An introduction to growth-fragmentations

An unfinished draft

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Abstract

Growth-fragmentation processes describe branching systems of particles, in which the mass of each particle may vary and split into smaller masses randomly as time passes. Such examples arise in several important models in statistical physics and large random structures.

The purpose of this lecture is to provide an elementary introduction to (binary) self-similar growth-fragmentation processes [5, 6]. We shall study a few fundamental martingales, characterize extinction or explosion, and present the long-time asymptotic behavior.

1 Cell systems and growth-fragmentations

Bertoin [6] developed a general construction of growth-fragmentations, which can be conveniently described as a cell system. Each cell may grow continuously and divide into two cells occasionally. These dynamics, both the growth and the splitting, are encoded by a càdlàg Markov process $X = (X(t), t \geq 0)$ on $[0, \infty)$ with no positive jumps, which shall be referred to as a **cell process**. Specifically, at initial time 0 there exists a single cell, called the *Eve*. As time proceeds, the size of Eve evolves according to the cell process X. At each jump time $t \geq 0$ of X with $\Delta X(t) = X(t) - X(t-) < 0$, a "daughter" cell with initial size $-\Delta X(t)$ is born. We stress that the Eve survives after this cell division. Each daughter follows the same dynamics as the Eve and evolves independently of the others.

This description can be made rigorous. Specifically, we shall index the cell system by the *Ulam-Harris tree* $\mathbb{U} := \bigcup_{n=0}^{\infty} \mathbb{N}^n$, with $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}^0 := \{\varnothing\}$ by convention. An element $u \in \mathbb{U}$ is a finite sequence of natural numbers $u = (n_1, \ldots, n_{|u|})$, where $|u| \in \mathbb{N}$ stands for the generation of u. Write $u_- = (n_1, \ldots, n_{|u|-1})$ for her mother and $uk = (n_1, \ldots n_{|u|}, k)$ for her k-th daughter with $k \in \mathbb{N}$. For each $u \in \mathbb{U}$, we shall build a process $\mathcal{X}_u$ that depicts the evolution of the size of the cell indexed by u as time passes, in the following way. For $y > 0$, write P_y for the law of X starting from $X(0) = y$. We fix an enumeration method for the jumps of càdlàg functions $[0, \infty) \to \mathbb{R}$. That is, whenever we have such a function f, we have a canonical way (by using this method) to list all jump times of f in a sequence $(t_i)_{i \geq 1}$ and refer to t_i as *the i-th jump of f*.

Definition 1.1 ([6]). *For $x > 0$, a* **cell system** *$\mathcal{X} := (\mathcal{X}_u,\ u \in \mathbb{U})$ driven by X, in which the Eve cell $\varnothing$ has the initial size x, is built by the following description.*

1. *We set the birth time of $\varnothing$ by $b_\varnothing := 0$ and let the Eve process $\mathcal{X}_\varnothing = (\mathcal{X}_\varnothing(t), t \geq 0)$ be of law P_x. Under P_x, the process $\mathcal{X}_\varnothing$ is possibly killed at a certain time $\zeta_\varnothing \in (0, \infty]$, and we write $\mathcal{X}_\varnothing(t) = \partial$ for any $t \geq \zeta_\varnothing$.*

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2. *For an individual $u \in \mathbb{U}$, suppose we have built $\mathcal{X}_u$. Say the i-th largest (in size) jump of $\mathcal{X}_u$ occurs at time t_i and has size $x_i := -\Delta\mathcal{X}_u(t)$. Then its i-th daughter ui is born at time $b_{ui} := b_u + t_i$ and ui's size process $\mathcal{X}_{ui} = (\mathcal{X}_{ui}(r), r \geq 0)$ has distribution P_{x_i} conditionally on $\mathcal{X}_u$, independent of the other individuals in the same generation. Let ζ_{ui} be the life time of ui.*

Write $\mathcal{P}_x$ for the law of this cell system $\mathcal{X}$ (recall that $x > 0$ indicates the initial size of the Eve $\varnothing$, i.e. $\mathcal{X}_\varnothing(0) = x$). The cell system can be viewed as a Crump-Mode-Jagers branching process [13], which infers that the probability distribution $\mathcal{P}_x$ indeed exists and is uniquely determined by the above description.

Definition 1.2 ([6]). *Let $\mathcal{X}$ be a cell system driven by X. For every $t \geq 0$, the multiset (that allows multiple instances of its elements) of the sizes of the cells alive at time t is*

$$\mathbf{X}(t) := \{\!\{\mathcal{X}_u(t - b_u) \ : \ u \in \mathbb{U}, b_u \leq t \leq b_u + \zeta_u\}\!\},$$

where b_u is the birth time of u. Then we call $\mathbf{X} := (\mathbf{X}(t), t \geq 0)$ a **(Markovian) growth-fragmentation process associated with the cell process** *X and we write $\mathbf{P}_x$ for the law of $\mathbf{X}$ under $\mathcal{P}_x$.*

Remark 1.3. *One can view a multiset $\mathcal{I}$ as a point measure $\sum_{i\in\mathcal{I}} \delta_i$, where δ stands for the Dirac mass.*

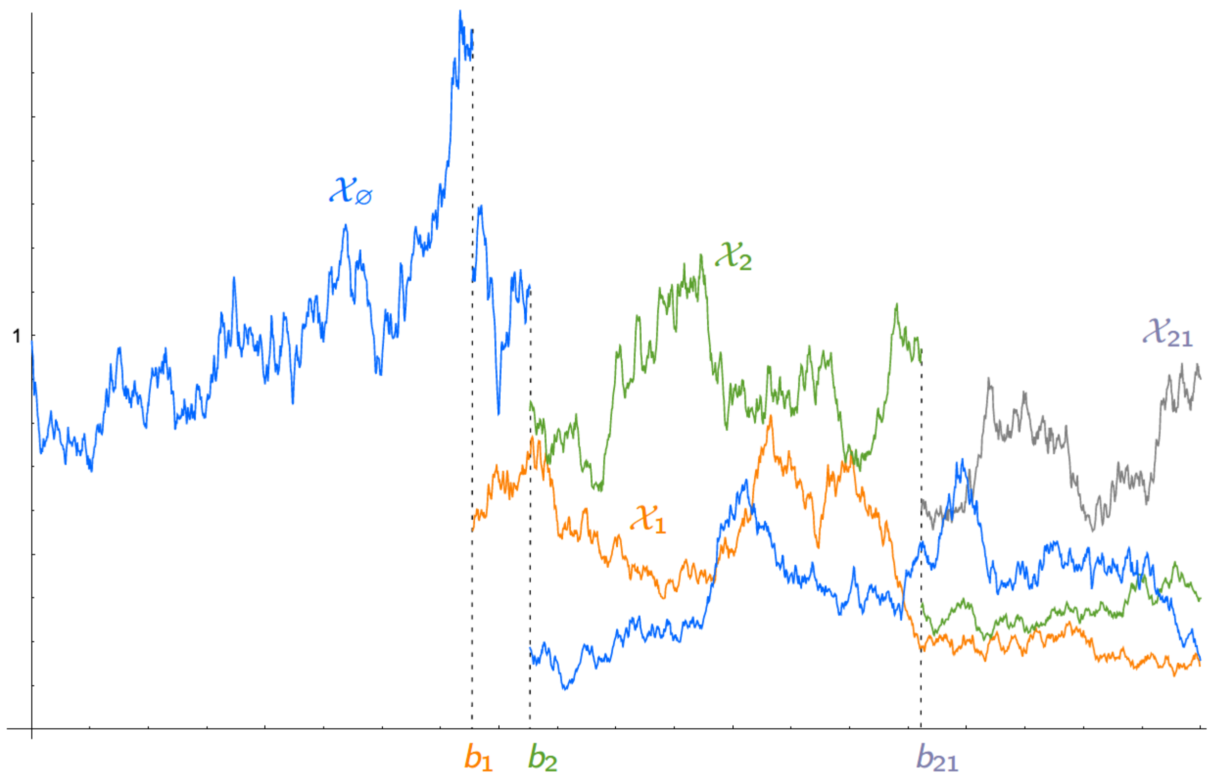

Figure 1: **A cell system (simulated by B.Dadoun)**

Theorem 1.4 (Theorem 1 in [6]). *Let $f : (0, \infty) \cup \{\partial\} \to (0, \infty)$ be a function such that $f(\partial) = 0$ and that*

$$\inf_{y\in(a,\infty)} f(y) < \infty \quad \textit{for every } a > 0.$$

Suppose that X satisfies

$$\mathrm{E}_x\left[f(X(t)) + \sum_{0\leq r\leq t} f(-\Delta X(r))\right] \leq f(x), \qquad \forall x > 0, t \geq 0. \tag{1.1}$$

Then f is called an excessive function *for* $\mathbf{X}$*, in the sense that for every $x > 0$, there is*

$$\mathbf{E}_x\left[\sum_{y\in\mathbf{X}(t)} f(y)\right] \leq f(x), \quad \textit{for all } t \geq 0.$$

Proof. We may assume that $\mathbf{X}$ is associated with a cell system $\mathcal{X}$ of law $\mathcal{P}_x$ and write $\mathcal{E}_x$ for mathematical expectation under $\mathcal{P}_x$. We will prove that the sequence

$$\Sigma(i) := \sum_{|u|\leq i, b_u \leq t} f(\mathcal{X}_u(t-b_u)) + \sum_{|v|=i, b_v\leq t}\sum_{b_v\leq r\leq t} f(-\Delta\mathcal{X}_v(r-b_v)), \quad i \in \mathbb{N}$$

is a non-negative super-martingale, then $\Sigma(\infty) = \lim_{i\to\infty}\Sigma(i)$ exists almost surely and $\Sigma(\infty) \geq \langle \mathbf{X}(t), f(s+t,\cdot)\rangle$. We thus deduce from Fatou's lemma that

$$\mathbf{E}_x\left[\langle \mathbf{X}(t), f\rangle\right] \leq \mathcal{E}_x\left[\Sigma(0)\right] = E_x\left[f(X(t)) + \sum_{0\leq r\leq t} f(-\Delta X(r))\right] \leq f(x),$$

where the last inequality derives from (1.1).

