Different Aspects of a Model for Random Fragmentation Processes

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Summary. This text surveys different probabilistic aspects of a model which is used to describe the evolution of an object that falls apart randomly as time passes. Each point of view yields useful techniques to establish properties of such random fragmentation processes.

Key words. **Fragmentation, branching process, Markov chain, multiplicative cascade, exchangeable partition.**

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1 Introduction

Fragmentation is natural phenomenon that can be observed at a great variety of scales. To give just a few examples, we may think of stellar fragments and meteoroids in Astrophysics, fractures and earthquakes in Geophysics, crushing in the mining industry, breaking of crystals in Crystallography, degradation of large polymer chains in Chemistry, fission of atoms in Nuclear Physics, fragmentation of a hard drive or files in Computer Science, ... In this text, we will be interested in situations where this phenomenon occurs randomly and repeatedly as time passes. Typically, we may imagine the evolution of blocks of mineral in a crusher.

Over many years, the importance of the activity on fragmentation in Physics (see e.g. [2, 21] and the references therein) has not been reflected in Probability Theory where the interest has been more irregular. The first significant works concerning its probabilistic aspects (strong laws) are due to Kolmogorov [38] and his student Filippov [29]. The systematic study of general fragmentation processes is quite recent, although some well-established areas of Probability Theory, such as branching random walks and multiplicative cascades, are clearly relevant to investigate a large family of fragmentation processes.

This text is intended as a survey on a stochastic model for fragmentation processes which is characterized by a few parameters. We shall present various point of views on this model, from which different properties depending on the values of the parameters are derived. In order to deal with models that can be studied mathematically, we are led to make hypotheses that may look at first sight somewhat stringent, but which are often fulfilled in applications. First, we suppose that the system has a memoryless evolution, i.e. its future only depends on its present state and not on its past. In other words, the system enjoys the Markov property. In particular, this excludes the possibility that an object might be more fragile (i.e. more likely to split) due to former shocks. Second, we assume that each fragment can be characterized by a real number that can be thought of as its size. This impedes to consider geometrical properties like the shape of a fragment. Third, we shall suppose that the branching property is fulfilled, in the sense that fragments split independently one of the other, or in other words, that the evolution of a given fragment does not depend on its environment. Finally, we shall assume that the process enjoys self-similarity, that is that the law of a fragmentation process started from a unique fragment of size $r > 0$ can be reduced by proper rescaling in space and time, to that when $r = 1$.

In order to give a formal definition, we introduce the state space of decreasing numerical sequences which tend to 0,

$$\mathcal{S} := \left\{ \mathbf{s} = (s_1, s_2, \ldots) : s_1 \geq s_2 \geq \ldots \geq 0 \text{ and } \lim_{n \to \infty} s_n = 0 \right\} .$$

A generic configuration $\mathbf{s}$ should be thought of as the ranked sequence of the sizes of some object that has been split. The space $\mathcal{S}$ is endowed with the uniform distance, so $\mathcal{S}$ is a complete metric separable space.

Let $X = (X(t), t \geq 0)$ denote a random process such that

$$X(t) = (X_1(t), X_2(t), \ldots) \in \mathcal{S} \qquad \forall t \geq 0, \ a.s.$$

We suppose that X is continuous in probability and Markovian. For every $r \geq 0$, we write $\mathbb{P}_r$ for its law started from the configuration $(r, 0, \ldots)$, and we further assume that for every $t \geq 0$, $\mathbb{P}_1(X_1(t) \leq 1) = 1$ and $\mathbb{P}_0(X_1(t) = 0) = 1$. In other words, the sizes of the pieces resulting at time t from an object with initial size 1 cannot exceed 1, and an object with 0 size cannot produce pieces with positive size.

Definition 1 *We call X a* self-similar fragmentation *if :*

(i) *There exists $\alpha \in \mathbb{R}$, called the* index of self-similarity*, such that for every $r > 0$, the distribution under $\mathbb{P}_1$ of the rescaled process $(rX(r^\alpha t), t \geq 0)$ is $\mathbb{P}_r$.*

(ii) *For every $\mathbf{s} = (s_1, s_2, \ldots) \in \mathcal{S}$, if $\left(X^{(i)}, i \in \mathbb{N}\right)$ is a sequence of independent processes such that $X^{(i)}$ has the law $\mathbb{P}_{s_i}$, and if $\tilde{X}(t)$ denotes the decreasing rearrangement of the family $\left(X_n^{(i)}(t) : i, n \in \mathbb{N}\right)$, then $\tilde{X} = (\tilde{X}(t), t \geq 0)$ is a version of X started from the configuration $\mathbf{s}$.*

The self-similarity and branching assumptions (i-ii) enable us to focus on the special case when the fragmentation starts from a single fragment with unit size, without losing generality. Thus, in the sequel, we whall implicitly work under the probability measure $\mathbb{P} := \mathbb{P}_1$.

