s. to a finite non-positive limit $M'(\infty)$,*
(ii) $\mathbb{E}(M'(\infty)) = -\infty$,
(iii) $\mathbb{P}(M'(\infty) < 0) = 1$.

Proof: The arguments for (i) and (ii) follow closely that for the proof of Theorem 1, in particular we use the same notation as there, and only provide details for the parts which have to be modified.

(i) Define for every $i \in \mathbb{N}$ and $s \leq t$, $\beta_{s,t}(i)$ as the unique block of $\Pi(s)$ containing $\Pi_i(t)$. For $a > 0$, let

$$\begin{cases} \Pi_i^{(a)}(t) = \Pi_i(t), & \text{if } |\beta_{s,t}(i)| \leq \exp\{a - s\Phi'(\bar{p})\} \text{ for every } s \leq t; \\ \Pi_i^{(a)}(t) = \emptyset, & \text{otherwise}\,. \end{cases}$$

The family $\left\{\Pi_i^{(a)}(t) : i \in \mathbb{N}\right\}$ obviously possesses asymptotic frequencies. Moreover, it should be plain that as t varies in $[0,\infty[$, this family of partitions is nested. We denote by $(\mathcal{H}(t))_{t\geq 0}$ the filtration generated by the process of their ranked asymptotic frequencies, so $(\mathcal{H}(t))_{t\geq 0}$ is another sub-filtration of $(\mathcal{G}(t))_{t\geq 0}$.

Because $\Phi'(\bar{p}) = \Phi(\bar{p})/(\bar{p}+1)$ and the martingale $M(\bar{p},t)$ converges to 0, $\mathbb{P}$-a.s., we have $\sup_{t\geq 0}\left\{\exp(t\Phi'(\bar{p}))|\Pi(t)|_1^{\downarrow}\right\} < \infty$, $\mathbb{P}$-a.s. It follows that

$$\lim_{a\to\infty} \mathbb{P}\left(\Pi_i^{(a)}(t) = \Pi_i(t) \text{ for all } i\in\mathbb{N} \quad \text{and for all } t\geq 0\right) = 1\,.$$

Thus, in order to prove the existence of a finite limit for M', it suffices to establish that if

$$M_a(t) := \sum_{i=1}^{\infty} \left(\log\left(1/|\Pi_i(t)|\right) - t\Phi'(\bar{p}) + a\right)\exp\left(t\Phi(\bar{p})\right)|\Pi_i^{(a)}(t)|^{\bar{p}+1} \tag{14}$$

then $\lim_{t\to\infty} M_a(t) =: M_a(\infty)$ exists $\mathbb{P}$-a.s. for every $a > 0$.

From now on, we fix $a > 0$. Since the process $\xi(t) - \Phi'(\bar{p})t$ has no negative jumps,

$$\mathcal{M}_a(t) \;:=\; (\xi(t) + a - t\Phi'(\bar{p}))\exp(-\bar{p}\xi(t) + t\Phi(\bar{p}))\mathbf{1}_{\{t<\zeta_a\}}$$

where $\zeta_a = \inf\{t\geq 0 : \xi(t) < t\Phi'(\bar{p}) - a\}$, can be viewed as a stopped (non-negative) $\mathbb{P}$-martingale. Its projection on the sub-filtration $(\mathcal{H}(t))_{t\geq 0}$ is $M_a(t)$, which therefore is a non-negative $(\mathbb{P},\mathcal{H}(t))$ martingale, and thus possesses a finite limit as $t\to\infty$, $\mathbb{P}$-a.s.

(ii) To prove that $\mathbb{E}(M'(\infty)) = -\infty$, it will be enough to prove that M_a is uniformly $\mathbb{P}$ integrable since $\mathbb{E}\left(M_a(t)\right) = a$ gives $\mathbb{E}(-M'(\infty)) \geq a$ for every $a > 0$.