So it remains to prove that $\Sigma(i)$ is a super-martingale. For every v with $|v| = i$, given $\mathcal{F}_{i-1} := \sigma(\mathcal{X}_u, |u| \leq i-1)$ we have by (1.1) that

$$\mathcal{E}_x\left[f(\mathcal{X}_v(t-b_v)) + \sum_{b_v\leq r\leq t} f(-\Delta\mathcal{X}_v(r-b_v)) \;\middle|\; \mathcal{F}_{i-1}\right] \leq f(\mathcal{X}_v(0)).$$

Summing over v of i-th generation on the event $\{t \geq b_v\}$, we get that

$$\begin{aligned}
&\mathcal{E}_x\left[\sum_{|v|=i, b_v\leq t} f(\mathcal{X}_v(t-b_v)) + \sum_{|v|=i, b_v\leq t}\sum_{b_v\leq r\leq t} f(-\Delta\mathcal{X}_v(r-b_v)) \;\middle|\; \mathcal{F}_{i-1}\right] \\
&\leq \sum_{|v|=i, b_v\leq t} f(\mathcal{X}_v(0)) = \sum_{|u|=i-1, b_u\leq t}\sum_{b_u\leq r\leq t} f(-\Delta\mathcal{X}_u(r-b_u)).
\end{aligned}$$

Adding $\sum_{|u|\leq i-1, b_u\leq t} f(\mathcal{X}_u(t-b_u))$ to both sides of inequality, we conclude that

$$\mathcal{E}_x\left[\Sigma(i) \mid \mathcal{F}_{i-1}\right] \leq \Sigma(i-1),$$

which means that $\Sigma(i)$ is a super-martingale. □

2 Homogeneous growth-fragmentation processes

In this section we focus on *homogeneous growth-fragmentations*. This case is closely related to *Lévy processes* (càdlàg processes with independent and stationary increments).

2.1 Lévy processes

We refer to [2, 14] for general theory of Lévy processes. Let ξ be a Lévy process without positive jumps, possibly killed at some independent time ζ with exponential distribution with parameter $k > 0$. Such a process is often referred to as a *spectrally negative Lévy process (SNLP)*. The distribution of the SNLP ξ is characterized by its Laplace exponent $\Phi : [0, \infty) \to \mathbb{R}$:

$$\mathbb{E}\left[\mathrm{e}^{q\xi(t)}\right] = \mathrm{e}^{\Phi(q)t}, \qquad \text{for all } q, t \geq 0.$$

It is well-known that the convex function Φ is can be expressed by the Lévy-Khintchine formula

$$\Phi(q) = -k + \frac{1}{2}\sigma^2 q^2 + cq + \int_{(-\infty,0)} (\mathrm{e}^{qz} - 1 + q(1 - \mathrm{e}^z))\,\Lambda(dz), \quad q \geq 0, \tag{2.1}$$

where $k \geq 0$ is the killing rate, $\sigma \geq 0$, $c \in \mathbb{R}$ and the Lévy measure Λ on $(-\infty, 0)$ satisfies

$$\int_{(-\infty,0)} (z^2 \wedge 1)\Lambda(dz) < \infty. \tag{2.2}$$

Then we say ξ is a SNLP with characteristics (σ, c, Λ, k). For every $t \geq 0$, let $\Delta\xi(t) := \xi(t) - \xi(t-) \leq 0$. The jump process $(t, \Delta\xi(t))_{t\geq 0}$ is a Poisson random measure with intensity $\mathrm{d}t \otimes \Lambda$, where $\mathrm{d}t$ denotes the Lebesgue measure.

The following statement is a consequence of the strong law of large numbers.

Lemma 2.1 ([14, Theorem 7.1 & 7.2]). *Let ξ be a SNLP. Then we have*

$$\lim_{t\to\infty} \frac{\xi(t)}{t} = \mathrm{E}[\xi(1)] = \Phi'(0+) \in [-\infty, \infty).$$

If $\Phi'(0+) = 0$, then ξ is oscillating, that is

$$\limsup_{t\to\infty} \xi(t) = \infty \quad \textit{and} \quad \liminf_{t\to\infty} \xi(t) = -\infty.$$

2.2 Homogeneous growth-fragmentation processes

For $x > 0$, denote by P_x the law of the homogeneous cell process

$$X(t) := x\mathrm{e}^{\xi(t)}, \quad t \geq 0.$$

Definition 2.2. *Let $\mathbf{X}$ be a Markovian growth-fragmentation (Definition 1.2) associated with X. Then we call $\mathbf{X}$ a* homogeneous growth-fragmentation.

Let us introduce an important function $\kappa\colon [0, \infty) \to (-\infty, \infty]$

$$\kappa(q) := \Phi(q) + \int_{(-\infty,0)} (1 - \mathrm{e}^z)^q \Lambda(dz), \quad q \geq 0. \tag{2.3}$$

We call κ the *cumulant* of ξ or X or $\mathbf{X}$. Let

$$\underline{q} := \inf\{q \geq 0 : \ \kappa(q) < \infty\},$$

then we have $\underline{q} \leq 2$ because of (2.2). Note that $\kappa(q) < \infty$ for all $q > \underline{q}$, and that κ is infinitely differentiable and strictly convex on $(\underline{q}, \infty)$. We stress that κ does not characterize the law of ξ; see [17, Lemma 2.1].

Theorem 2.3 ([5]). *Let $\mathbf{X}$ be a homogeneous growth-fragmentation with cumulant κ. For $q > \underline{q}$, there is*

$$\mathbf{E}_x\left[\sum_{y\in\mathbf{X}(t)} y^q\right] = x^q \exp(\kappa(q)t), \qquad \textit{for all } t \geq 0. \tag{2.4}$$

Proof. For simplicity, let us assume the support of the measure Λ is included in $[-\log 2, 0)$. For the general case the arguments are quite similar but the notations become heavier; see [17, Proposition 2.15].

For $\epsilon > 0$, define two independent Lévy processes $\xi^{[\epsilon]}$ and $\eta^{[\epsilon]}$, where $\eta^{[\epsilon]}$ is a compound Poisson process with (finite) Lévy measure $\Lambda|_{(-\infty,\log(1-\epsilon))}$ and Laplace exponent $\Phi_\eta^{[\epsilon]}(q) := \int_{(-\infty,\log(1-\epsilon))}(1-\mathrm{e}^z)^q\Lambda(\mathrm{d}z)$, and $\xi^{[\epsilon]}$ a Lévy process with Laplace exponent $\Phi_\xi^{[\epsilon]}(q) := \Phi(q) - \Phi_\eta^{[\epsilon]}(q)$. Then the process $\xi^{[\epsilon]} + \eta^{[\epsilon]}$ has the same law as ξ.

Let us truncate the cell system at level $\epsilon > 0$ in the following way. For every $u \in \mathbb{U}$, at each jump $t_i \geq 0$ of the process $\mathcal{X}_u$, we kill the child ui, as well as its descendants, if and only if

$$\frac{|\Delta\mathcal{X}_u(t_i)|}{\mathcal{X}_u(t_i-)} \leq \epsilon \Leftrightarrow |\Delta\xi(t)| \leq \log(1-\epsilon).$$

Then the dynamics of the truncated system has the following description. It starts with an initial fragment whose size evolves according to the process $\exp(\xi^{[\epsilon]})$. We run an independent Lévy process $\eta^{[\epsilon]}$. At the first jump time τ of $\eta^{[\epsilon]}$, this fragment is replaced by two particles with respective initial sizes

$$y_1 := \mathrm{e}^{\xi^{[\epsilon]}(\tau)}\mathrm{e}^{\Delta\eta^{[\epsilon]}(\tau)} \text{ and } y_2 := \mathrm{e}^{\xi^{[\epsilon]}(\tau)}(1-\mathrm{e}^{\Delta\eta^{[\epsilon]}(\tau)}).$$

The two children continue to evolve in a similar way, independent one of the other. Let $\mathbf{X}^{[\epsilon]}$ be the growth-fragmentation associated with this truncated system, then one can check that

$$\mathbf{X}^{[\epsilon]}(t+\tau) = y_1\mathbf{X}^{[\epsilon]}(t) \uplus y_2\mathbf{X}^{[\epsilon]}(t),$$

where $\uplus$ denotes the multiset sum.

From properties of the compound Poisson process $\eta^{[\epsilon]}$, we know that τ has exponential distribution with parameter $\beta := \Lambda\big((-\infty, \log(1-\epsilon))\big) < \infty$, and $\Delta\eta^{[\epsilon]}(\tau)$ has distribution $\Lambda\big(\cdot\ |(-\infty,\log(1-\epsilon))\big)$. we have

$$\begin{aligned} m(q,t) &:= \mathbf{E}_1\Big[\sum_{y\in\mathbf{X}^{[\epsilon]}(t)} y^q\Big] \\ &= \mathcal{P}(\tau > t)\mathrm{E}[\exp(\xi^{[\epsilon]}(t))] + \int_0^t \beta\mathrm{e}^{-\beta s}\mathcal{E}_1[y_1^q m(q,t-s) + y_2^q m(q,t-s)]\mathrm{d}s \\ &= \mathrm{e}^{-\beta t}\exp(\Phi_\xi^{[\epsilon]}(q)t) + \int_0^t \beta\mathrm{e}^{-\beta s}\exp(\Phi_\xi^{[\epsilon]}(q)s)m(q,t-s)\beta^{-1}\int_{(-\infty,\log(1-\epsilon))}(\mathrm{e}^{qz} + (1-\mathrm{e}^z)^q)\Lambda(\mathrm{d}z)\mathrm{d}s. \end{aligned}$$

Solving this equation yields $m(q,t) = \mathrm{e}^{\kappa^{[\epsilon]}(q)t}$, where

$$\kappa^{[\epsilon]}(q) := \Phi_\xi^{[\epsilon]}(q) + \int_{(-\infty,\log(1-\epsilon))}(\mathrm{e}^{qz} + (1-\mathrm{e}^z)^q)\Lambda(\mathrm{d}z) = \Phi(q) + \int_{(-\infty,\log(1-\epsilon))}((1-\mathrm{e}^z)^q)\Lambda(\mathrm{d}z).$$

Letting $\epsilon \downarrow 0+$, the monotone convergence completes the proof. □

Rearranging the elements of $\mathbf{X}(t)$ in decreasing order, we denote the obtained decreasing sequence by

$$X_1(t) \geq X_2(t) \geq \ldots \geq 0.$$

The notation shall be adopted throughout the rest of this note.

Proposition 2.4. *The following properties holds for a homogeneous growth-fragmentation* $\mathbf{X}$*:*

(P1) (Temporal branching property) For $s \geq 0$, write $\mathbf{X}(s) = \{\!\{X_1(s), X_2(s), \cdots\}\!\}$. Then conditionally on $\sigma(\mathbf{X}(r), r \leq s)$, the distribution of the process $\mathbf{X}(t+s)_{t\geq 0}$ is the same as the (multiset) sum of a sequence of independent growth-fragmentations $(\mathbf{X}^{[i]})_{i\geq 1}$, where each $\mathbf{X}^{[i]}$ has distribution $\mathbf{P}_{X_i(s)}$.

(P2) (Homogeneity) For $x > 0$, the process $\{\!\{xy, y \in \mathbf{X}(t)\}\!\}_{t\geq 0}$ Under $\mathbf{P}_1$ has the law of $\mathbf{P}_x$.

Proof. It is easy to see that the properties hold for the truncated growth-fragmentation $\mathbf{X}^{[\epsilon]}$ as in the proof of Theorem 2.3. Further, for every $t \geq 0$, we have

$$\lim_{\epsilon\to 0+} \mathbf{E}_x[\sum_{y\in\mathbf{X}(t)\setminus\mathbf{X}^{[\epsilon]}(t)} y^q] = 0.$$

Then the temporal branching property can be transferred to $\mathbf{X}$ by letting $\epsilon \to 0+$. □

2.3 The additive martingale

For $\omega > \underline{q}$ (such that $\kappa(\omega) < \infty$), it follows from Theorem 2.3 that the following process under $\mathbf{P}_x$ has a constant mean value 1:

$$M(\omega, t) := x^{-\omega} e^{-t\kappa(\omega)} \sum_{i\geq 1} X_i(t)^{\omega}, \qquad t \geq 0$$

Then it follows from the temporal branching property that $M(\omega, \cdot)$ is a martingale with respective to the filtration $\mathbf{F}_t := \sigma(\mathbf{X}(s), s \leq t)$.

Let us to define a change of measure: for every $A \in \mathbf{F}_t$,

$$\mathbf{P}_x^{\omega,t}(A) := \mathbf{E}_x(M(\omega, t)\mathbb{1}_A).$$

Then the martingale property of $M(\omega, \cdot)$ ensures that $\mathbf{P}_x^{\omega}$ is indeed a probability on $\mathbf{F}_t$, with consistency: for every $s \leq t$ and $A \in \mathbf{F}_s$, there is the identity

$$\mathbf{P}_x^{\omega,t}(A) = \mathbf{P}_x^{\omega,s}(A).$$

By Kolmogorovs theorem, there exists a probability measure $\mathbf{P}_x^{\omega}$ on $\mathbf{F}_\infty$ such that $\mathbf{P}_x^{\omega}|_{\mathbf{F}_t} = \mathbf{P}_x^{\omega,t}$.

To describe the law of $\mathbf{X}$ under the new measure $\mathbf{P}_x^{\omega}$, we introduce a new cell system $\mathcal{Y} := (\mathcal{Y}_u, u \in \mathbb{U})$ constructed in follow manner.

- The Eve cell is born at the birth time $b_\varnothing = 0$, and the Eve process is given by $\mathcal{Y}_\varnothing(t) := x\mathrm{e}^{\eta(t)}$, with $x > 0$ and η a (non-killed) SNLP with Laplace exponent $\Phi^{\omega}(\cdot) := \kappa(\cdot + \omega) - \kappa(\omega)$. More precisely, the Lévy process η has characteristics $(\sigma, c_\omega, \Lambda_\omega, 0)$, where

$$c_\omega := c + \sigma^2\omega + \int_{(-\infty,0)} \Big((1-\mathrm{e}^z) - (\mathrm{e}^{\omega z}(1-\mathrm{e}^z) + (1-\mathrm{e}^z)^{\omega}\mathrm{e}^z\Big)\Lambda(\mathrm{d}z),$$

  and the Lévy measure Λ_ω on $(-\infty, 0)$ is defined such that for every bounded measurable function g on $(-\infty, 0)$ there is

$$\int_{(-\infty,0)} g(z)\Lambda_\omega(dz) = \int_{(-\infty,0)} \big(\mathrm{e}^{\omega z}g(z) + (1-\mathrm{e}^z)^{\omega}g(\log(1-\mathrm{e}^z))\big)\Lambda(\mathrm{d}z).$$

- Regeneratively, given $\mathcal{Y}_u$ and b_u, say t_i is the i-th jump of $\mathcal{Y}_u$ with $y_i := -\Delta\mathcal{Y}_u(t_i) > 0$, then $ui \in \mathbb{U}$ is born at time $b_{ui} := b_u + t_i$, and $\mathcal{Y}_{ui}$ has the law of $(y_i\mathrm{e}^{\xi(t)}, t \geq 0)$, independent of the other $\mathcal{Y}_{uj}$ with $j \neq i$.

Write $\mathcal{Q}_x$ for the law of $\mathcal{Y}$ and let $\mathbf{Y}(t) := \{\!\{\mathcal{Y}_u(t-b_u) : u \in \mathbb{U}, t \geq b_u\}\!\}$.

Theorem 2.5 (Spinal Decomposition [8, 18]). *For every $x > 0$, the process $\mathbf{Y}$ under $\mathcal{Q}_x$ has the same distribution as $\mathbf{X}$ under $\mathbf{P}_x^{\omega}$.*

This theorem has been proven in [8] for the case with finite dislocation rate; a complete proof of this theorem is given in [18, Theorem 5.2].

For every $x > 0$, the (non-negative) additive martingale $M(\omega, t)$ converges $\mathbf{P}_x$-almost surely to a limit $M(\omega, \infty)$ as $t \to \infty$. The following result shows how the uniform integrability of $M(\omega, \cdot)$ depends on the value of ω. Let $\mathbf{P}_x^*(\cdot) := \mathbf{P}_x(\cdot\ |\text{non-extinction})$, where the *non-extinction* event refers to $\{\mathbf{X}(t) \neq \emptyset$ for all $t \geq 0\}$. Note that $\mathbf{P}_x(\text{non-extinction}) = 1$ whenever $k = 0$ or $\Lambda((-\infty, 0)) = \infty$.

Theorem 2.6 ([12, Theorem 2.3]). *Let $x > 0$ and $\bar{q} := \sup\{q \geq \underline{q} : q\kappa'(q) - \kappa(q) < 0\} \in [\underline{q}, \infty]$.*

1. *If $\omega \in [\bar{q}, \infty)$, then $M(\omega, \infty)$ is $\mathbf{P}_x$-almost surely equal to zero.*
2. *If $\omega \in (\underline{q}, \bar{q})$, then $\mathbf{E}_x[M(\omega, \infty)] = 1$ and $M(\omega, \infty)$ is $\mathbf{P}_x^*$-almost surely strictly positive.* [1]

Proof. This statement was proved by Dadoun [12] via a reduction to branching random walks; here we offer a direct proof by using the spine decomposition. Let us consider the system $\mathcal{Y}$ with distribution $\mathcal{Q}_x$ and let

$$M(\omega, t) = \sum_{u \in \mathbb{U}} \mathcal{Y}_u(t - b_u) \mathbb{1}_{\{b_u \leq t < b_u + \zeta_u\}};$$

then $\mathbf{Y}$ has law $\mathbf{P}_x^{\omega}$ under $\mathcal{Q}_x$. We omit the subscript x as it is clear.

1. By a fundamental result in measure theory (see e.g. [1, Corollary 1]), it suffices to prove that

   $$\limsup_{t \to \infty} M(\omega, \infty) = \infty, \quad \mathbf{P}^{\omega}\text{-a.s.}$$

   As $\omega \geq \bar{q}$, there is $\kappa'(\omega)\omega \geq \kappa(\omega)$. Then for every $t \geq 0$, we have

   $$M(\omega, t) \geq \exp\big(-\kappa(\omega)t + \omega\eta(t)\big) \geq \exp\Big(\omega\big(\eta(t) - \kappa'(\omega)(t)\big)\Big).$$

   Since by Lemma 2.1 there is $\limsup_{t \geq 0} \frac{\eta(t) - \kappa'(\omega)t}{t} = +\infty$, the claim follows.