Let us introduce the tilted probability measure $\mathbb{Q}$ on Ω given by

$$d\mathbb{Q}_{|\mathcal{G}(t)} \;=\; a^{-1}\mathcal{M}_a(t)\, d\mathbb{P}_{|\mathcal{G}(t)}\,; \tag{15}$$

so we also have

$$d\mathbb{Q}_{|\mathcal{H}(t)} \;=\; a^{-1}M_a(t)\, d\mathbb{P}_{|\mathcal{H}(t)}\,.$$

We have to check that $\liminf_{t\to\infty} M_a(t) < \infty$, $\mathbb{Q}$-a.s. A main ingredient to that end is the fact that the process

$$\lambda(t) \;:=\; (\xi(t) + a - t\Phi'(\bar{p}))\,\mathbf{1}_{\{t<\zeta_1\}}\,, \qquad t\geq 0$$

is, under $\mathbb{P}$, a Lévy process with no negative jumps started from a and stopped when it becomes negative, and under $\mathbb{Q}$, a centered Lévy process with no negative jumps started from a and conditioned to stay positive forever (see for instance [10]). In particular, under $\mathbb{Q}$, we may get rid of the indicator function $\mathbf{1}_{\{\cdot<\zeta_a\}}$ as it equals 1 a.s., and it is easily seen that

$$\inf\left\{\lambda(t), t\geq 0\right\} \;>\; 0 \quad \text{and} \quad \lim_{t\to\infty}\frac{\log\lambda(t)}{\log t} \;=\; 1/2 \qquad \mathbb{Q}\text{-a.s.} \tag{16}$$

As a consequence,

$$\lim_{t\to\infty}\left(\log\left(1/|\Pi_1(t)|\right) - t\Phi'(\bar{p}) + a\right)\exp\left(t\Phi(\bar{p})\right)|\Pi_1(t)|^{\bar{p}+1} \;=\; 0 \qquad \mathbb{Q}\text{-a.s.},$$

which enables us to focus henceforth on

$$\begin{aligned}\tilde{M}(t) &= M_a(t) - \left(\log\left(1/|\Pi_1(t)|\right) - t\Phi'(\bar{p}) + a\right)\exp\left(t\Phi(\bar{p})\right)|\Pi_1(t)|^{\bar{p}+1}\\ &= \sum_{i=2}^{\infty}\left(\log\left(1/|\Pi_i(t)|\right) - t\Phi'(\bar{p}) + a\right)\exp\left(t\Phi(\bar{p})\right)|\Pi_i(t)|^{\bar{p}+1} \qquad \mathbb{Q}\text{-a.s.},\end{aligned}$$

Next, we compute the conditional expectation of this quantity given $\mathcal{G}_1(t)$ by calculations similar to those in the proof of Theorem 1. We get

$$\mathbb{Q}(\tilde{M}(t) \mid \mathcal{G}_1(t)) \;=\; \sum_{r\in\mathcal{D}_1\cap[0,t]} \exp\{-(\bar{p}+1)(\lambda(r-)-a)\}\,\Sigma(r)\,,$$

where

$$\Sigma(r) \;=\; \sum_{j=2}^{\infty} (\lambda(r-) - \log|\pi_j(r)|)\,|\pi_j(r)|^{\bar{p}+1}\,.$$

Then we have to compute the $(\mathbb{Q},\mathcal{G}_1(t))$-predictable compensator corresponding to the point process $\{\Sigma(r), r\in\mathcal{D}_1\}$, and we find

$$\begin{aligned}
&\lambda(r-)^{-1}\int_{\mathcal{P}}\mu(d\pi)\,(\lambda(r-)-\log|\pi_1|)\,|\pi_1|^{\bar{p}}\left(\sum_{j=2}^{\infty}(\lambda(r-)-\log|\pi_j|)\,|\pi_j|^{\bar{p}+1}\right)\\
=\;\;&\lambda(r-)^{-1}\int_{\mathcal{S}}\nu(ds)\left\{\left(\sum_{j=1}^{\infty}(\lambda(r-)-\log|s_j|)\,|s_j|^{\bar{p}+1}\right)^2-\sum_{j=1}^{\infty}(\lambda(r-)-\log|s_j|)^2\,|s_j|^{2\bar{p}}\right\}.
\end{aligned}$$

Using the fact that $\bar{p}>0$, it is easily seen that this quantity can be bounded from above by $C\,(\lambda(r-)+1+1/\lambda(r-))$ for some constant C that depends only on ν. So, all that we need now is to verify that the integral

$$\int_0^{\infty}(\lambda(r)+1+1/\lambda(r))\exp\{-(\bar{p}+1)(\lambda(r)-a)\}\,dr$$

converges $\mathbb{Q}$-a.s., which is immediate from (16).