2. By [19, Lemma 4.2] it suffices to show that

   $$\liminf_{t \to \infty} \mathcal{Q}\big[M(\omega, t) \big| \mathcal{G}_{\infty}\big] < \infty, \quad \mathcal{Q}\text{-a.s.},$$

   where $\mathcal{G}_{\infty} := \sigma(\eta(t), t \geq 0)$.

   For every $i \geq 1$, by the independence between the sub-population $(\mathcal{Y}_{iv}, v \in \mathbb{U})$ and $\mathcal{G}_{\infty}$, we have the identity

   $$\mathcal{Q}\big[\sum_{v \in \mathbb{U}} \mathcal{Y}_{iv}(t - b_{iv})^{\omega} \big| \mathcal{G}_{\infty}\big] = \mathcal{Y}_i(0)^{\omega} \mathrm{e}^{\kappa(\omega)(t - b_i)} = \Delta\mathcal{Y}_{\varnothing}(b_i)^{\omega} \mathrm{e}^{\kappa(\omega)(t - b_i)}.$$

[1] We are in the conservative case; so the condition (2.7) in [12] is satisfied. Further, since $\omega\kappa'(\omega) < \kappa(\omega)$ and by convexity that $\kappa(\omega) - \kappa(0) \leq \omega\kappa'(\omega)$, we have $\kappa(0) \in (0, \infty]$.

We hence deduce that

$$\mathcal{Q}\big[M(\omega,t)\big|\mathcal{G}_\infty\big] = e^{-\kappa(\omega)t+\omega\eta(t)} + \mathrm{e}^{-\kappa(\omega)t}\sum_{i\geq 1}\mathcal{Q}\big[\sum_{v\in\mathbb{U}}\mathcal{Y}_{iv}(t-b_{iv})^{\omega}\big|\mathcal{G}_\infty\big]$$
$$= e^{-\kappa(\omega)t+\omega\eta(t)} + \sum_{0<s\leq t} e^{-\kappa(\omega)s}\mathrm{e}^{\omega\eta(s-)}(1-\mathrm{e}^{\Delta\eta(s)})^{\omega}.$$

Let $Y_n := \sum_{n-1<s\leq n}\mathrm{e}^{-\kappa(\omega)(s-(n-1))}\mathrm{e}^{\omega[\eta(s-)-\eta(n-1)]}(1-\mathrm{e}^{\Delta\eta(s)})^{\omega}$. By the definition of Lévy processes, $(Y_n)_{n\geq 0}$ are i.i.d. random variables, with finite mean value

$$\mathrm{E}[Y_1] = \int_0^1 \mathrm{e}^{(\kappa(2\omega)-2\kappa(\omega))s}\mathrm{d}s\int_{(-\infty,0)}(1-\mathrm{e}^{z})^{\omega}\Lambda(\mathrm{d}z) < \infty.$$

For any interger $n > t$, observe that

$$\mathcal{Q}\big[M(\omega,t)\big|\mathcal{G}_\infty\big] \leq e^{-\kappa(\omega)t+\omega\eta(t)} + Y_1 + \sum_{i=1}^{n-1}\exp\Big(i\big(-\kappa(\omega)+\omega\frac{\eta(i)}{i}+\frac{\log Y_{i+1}}{i}\big)\Big).$$

By the law of large numbers we have $\frac{\log Y_{i+1}}{i}\to 0$ a.s. We also have by Lemma 2.1 that $\omega\frac{\eta(i)}{i}\to\omega\kappa'(\omega) < \kappa(\omega)$ a.s. Then the claim follows.

□

3 Self-similar growth-fragmentation processes

3.1 Lamperti transform

To construct self-similar growth-fragmentations, we first provide some background [14, Sec 13] on positive self-similar Markov processes and Lamperti's representation of the latter.

Let $\alpha\in\mathbb{R}$ and ξ be a SNLP. We define a time-change by

$$\tau^{(\alpha)}(t) := \inf\left\{r\geq 0 \ : \ \int_0^r \exp(-\alpha\xi(s))\mathrm{d}s \geq t\right\},\quad t\geq 0.$$

For every $x>0$, denote by P_x the law of the process

$$X^{(\alpha)}(t) := x\exp\Big(\xi\big(\tau^{(\alpha)}(x^{\alpha}t)\big)\Big),\quad t\geq 0, \tag{3.1}$$

with convention $X^{(\alpha)}(t) := \partial$ whenever

$$t\geq\zeta^{(\alpha)} := x^{-\alpha}\int_0^\infty \exp(-\alpha\xi(s))\mathrm{d}s.$$

For every $\gamma>0$, one can deduce from (3.1) that

the law of $(\gamma X^{(\alpha)}(\gamma^{\alpha}t), t\geq 0)$ under P_x is $\mathrm{P}_{\gamma x}$.

So we call $X^{(\alpha)}$ a *positive self-similar Markov process (pssMp) with index α*. If the SNLP ξ has characteristics (σ,c,Λ,k) and Laplace exponent Φ as in (2.1), then we say that $X^{(\alpha)}$ has characteristics $(\sigma,c,\Lambda,k,\alpha)$, or simply

(Φ, α). Using (3.1) we also have the following identity:

$$X^{(\alpha)}(t) = X^{(0)}(T^{(\alpha)}(t)) = X^{(0)}\Big(\int_0^t X^{(\alpha)}(s)^{\alpha}\mathrm{d}s\Big), \qquad t \geq 0,$$

where $T^{(\alpha)}(t) := \inf\big\{r \geq 0 \ : \ \int_0^r X^{(0)}(s)^{-\alpha}\mathrm{d}s \geq t\big\}$. Note that for every fixed t, $T^{(\alpha)}(t)$ is a stopping time with respect to the filtration $F_t := \sigma(\xi(s), s \leq t) = \sigma(X^{(0)}(s), s \leq t)$.

Let us gather a few useful results on pssMps.

Lemma 3.1. *Let $X^{(\alpha)}$ be a pssMp with characteristics (Φ, α). Then the process $\tilde{X} := (X^{(\alpha)})^{-1}$ is a pssMp with characteristics $(-\Phi, -\alpha)$.*

Proof.

$$\tilde{X}(t) = (X^{(\alpha)}(t))^{-1} = x^{-1}\exp\Big(-\xi\big(\tau^{(\alpha)}(tx^{\alpha})\big)\Big) = x^{-1}\exp\Big(-\xi\big(\tilde{\tau}^{(-\alpha)}(t(x^{-1})^{-\alpha})\big)\Big),$$

where $\tilde{\tau}^{(-\alpha)}(t) := \inf\big\{r \geq 0 \ : \ \int_0^r \exp(-(-\alpha)(-\xi)(s))\mathrm{d}s \geq t\big\}$. □

Lemma 3.2. *Suppose that $\Phi(q) < 0$, then*

$$\mathrm{E}_x\Big[\int_0^{\zeta^{(\alpha)}} (X^{(\alpha)}(t))^{q+\alpha}\mathrm{d}t\Big] = -x^q\frac{1}{\Phi(q)};$$

otherwise, the above expected value is ∞.

Proof. Using Lamperti transform, we have

$$\begin{aligned}
\mathrm{E}_x\Big[\int_0^{\zeta} (X^{(\alpha)}(t))^{q+\alpha}\mathrm{d}t\Big] &= \mathrm{E}\Big[\int_0^{\infty} x^{q+\alpha}\exp\big((q+\alpha)\xi(\tau(x^{\alpha}t))\big)\mathrm{d}t\Big] \\
&= \mathrm{E}\Big[\int_0^{\infty} x^{q}\exp\big((q+\alpha)\xi(\tau(s))\big)\mathrm{d}s\Big] \\
&= \mathrm{E}\Big[\int_0^{\infty} x^{q}\exp\big(q\xi(r)\big)\mathrm{d}r\Big] \\
&= \int_0^{\infty} x^{q}\exp\big(\Phi(q)r\big)\mathrm{d}r.
\end{aligned}$$

□

Lemma 3.3 ([9, Theorem 1]). *Let $X^{(\alpha)}$ be a pssMp with characteristics (Φ, α). Suppose that ξ is not arithmetic (i.e. there is not $r > 0$ such that $P(\xi(t) \in r\mathbb{Z})$ for all $t \geq 0$). If $\alpha < 0$, $\Phi(0) = 0$, and $\Phi'(0+) \in (0, \infty)$, then as $x \to 0+$, the probability measure $P_x^{(\alpha)}$ converge in the sense of finite-dimensional distributions to a probability measure denoted by $P_0^{(\alpha)}$. The process Y under $P_0^{(\alpha)}$ is a self-similar process with càdlàg path and no positive jumps, and $\lim_{t\to\infty} Y(t) = +\infty$, $P_0^{(\alpha)}$ a.s. Further, under $P_0^{(\alpha)}$ there is*

$$\mathrm{E}_0[f(Y(t)^{\alpha})] = -\frac{1}{\alpha\Phi'(0+)}\mathrm{E}[I^{-1}f(t/I)],$$

where $I := \int_0^{\infty} \mathrm{e}^{\alpha\xi(s)}\mathrm{d}s$.

The following statement is a useful corollary of Lemma 3.3.

Lemma 3.4. *Let $X^{(\alpha)}$ be a pssMp with characteristics (Φ,α), associated with ξ. Suppose that $\alpha>0$, $\Phi(0)=0$, and $\Phi'(0+)\in(-\infty,0)$. As $t\to\infty$, the random variable $t^{\frac{1}{\alpha}}X^{(\alpha)}(t)$ converges in distribution to a law ρ given by*

$$\int_{(0,\infty)} f(y)\rho(\mathrm{d}y) := -\frac{1}{\alpha\Phi'(0+)}\mathrm{E}[I^{-1}f(I^{\frac{1}{\alpha}})],$$

where $I:=\int_0^\infty \mathrm{e}^{\alpha\xi(s)}\mathrm{d}s$.

Proof. Applying Lemma 3.3 to $(X^{(\alpha)})^{-1}$ gives the result. □

3.2 Self-similar growth-fragmentation processes and excessive functions

Definition 3.5. *Let $X^{(\alpha)}$ be a pssMp with characteristics (Φ,α), $\mathcal{X}^{(\alpha)}$ be a cell system generated by X, and $\mathbf{X}^{(\alpha)}$ be its associated growth-fragmentation. Then we call $\mathbf{X}^{(\alpha)}$ a* ***self-similar growth-fragmentation process*** *driven by $X^{(\alpha)}$.*

Recall that the cumulant κ of ξ is defined by (2.3), then κ is also called the *cumulant* of the pssMp X.

Theorem 3.6 ([6, Theorem 2]). *Suppose that*

$$\text{there exists } q>0 \text{ such that } \kappa(q)\le 0. \tag{3.2}$$

Then

$$\mathbf{E}_x\Big[\sum_{y\in\mathbf{X}^{(\alpha)}(t)} y^q\Big] = \mathcal{E}_x\Big[\sum_{u\in\mathbb{U}}\mathcal{X}_u^{(\alpha)}(t-b_u)^q\Big]\le x^q, \qquad \text{for all } t\ge 0.$$

We say the function $y\mapsto y^q$ is excessive *for $\mathbf{X}^{(\alpha)}$.*

Proof of Theorem 3.6. We first prove that for every $t\ge 0$

$$\mathrm{E}_x\Big[X^{(\alpha)}(t)^q+\sum_{0\le s\le t}|\Delta X^{(\alpha)}(s)|^q\Big]\le x^q.$$

Let us start with the case $\alpha=0$, where we have $X^{(0)}(t)=x\mathrm{e}^{\xi(t)}$. Since $(t,\Delta\xi(t))_{t\ge 0}$ is a Poisson random measure with intensity $\mathrm{d}t\otimes\Lambda(\mathrm{d}z)$, the compensation formula leads to

$$\mathrm{E}_x\Big[\sum_{0\le s\le t}|\Delta X^{(0)}(s)|^q\Big]=\mathrm{E}_x\Big[\sum_{0\le s\le t}\mathrm{e}^{q\xi(s-)}(1-\mathrm{e}^{\Delta\xi(s)})\Big]=\int_0^t\Big(\mathrm{E}_x[\mathrm{e}^{q\xi(s-)}]\int_{(-\infty,0)}(1-\mathrm{e}^z)\Lambda(\mathrm{d}z)\Big)\mathrm{d}s.$$

An easy calculation leads to

$$\mathrm{E}_x\Big[X^{(0)}(t)^q+\sum_{0\le s\le t}|\Delta X^{(0)}(s)|^q\Big]=\Big(1-\frac{\kappa(q)}{\Phi(q)}(1-\mathrm{e}^{\Phi(q)t})\Big)x^q\le x^q. \tag{3.3}$$

By the Markov property of the process $X^{(0)}$, we hence know that $X^{(0)}(t)^q+\sum_{0\le s\le t}|\Delta X^{(0)}(s)|^q$ is a supermartingale with respect to the filtration $F_t:=\sigma(\xi(s),s\le t)$. Finally, for $\alpha\neq 0$, recall that the Lamperti time-change $T^{(\alpha)}(t)$ is a F_t-stopping time, using the optional stopping theorem (for supermartingales) yields

$$\mathrm{E}_x\Big[X^{(\alpha)}(t)^q+\sum_{0\le s\le t}|\Delta X^{(\alpha)}(s)|^q\Big]=\mathrm{E}_x\Big[X^{(0)}(T^{(\alpha)}(t))^q+\sum_{0\le s\le T^{(\alpha)}(t)}|\Delta X^{(0)}(s)|^q\Big]\le x^q.$$

The the claim follows from Theorem 1.4. □

Proposition 3.7 ([6, Theorem 2]). *Suppose that (3.2) holds. Then $\mathbf{X}^{(\alpha)}$ satisfies the branching property (P1) and the following self-similarity: For $x > 0$, the re-scaled process $(x\mathbf{X}^{(\alpha)}(x^\alpha t))_{t\geq 0}$ under $\mathbf{P}_1$ has distribution $\mathbf{P}_x$.*

Proof. The proof of temporal branching property is similar as in the homogeneous case. For the self-similarity, consider the system

$$\mathcal{X}'_u(t) := x\mathcal{X}_u(x^\alpha t), \quad b'_u := x^{-\alpha} b_u, \qquad u \in \mathbb{U}, t \geq 0.$$

Then by the self-similarity of $\mathcal{X}_\varnothing$, the process $\mathcal{X}'_\varnothing$ has law P_x. The jump sequence of $\mathcal{X}'_\varnothing$ is given by $(b'_j)_{j\geq 1}$, and there is the identity $-\Delta\mathcal{X}'_\varnothing(b_j) = \mathcal{X}'_j(0)$. So the law of $(\mathcal{X}'_u, b'_u)_{|u|=1}$ under $\mathcal{P}_1$ has the law of $(\mathcal{X}_u, b_u)_{|u|=1}$ under $\mathcal{P}_x$. By iterating this argument we complete the proof. □

3.3 Extinction

Let $\mathcal{X}^{(0)}$ be a homogeneous cell system associated with the cell process $X^{(0)}(t) = x\mathrm{e}^{\xi(t)}$ and $\mathcal{P}_x^{(0)}$ be its distribution. We can construct all cell systems $(\mathcal{X}^{(\alpha)}, \alpha \in \mathbb{R})$ in the same probability space as $\mathcal{X}^{(0)}$ by using the Lamperti transforms. Specifically, for every $u \in \mathbb{U}$, let

$$\mathcal{X}_u^{(\alpha)}(t) := \mathcal{X}_u^{(0)}(\tau_u^{(\alpha)}(t)),$$

with $\tau_u^{(\alpha)}(t) := \inf\left\{r \geq 0 : \int_0^r \mathcal{X}_u^{(0)}(s)^{-\alpha}\mathrm{d}s \geq t\right\}$.

Theorem 3.8 ([6, Corollary 3]). *Suppose that $\alpha < 0$ and that $\kappa(q) < 0$ for some $q > 0$. Then the extinction time*

$$\inf\{t \geq 0 : \mathbf{X}^{(\alpha)}(t) = \emptyset\}$$

is $\mathbf{P}_x$-a.s. finite for every $x > 0$.

Proof. For every $u \in \mathbb{U}$ and $i \leq |u|$, let $u_i \in \mathbb{N}^i$ be the ancestor of u at the i-th generation. Consider the ancestral lineage of u:

$$\mathcal{Y}_u^{(\alpha)}(t) := \mathcal{X}_{u_i}^{(\alpha)}(t - b_{u_i}), \qquad t \in [b_{u_i}, b_{u_i} + \zeta_{u_i}) \text{ for } i \leq |u|.$$

By convention $\mathcal{Y}_u^{(\alpha)}(t) := \partial$ for $t \geq b_u + \zeta_u$. We see that $\mathcal{Y}^{(\alpha)}$ is also related to $\mathcal{Y}^{(0)}$ by the Lamperti transformation:

$$\mathcal{Y}_u^{(\alpha)}(t) = \mathcal{Y}_u^{(0)}(T_u^{(\alpha)}(t)),$$

with $T_u^{(\alpha)}(t) := \inf\left\{r \geq 0 : \int_0^r \mathcal{Y}_u^{(0)}(s)^{-\alpha}\mathrm{d}s \geq t\right\}$. In particular, we have the identity

$$b_u^{(\alpha)} + \zeta_u^{(\alpha)} = \int_0^\infty \mathcal{Y}_u^{(0)}(s)^{-\alpha}\mathrm{d}s.$$

Let $X_1^{(0)}(t) = \max\{\mathcal{X}_u(t - b_u), u \in \mathbb{U}\} = \max\{\mathcal{Y}_u(t), u \in \mathbb{U}\}$, and $C := \sup\left\{\mathrm{e}^{-\kappa(q)t}X_1^{(0)}(t)^q : \ t \geq 0\right\}$. Then $C < \infty$ by the convergence of the martingale $M(q,t)$ (Theorem 2.6). We hence have

$$\int_0^\infty \mathcal{Y}_u^{(0)}(s)^{-\alpha}\mathrm{d}s \leq \int_0^\infty C^{-\frac{\alpha}{q}}\mathrm{e}^{-\alpha\frac{\kappa(q)}{q}s}\mathrm{d}s = C^{-\alpha}\frac{q}{\alpha\kappa(q)} < \infty,$$

which entails that

$$\inf\{t \geq 0 : \mathbf{X}^{(\alpha)}(t) = \emptyset\} = \sup\left\{b_u^{(\alpha)} + \zeta_u^{(\alpha)} : \ u \in \mathbb{U}\right\} < C^{-\frac{\alpha}{q}}\frac{q}{\alpha\kappa(q)}.$$

We complete the proof. ◻

3.4 Local explosion

When (3.2) is not satisfied, the self-similar growth-fragmentation $\mathbf{X}^{(\alpha)}$ with index $\alpha \neq 0$ explodes in finite time.