(iii) To ease the reading, let us denote

$$Y_i(t) = \exp(t\Phi(\bar{p}))\left(|\Pi(t)|_i^{\downarrow}\right)^{\bar{p}+1}\,.$$

We first remark that for all $i\in\mathbb{N}$ $\lim_{t\to\infty}Y_i(t)=0$, $\mathbb{P}$-a.s., and we deduce from (14) that $M_a(t)\le -M'(t)+aM(\bar{p},t)$, for t large enough. Taking the limit as $t\to\infty$, we get

$$M_a(\infty)\le -M'(\infty)\,,\qquad \mathbb{P}\text{-a.s.,} \tag{17}$$

which proves that $-M'(\infty)\ge 0$ $\mathbb{P}$-a.s.

From the fragmentation property at time 1, we may express $M'(1+t)$ in the form

$$-M'(1+t) \;=\; \sum_{i,j} Y_i(1)Y_{i,j}(t)\log\frac{1}{Y_i(1)Y_{i,j}(t)}\,,$$

where $\{Y_{i,j}(\cdot), j\in\mathbb{N}\}$ for $i=1,\dots$ are independent copies of $\{Y_j(\cdot), j\in\mathbb{N}\}$, which are also independent of $\mathcal{G}(1)$. This yields

$$-\,M'(1+t) \;=\; \sum_i Y_i(1)\,(-M_i'(t)) + \sum_i\left(Y_i(1)\log\frac{1}{Y_i(1)}\right)M_i(t) \tag{18}$$

where $\{M_i(\cdot), i \in \mathbb{N}\}$ (respectively, $\{M'_i(\cdot), i \in \mathbb{N}\}$) are independent copies of $M(\bar{p}, \cdot)$ (respectively, of $M'(\cdot)$) and independent of $\mathcal{G}(1)$. To get rid of the last infinite random combination of martingales converging to zero, we first establish the following technical result :

$$\lim_{t\to\infty} \sum_i \left(Y_i(1) \log \frac{1}{Y_i(1)} \right) M_i(t) \ = \ 0 \qquad \text{in probability under } \mathbb{P}. \tag{19}$$

Indeed, because the $|\Pi(1)|_i^{\downarrow}$, $i \in \mathbb{N}$ are ranked in the decreasing order, and their sum is at most 1, we have $|\Pi(1)|_i^{\downarrow} \leq 1/i$ for every i, and thus $Y_i(1) < 1$ for $i > \mathrm{e}^{\Phi'(\bar{p})}$. The series $-M'(1) = \sum_i Y_i(1) |\log Y_i(1)|$ is absolutely convergent (and in L^1). Therefore, for every $\epsilon > 0$ there exists $k > \mathrm{e}^{\Phi'(\bar{p})}$ such that

$$\mathbb{E}\left(\sum_{k+1}^{\infty} Y_i(1) |\log Y_i(1)| \right) \leq \varepsilon^2 .$$

Since $\mathbb{E}(M_i(t)) = 1$ for all i, the Markov inequality enables us to write

$$\mathbb{P}\left(\sum_{k+1}^{\infty} \left(Y_i(1) |\log Y_i(1)| \right) M_i(t) > \epsilon \right) \leq \epsilon .$$

Since the sum of the k remaining terms converges $\mathbb{P}$-a.s. to 0, the claim (19) is proved.

Now we are able to complete the proof of (ii). Assume that $\mathbb{P}(M'(\infty) = 0) > 0$. From (18) and (19) we may write

$$M'(\infty) = Y_1(1) M'_1(\infty) + Y_2(1) M'_2(\infty) + B$$

where $B = \lim_t \sum_{i=3}^{\infty}$ is independent of $(M'_1(\infty), M'_2(\infty))$ conditionally on $\mathcal{G}_1$. Since $\mathbb{P}(M'(\infty) \leq 0) = 1$, this entails $\mathbb{P}(B \leq 0) = 1$ and

$$M'(\infty) \leq Y_1(1) M'_1(\infty) + Y_2(1) M'_2(\infty) .$$

This implies $\mathbb{P}(M'(\infty) = 0) \leq \mathbb{P}(M'(\infty) = 0)^2$, so we would have $\mathbb{P}(M'(\infty) = 0) = 1$, which contradicts (ii). ■

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