Theorem 3.9 ([8, Theorem 2.3]). *Suppose that $\alpha \neq 0$ and that*

$$\kappa(q) > 0, \text{ for every } q \geq 0. \tag{3.4}$$

Then for every $0 < a < a'$, there exists almost surely a random time $T > 0$, such that $\mathbf{X}^{(\alpha)}(T)$ has infinitely many elements in the interval (a, a').

The proof of Theorem 3.9 is based on the following two lemmas.

Lemma 3.10. *Under condition (3.4), κ reaches its minimum of $[0, \infty)$, at $q_m \in (0, \infty)$.*

Lemma 3.11 ([8, Lemma 3.5]).

1. *Suppose that $\alpha < 0$. Fix $0 < a < a'$, then there exists $0 < t < t'$ such that*
$$\liminf_{x \to 0+} \mathbf{P}_x \left(\mathbf{X}^{(\alpha)}(r) \cap (a, a') \neq \emptyset \text{ for all } t \leq r \leq t' \right) > 0$$
2. *Suppose that $\alpha > 0$. Fix $0 < a < a'$, then there exists $0 < t < t'$ such that*
$$\liminf_{x \to \infty} \mathbf{P}_x \left(\mathbf{X}^{(\alpha)}(r) \cap (a, a') \neq \emptyset \text{ for all } t \leq r \leq t' \right) > 0$$

Proof of Theorem 3.9. We first prove for the case $\alpha < 0$.

Let us consider the minimum q_m as in Lemma 3.10. There is $\kappa'(q_m) = 0 < \frac{\kappa(q_m)}{q_m}$, which infers that $q_m \in (\underline{q}, \bar{q})$ ($\bar{q}$ is defined in Theorem 2.6). Then can choose $q_- < q_m < q_+$, such that $q_-, q_+ \in (\underline{q}, \bar{q})$, and $\kappa'(q_-) < 0 < \kappa'(q_+)$. Let $\mathbf{P}_x^-$ and $\mathbf{P}_x^+$ be the measures defined as in Theorem 2.5 by using $M(q_-, \cdot)$ and $M(q_+, \cdot)$ respectively. By Theorem 2.6, both these two martingales are uniform integrable, so we have

$$\mathbf{P}_x^-(A) = \mathbf{E}_x[M(q_-, \infty)\mathbb{1}_A], \quad \forall A \in \sigma(\mathbf{X}(t), t \geq 0),$$

and a similar result for $\mathbf{P}_x^+$. We also know from Theorem 2.6 that $M(q_-, \infty)$ is strictly positive conditionally on no sudden death, hence the law of $\mathbf{X}^{(\alpha)}$ under $\mathbf{P}_x$ conditionally on no sudden death is equivalent to that under $\mathbf{P}_x^-$.

Let $\mathcal{Y}$ be a cell system as in Theorem 2.5, and $\mathcal{Y}^{(\alpha)}$ be the Lamperti transform of $\mathcal{Y}$. Let $\mathbf{Y}^{(\alpha)}$ be its associated growth-fragmentation. Then under $\mathbf{P}_x^-$, $\mathbf{X}^{(\alpha)}$ has the same distribution as $\mathbf{Y}^{(\alpha)}$.

Let us consider $\mathcal{Y}^{(\alpha)}$. The Eve process $\mathcal{Y}_\varnothing^{(\alpha)}$ is a pssMp with characteristics (Φ_-, α), where $\Phi_-(\cdot) := \kappa(\cdot + q_-) - \kappa(q_-)$. Since $\alpha < 0$ and $\Phi_-'(0+) < 0$, we have $\zeta_\varnothing^{(\alpha)} < 0$ and $\mathcal{Y}_\varnothing^{(\alpha)}(\zeta_\varnothing^{(\alpha)}-) = 0$. So there exists a subsequence of jump times of $\mathcal{Y}_\varnothing^{(\alpha)}$, denoted by $(t_{k_i})_{i \geq 1}$, such that

$$\lim_{i \to \infty} t_{k_i} = \zeta_\varnothing^{(\alpha)} \text{ and } \lim_{i \to \infty} \Delta \mathcal{Y}_\varnothing^{(\alpha)}(t_{k_i}) = 0.$$

Next, consider the sub-populations $\mathcal{X}_{k_i}^{(\alpha)}$ generated at (t_{k_i}). For every $0 < a < a'$, using Lemma 3.11 and the Borel-Cantelli lemma, conditionally on $\mathcal{Y}_\varnothing^{(\alpha)}$, almost surely there exists infinitely many i such that $\mathbf{X}_{k_i}^{(\alpha)}(r) \cap (a, a') \neq \emptyset$ for all $r \in [t, t']$. Then for any $r \in (t, t')$, the multiset $\mathbf{X}^{(\alpha)}(\zeta_\varnothing^{(\alpha)} + r)$ has infinitely many elements in (a, a'), $\mathbf{P}_x^-$ almost surely. The equivalence between $\mathbf{P}_x^-$ and $\mathbf{P}_x$ conditionally on no sudden death completes the proof (for the case $\alpha < 0$).

To prove for the case $\alpha > 0$, we just consider $\mathbf{P}^+$ and use the very similar arguments. □

We finally prove the two lemmas. [TODO]

4 Martingales in self-similar growth-fragmentations

The object of this section is to establish two temporal martingales for a self-similar growth-fragmentations. We fix $\alpha \in \mathbb{R}$ in this section and omit the superscript $\cdot^{(\alpha)}$ for simplicity.

4.1 The genealogical martingales

We observe that $\mathcal{Z} := (\mathcal{Z}_u := -\log \mathcal{X}_u(0), u \in \mathbb{U})$ is a branching random walk, in the sense that for any $u \in \mathbb{U}$, there is $(\mathcal{Z}_{ui} - \mathcal{Z}_u)_{i\geq 1} \overset{d}{=} (\mathcal{Z}_i - \mathcal{Z}_\varnothing)_{i\geq 1}$. The Laplace transform of the point process of the first generation is

$$m(q) := x^{-q}\mathcal{E}_x\Big[\sum_{|u|=1} \mathrm{e}^{-q(-\log \mathcal{X}_u(0))}\Big] = \mathrm{E}_1\Big[\sum_{0<s<\zeta} |\Delta X(s)|^q\Big] = 1 - \frac{\kappa(q)}{\Phi(q)}, \qquad \text{if } \Phi(q) < 0 \text{ and } \kappa(q) < \infty. \quad (4.1)$$

where the last equality is obtained by a similar calculation as (3.3). By the branching random walk property,

$$\mathcal{M}(q,n) := x^{-q} m(q)^{-n} \sum_{|u|=n} \mathrm{e}^{-q\mathcal{Z}_u} = x^{-q} m(q)^{-n} \mathcal{X}_u(0)^q, \quad n \geq 1$$

is a martingale, known as the *additive martingale* of the branching random walk $\mathcal{Z}$. We recall two useful results for general branching random walks.

Lemma 4.1 ([10, Theorem A], [11, Theorem 1]). *Let $\mathcal{M}(q,\cdot)$ be the additive martingale of a certain branching random walk.*

1. *We have $\mathbb{E}[\mathcal{M}(q,\infty)] = 1$ if and only if*
$$\mathbb{E}[\mathcal{M}(q,1)\log^+ \mathcal{M}(q,1)] < \infty \text{ and } m(q)\exp(-qm'(q)/m(q)) > 1. \quad (4.2)$$

2. *Suppose that there exists $\gamma \in (1,2]$, such that $\mathbb{E}[\mathcal{M}(q,1)^\gamma] < \infty$, and that for some $\theta \in (1,\gamma]$, there is $\frac{m(\theta q)}{m(q)^\theta} < 1$. Note that this condition entails (4.2). Then $\mathcal{M}(q,\infty)$ converges a.s. and in L^θ.*

Lemma 4.2. *Suppose that*

(H1) *there exists $\omega_+ > 0$ such that $\kappa(\omega_+) = 0$ and $\kappa'(\omega_+) > 0$.*

Then $m(\omega_+) = 1$, and the martingale

$$\mathcal{M}^+(n) := \mathcal{M}(\omega_+, n) = x^{-\omega_+} \sum_{|u|=n} \mathrm{e}^{-\omega_+\mathcal{Z}_u}$$

converges $\mathcal{P}_x$-a.s. to 0.

Proof. Since $\Phi(\omega_+) < \kappa(\omega_+) < 0$ and $\kappa'(\omega_+) > 0$, we have $m'(\omega_+) = -\frac{\kappa'(\omega_+)}{\Phi(\omega_+)} > 0$. Then

$$m(\omega_+)\exp(-\omega_+ m'(\omega_+)/m(\omega_+)) < 1,$$

and thus $\mathcal{M}^+$ converges $\mathcal{P}_x$-a.s. to 0. □

Lemma 4.3. *Suppose that* (H1) *holds and that*

(H2) *(Cramér's condition) there exists* $\omega_- \in (0, \omega_+)$ *such that* $\kappa(\omega_-) = 0$ *and* $\kappa'(\omega_-) > -\infty$.

Then for any $p \in [1, \frac{\omega_+}{\omega_-})$*, the martingale*

$$\mathcal{M}^-(n) := \mathcal{M}(\omega_-, n) = x^{-\omega_-} \sum_{|u|=n} \mathrm{e}^{-\omega_- \mathcal{Z}_u}$$

converges $\mathcal{P}_x$*-a.s. and in* $L^p(\mathcal{P}_x)$*. Its terminal value* $\mathcal{M}^-(\infty)$ *is* $\mathcal{P}^*$*-a.s. strictly positive.*

Note that if (3.2) holds and ξ is not the negative of a subordinator, then (H1) is satisfied.

Proof. We first check that $\mathcal{M}^-(1)$ is in L^{ω_+/ω_-}. By the Lamperti transform, it is clear that we only need to prove for the case $\alpha = 0$, i.e. $X(t) = x\mathrm{e}^{\xi(t)}$.

Using the compensation formula, we find that the following process is a F_t-martingale:

$$\sum_{0<s\leq t} |\Delta X(s)|^{\omega_-} + \Phi(\omega_-) \int_0^t X^{\omega_-}(s)\mathrm{d}s.$$

This martingale is purely discontinuous with quadratic variation $\sum_{0<s\leq t} |\Delta X(s)|^{2\omega_-}$. We further have by [16, Lemma 3.1] that $\int_0^\infty e^{q\xi(t)}dt$ is in $L^{\omega_+/q}$ for every $q \in (0, \omega_+]$, since $\mathrm{E}(\mathrm{e}^{\omega_+\xi(1)}) = \mathrm{e}^{\Phi(\omega_+)} < 1$. By this observation and the Burkholder-Davis-Gundy inequality, in order to prove $\mathcal{M}^-(1)$ is in L^{ω_+/ω_-}, we only need to check that $\sum_{0<s<\infty} |\Delta X(s)|^{2\omega_-}$ is in $L^{\omega_+/2\omega_-}$. Applying the same arguments as above to the martingale

$$\sum_{0<s\leq t} |\Delta X(s)|^{2\omega_-} + \Phi(2\omega_-) \int_0^t X^{2\omega_-}(s)\mathrm{d}s,$$

then it suffices to prove that $\sum_{0<s<\infty} |\Delta X(s)|^{4\omega_-}$ is in $L^{\omega_+/4\omega_-}$. By iteration, it suffices to prove that there exists $k \geq 1$ that $\sum_{0<s<\infty} |\Delta X(s)|^{2^k\omega_-}$ is in $L^{\omega_+/(2^k\omega_-)}$.

Choose k large enough such that $\omega_+/2^k\omega_- \leq 1$, then Jensen's inequality infers that

$$\left(\sum_{0<s<\infty} |\Delta X(s)|^{2^k\omega_-} \right)^{\omega_+/2^k\omega_-} \leq \sum_{0<s<\infty} |\Delta X(s)|^{\omega_+}.$$

By (4.1), the right-hand-side has finite mean value. The claim is proven. □

4.2 Many-to-one formula

For the self-similar case with index $\alpha \neq 0$, there does not exist a cumulant in the sense of (2.4). Nevertheless, to describe the mean value of the particles, one can use an *one particle picture* developed in [7], which extends Corollary 2 in [3] for self-similar (pure) fragmentations. Suppose that (H1) holds and let

$$\Phi^+(q) := \kappa(q + \omega_+), \qquad q \geq 0.$$

It is known that there exists a self-similar Markov process Y^+ with characteristics (Φ^+, α).

Theorem 4.4 (Many-to-one formula [7, Theorem 3.5]). *Let g be a non-negative measurable function on $(0,\infty)\cup\{\partial\}$ with $g(\partial)=0$. Then for every $x>0$ and $t\geq 0$, there is the identity*

$$\mathbf{E}_x\left[\sum_{i=1}^{\infty} g(X_i(t))X_i(t)^{\omega_+}\right] = x^{\omega_+}E_x\left[g(Y^+(t))\right],$$

where E_x denotes the mathematical expectation under the law of Y^+ started from $Y^+(0)=x$. Further, similar result holds for ω_-.

To prove the theorem we need the following formula.

Lemma 4.5. *Suppose that $\kappa(q)<0$. Then*

$$\mathbf{E}_x\Big[\int_0^{\infty}\Big(\sum_{i=1}^{\infty}X_i(t)^{q+\alpha}\Big)\mathrm{d}t\Big] = -\frac{1}{\kappa(q)}x^q.$$

Proof. By first using Lemma 3.2 and then (4.1), we have

$$\begin{aligned}\mathbf{E}_x\Big[\int_0^{\infty}\Big(\sum_{i=1}^{\infty}X_i(t)^{q+\alpha}\Big)\mathrm{d}t\Big] &= \mathcal{E}_x\Big[\sum_{u\in\mathbb{U}}\int_0^{\zeta_u}\mathcal{X}_u(t)^{q+\alpha}\mathrm{d}t\Big]\\ &= \mathcal{E}_x\Big[\sum_{u\in\mathbb{U}}\mathcal{X}_u(0)^q\frac{-1}{\Phi(q)}\Big]\\ &= \frac{-1}{\Phi(q)}x^q\sum_{n=0}^{\infty}(1-\frac{\kappa(q)}{\Phi(q)})^n.\end{aligned}$$

The series is convergent whenever $\kappa(q)<0$ (then $\Phi(q)<\kappa(q)<0$), and the sum is $\frac{\Phi(q)}{\kappa(q)}$, which leads to the desire result. □

Proof of Theorem 4.4. For any $\theta>0$ such that $\kappa(\theta)<0$, let $\tilde{\Phi}(q):=\kappa(q+\theta)$, for all $q\geq 0$. Then $\tilde{\Phi}$ is the Laplace exponent of a killed Lévy process with killing rate $-\kappa(\theta)>0$. Define

$$\langle\tilde{\rho}_t(x,\cdot),f\rangle := x^{-\theta}\mathbf{E}_x\left[\sum_{i=1}^{\infty}f(X_i(t))X_i(t)^{\theta}\right].$$

We shall prove that $\tilde{\rho}_t$ is the transition kernel of a pssMp with characteristics $(\tilde{\Phi},\alpha)$. We first notice that the Chapman-Kolmogorov equation holds:

$$\langle\tilde{\rho}_{t+s}(x,\cdot),f\rangle = \int_{(0,\infty)}\langle\tilde{\rho}_t(y,\cdot),f\rangle\,\tilde{\rho}_s(x,\mathrm{d}y).$$

So $\tilde{\rho}_t$ is a sub-probability Markovian kernel. Next, we have the self-similarity:

$$\langle\tilde{\rho}_t(x,\cdot),f\rangle = \langle\tilde{\rho}_{x^{\alpha}t}(1,\cdot),f(x\cdot)\rangle.$$

Further, for every q such that $\tilde{\Phi}(q)=\kappa(q+\theta)<0$, by Lemma 4.5 there is

$$\int_0^{\infty}\mathrm{d}t\int_{(0,\infty)}y^{q+\alpha}\tilde{\rho}_t(1,\mathrm{d}y) = x^{-\theta}\mathbf{E}_1\Big[\int_0^{\infty}\Big(\sum_{i=1}^{\infty}X_i(t)^{\theta+q+\alpha}\Big)\mathrm{d}t\Big] = -\frac{1}{\kappa(\theta+q)} = -\frac{1}{\tilde{\Phi}(q)}.$$

Then it follows essentially from Lamperti's characterization [15] that $\tilde{\rho}_t$ is the transition kernel of a pssMp with characteristics $(\tilde{\Phi},\alpha)$; see [7, Lemma 3.6] for details.

We end the proof by the change of measure: since $\tilde{\Phi}(q) = \Phi^+(q + \theta - \omega_+)$, there is the identity

$$\langle \tilde{\rho}_t(x, \cdot), f\rangle = \mathrm{E}[f(Y^+(t))(Y^+(t))^{\theta-\omega_+}],$$

which induces the desired many-to-one formula. □

4.3 Two temporal martingales

Theorem 4.6 ([7, Corollary 3.7]). *Suppose that (H1) and (H2) hold.*

1. *If $\alpha \le 0$, then under $\mathbf{P}_x$,*

$$M^+(t) := x^{-\omega_+}\sum_{i=1}^{\infty} X_i(t)^{\omega_+}, \qquad t \ge 0$$

is a martingale with respect to $\mathbf{F}_t := \sigma(\mathbf{X}(s), 0 \le s \le t)$.

2. *If $\alpha > 0$, then M^+ is a supermartingale with*

$$\mathbf{E}_x\Big[M^+(t)\Big] \sim c t^{-(\omega_+-\omega_-)/\alpha} x^{\omega_+-\omega_-}, \qquad \textit{as } t \to \infty.$$

Proof. Applying the many-to-one formula entails that

$$\mathbf{E}_x[M^+(t)] = \mathrm{P}_x(Y^+(t) \in (0,\infty)) = \mathrm{P}_x(\zeta^+ > t) = \mathrm{P}\left(x^{-\alpha}\int_0^{\infty} \exp(-\alpha\xi^+(s))\mathrm{d}s > t\right),$$

where ξ^+ is a SNLP with Laplace exponent $\kappa(\cdot + \omega_+)$. As $\frac{\xi^+(t)}{t} \to \kappa'(\omega_+) > 0$, if $\alpha \le 0$ then the integral is almost surely infinite, which yields $\mathbf{E}_x[M^+(t)] = 1$. This is enough to deduce that M^+ is a martingale by the temporal branching property.

On the other hand, when $\alpha > 0$, since

$$\mathrm{E}[\mathrm{e}^{(\omega_- -\omega_+)\eta^+(1)}] = 1 \text{ and } \mathrm{E}[|\eta^+(1)|\mathrm{e}^{(\omega_- -\omega_+)\eta^+(1)}] = \mathrm{E}[|\eta^-(1)|] < \infty,$$

then results in [16] entail that

$$\mathrm{P}\left(\int_0^{\infty} \exp(-\alpha\xi^+(s))\mathrm{d}s > t\right) \sim c t^{-(\omega_+-\omega_-)/\alpha}.$$

We complete the proof. □

Theorem 4.7 ([7, Corollary 3.9, Theorem 3.10]). *Suppose that (H1) and (H2) hold.*

1. *If $\alpha \ge 0$, then*

$$M^-(t) := x^{-\omega_-}\sum_{i=1}^{\infty} X_i(t)^{\omega_-}, \qquad t \ge 0$$

is a uniformly integrable $\mathbf{F}_t$-martingale under $\mathbf{P}_x$, and M^- is bounded in $L^p(\mathbf{P}_x)$ for any $p \in [1, \frac{\omega_+}{\omega_-})$. Further, there is $M^-(\infty) = \mathcal{M}^-(\infty)$ under $\mathcal{P}_x$.

2. *If $\alpha < 0$, then M^- is a supermartingale with*

$$\mathbf{E}_x\Big[M^-(t)\Big] \sim c' t^{(\omega_+-\omega_-)/\alpha} x^{\omega_+-\omega_-}, \qquad \textit{as } t \to \infty.$$

Proof. We shall only prove that when $\alpha \geq 0$, there is $M^-(\infty) = \mathcal{M}^-(\infty)$ under $\mathcal{P}_x$. Then M^- is bounded in $L^p(\mathbf{P}_x)$ for any $p \in [1, \frac{\omega_+}{\omega_-})$ by Theorem 2.6. The other parts can be deduced by similar arguments as in M^+ case.

Let us introduce

$$\bar{\mathbf{X}}(t) := \{\!\{(\mathcal{X}_u(t - b_u), |u|) \ : \ u \in \mathbb{U}, b_u \leq t < b_u + \zeta_u\}\!\},$$

and define a new filtration $\bar{\mathbf{F}}_t := \sigma(\bar{\mathbf{X}}(s), s \leq t)$. Then a variation of the branching property still holds [TODO] ; see [7, Lemma 3.2] for details. By this branching property and (4.1), we have that

$$\mathcal{P}_x\Big[\sum_{|u|=n} \mathbb{1}_{\{b_u \leq t\}} \sum_{t \leq s < b_u + \zeta_u} |\Delta \mathcal{X}_u(s - b_u)|^{\omega_-} |\bar{\mathbf{F}}_t\Big] = \sum_{|v| \leq n} \mathbb{1}_{\{b_v \leq t < b_v + \zeta_v\}} \mathcal{X}_u(t - b_v)^{\omega_-}.$$

Then we have by the cell system construction that

$$\mathcal{P}_x[\mathcal{M}^-(n)|\bar{\mathbf{F}}_t] = \mathcal{P}_x\Big[\sum_{|u|=n+1} \mathcal{X}_u(0)^{\omega_-} |\bar{\mathbf{F}}_t\Big] = \sum_{|u|=n+1} \mathbb{1}_{\{b_u \leq t\}} \mathcal{X}_u(0)^{\omega_-} + \sum_{|v| \leq n} \mathbb{1}_{\{b_v \leq t < b_v + \zeta_v\}} \mathcal{X}_v(t - b_v)^{\omega_-}.$$

Letting $n \to \infty$, recall that $\mathcal{M}^-(n)$ converges to $\mathcal{M}^-(\infty)$ in L^p for any $p \in [1, \frac{\omega_+}{\omega_-})$, then

$$\mathcal{P}_x[\mathcal{M}^-(\infty)|\bar{\mathbf{F}}_t] \geq \sum_{|u| \in \mathbb{U}} \mathbb{1}_{\{b_u \leq t < b_u + \zeta_u\}} \mathcal{X}_u(t - b_u)^{\omega_-} = M^-(t).$$

On the other hand, by the many-to-one formula we have

$$\mathbf{E}_x[M^-(t)] = \mathrm{P}_x(Y^-(t) \in (0, \infty)) = \mathrm{P}_x(\zeta^- > t) = \mathrm{P}_x\left(x^{-\alpha} \int_0^\infty \exp(-\alpha \xi^-(s)) \mathrm{d}s > t\right),$$

where ξ^- is a SNLP with Laplace exponent $\kappa(\cdot + \omega_-)$. When $\alpha \geq 0$, we have $\mathbf{E}_x[M^-(t)] = 1$. So $\mathbf{E}_x[M^-(t)] = \mathcal{E}_x[\mathcal{M}^-(\infty)]$ and we hence conclude that $\mathcal{P}_x[\mathcal{M}^-(\infty)|\bar{\mathbf{F}}_t] = M^-(t)$. This completes the proof. □

4.4 Asymptotics of self-similar growth-fragmentations with $\alpha > 0$

We finally use Theorem 4.7 to establish the following statement.

Theorem 4.8 ([12, Theorem 3.4]). *Let η^- be a SNLP with Laplace exponent $\Phi^-(\cdot) := \kappa(\cdot + \omega_-)$. Suppose that η^- is not arithmetic, that is there is no $r > 0$ such that $\mathrm{P}(\eta^-(t) \in r\mathbb{Z}) = 1$ for all $t \geq 0$. Let $I := \int_0^\infty \exp(\alpha \eta^-(s))\mathrm{d}s$ and define a measure ρ on $(0, \infty)$: for every f with compact support on $(0, \infty)$,*

$$\int_0^\infty f(y)\rho(\mathrm{d}y) := -\frac{1}{\alpha \kappa'(\omega_-)} \mathrm{E}[I^{-1} f(I^{\frac{1}{\alpha}})].$$

Then ρ is a probability measure, and we have for every $p \in (1, \frac{\omega_=}{\omega_-})$,

$$\lim_{t \to \infty} \sum_{i=1}^\infty X_i(t)^{\omega_-} f(t^{\frac{1}{\alpha}} X_i(t)) = M^-(\infty) \int_0^\infty f(y)\rho(\mathrm{d}y) \qquad \textit{in } L^p(\mathbf{P}_1).$$

Proof. Since $(\Phi^-)'(0+) = \kappa'(\omega_-) < 0$ and $\alpha > 0$, we have by Lemma 3.4 that ρ is indeed a well-defined probability measure.

Let $H(t + t^2) := \sum_{i=1}^\infty X_i(t + t^2)^{\omega_-} f((t + t^2)^{\frac{1}{\alpha}} X_i(t + t^2))$. By the branching property, we can write

$$\mathbf{E}_1[H(t + t^2)|\mathbf{F}_t] = \sum_{i=1}^\infty X_i(t)^{\omega_-} A_i(t)$$

with

$$A_i(t) := \sum_{j=1}^{\infty} X_{i,j}(X_i(t)^\alpha t^2) f((t+t^2)^{\frac{1}{\alpha}} X_i(t) X_{i,j}(X_i(t)^\alpha t^2)),$$

where $\mathbf{X}^{[i]} := (X_{i,1}(t), X_{i,2}(t), \ldots)_{t\geq 0}$ are i.i.d. copies of $\mathbf{X}$, further independent of $\mathbf{F}_t$. By the many-to-one formula, we deduce that

$$\mathbf{E}_1[A_i(t)] = \mathrm{E}_1\left[f\left((1+t^{-1})^{\frac{1}{\alpha}} x_i t^{\frac{2}{\alpha}} Y^-(x_i^\alpha t^2)\right)\right]\Big|_{x_i = X_i(t)}.$$

For every i, by Lemma 3.4 we have $x_i t^{\frac{2}{\alpha}} Y^-(x_i^\alpha t^2)$ converges to ρ weakly as $t\to\infty$. So we have for all $x_i^\alpha t^2 > t^{\frac{1}{2}}$ i.e. $x_i > t^{-3/2\alpha}$ uniformly,

$$\lim_{t\to\infty} \mathrm{E}_1\left[f\left((1+t^{-1})^{\frac{1}{\alpha}} x_i t^{\frac{2}{\alpha}} Y^-(x_i^\alpha t^2)\right)\right] = \int_0^\infty f(y)\rho(\mathrm{d}y).$$

On the other hand, the many-to-one formula leads to

$$\mathbf{E}_1[\sum_{i=1}^{\infty} X_i(t)^{\omega-} \mathbb{1}_{\{X_i(t)\leq t^{-3/2\alpha}\}}] = \mathrm{E}_1[t^{1/\alpha} Y^-(t) < t^{1/2\alpha}],$$

which tends to 0. Since M^- is bounded in $L^p(\mathbf{P}_1)$, the convergence also holds in $L^p(\mathbf{P}_1)$. Summarizing, we conclude that

$$\lim_{t\to\infty} \mathbf{E}[H(t+t^2)|\mathbf{F}_t] = M^-(\infty)\int_0^\infty f(y)\rho(\mathrm{d}y) \qquad \textit{in } L^p(\mathbf{P}_1).$$

It remains to prove that

$$\lim_{t\to\infty} H(t+t^2) - \mathbf{E}[H(t+t^2)|\mathbf{F}_t] = 0 \qquad \textit{in } L^p(\mathbf{P}_1).$$

Since M^- is bounded in $L^p(\mathbf{P}_1)$, using Doob's maximal inequality, we have that $\|f\|_\infty \sup_{t\geq 0}\sum_{j=1}^\infty X_{i,j}(t)^{\omega-}$ is in $L^p(\mathbf{P}_1)$. We can also obtain from the many-to-one formula that

$$\lim_{t\to\infty} \mathbf{E}_1[\sum_{i=1}^{\infty} X_i(t)^{p\omega-}] = 0.$$

Therefore, by a variation of law of large numbers [4, Lemma 1.5], we prove the claim. □

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