It is intuitively obvious that the behavior of a self-similar fragmentation should depend crucially on the sign of the index of self-similarity. Roughly, since fragments get smaller as time passes, the rate of dislocations decreases when the index is positive, whereas it increases when the index is negative. In the critical case when $\alpha = 0$, the fragmentation process is called *homogeneous*, as the rates at which fragments split do not depend of their sizes. Homogeneous fragmentations are of course easier to study, and more results are known for homogeneous fragmentations than for general ones.

In the case when splits occur discretely, that is when each fragment remains unchanged for some random time and then splits, fragmentation processes can be described in terms of branching Markov chains in continuous time. Further, they can be endowed with a discrete genealogical structure, which enables us to develop their study using general techniques from the theory of multiplicative cascades. Plainly, this approach fails when splits occur continuously, that is when each fragment breaks down instantaneously. In the sixth section of this paper, we shall circumvent this fundamental difficulty by performing a spatial discretization. Specifically, Kingman's theory of exchangeable random partitions provides the right framework for studying a fundamental special class of fragmentation processes, called exchangeable. This approach enables us to reveal the fine structure of exchangeable fragmentation and to extend results which were proven in the discrete setting to this general setting. Finally, the last section of this paper is devoted to a rather informal discussion of the duality relation involving time-reversal, which exists between certain pairs of fragmentation and coalescent processes.

2 Fragmentations as branching Markov chains

2.1 Construction of fragmentation chains

A self-similar fragmentation process is called a (self-similar fragmentation) *chain* if the first dislocation time

$$T := \inf \{t > 0 : X(t) \neq (1, 0, \ldots)\}$$

is strictly positive a.s. (recall that we implicitly assume that at the initial time there is a single fragment with unit size). In this situation, the Markov property forces T to have an exponential distribution, say with parameter $c \geq 0$. Excluding implicitly the degenerate case when $c = 0$ (i.e. $T = \infty$ a.s.), we may and will henceforth focus on the case when $c = 1$ for the sake of simplicity, as the general case can be reduced to the former by a linear time-substitution. We write ν for the distribution of $X(T)$ under $\mathbb{P}_1$. So ν is a probability measure on $\mathcal{S}$ with $\nu(\{(1, 0, \ldots)\}) = 0$, which we call the *dislocation law* of X.

We can then think of X as the evolution of a non-interacting particle system in $]0, \infty[$, in which each fragment with positive size is viewed as a particle and different particles have independent evolutions. Specifically, a particle with size $r > 0$ lives for an exponential time with parameter r^α. Then it disappears and is replaced by a cloud of smaller particles, say $(r_1, \ldots) \in \mathcal{S}$, such that the sequence of ratios $(r_1/r, \ldots)$ has the fixed law ν and is independent of the lifetime of the particle r. This description makes sense only when $r > 0$; however by assumption, the possible children of a particle with size 0 have all size 0. Particles with size 0 play no role, and the evolution is thus well-defined in all cases. It should be intuitively

clear from this description that the law of the system is entirely determined by the dislocation measure ν and the index of self-similarity α.

Conversely, given a real number $\alpha \in \mathbb{R}$ and a probability measure ν on $\mathcal{S}$ which fulfils mild conditions, it is easy to construct a self-similar fragmentation chain with index α and dislocation law ν. Roughly, the idea is to focus first on particles with size at least $\varepsilon > 0$, and to built the restricted system using the standard theory of Markov chains in continuous time. Then one observes a compatibility property for different values of the threshold parameter $\varepsilon > 0$, which allows us to take a projective limit as $\varepsilon \to 0$.

More precisely, assume that $\nu(\{\mathbf{s} \in \mathcal{S} : s_1 > 1\}) = 0$, $\nu(\{1, 0, \ldots)\}) = 0$, and

$$\int_{\mathcal{S}} \# \{i : s_i > \varepsilon\} \, \nu(d\mathbf{s}) < \infty \qquad \text{for every } \varepsilon > 0. \tag{1}$$

For every $\varepsilon > 0$, one can then construct [1] a branching Markov chain in continuous time, say $X^{(\varepsilon)}$, with values in the space of finite atomic measures on $]\varepsilon, \infty[$ and governed by the following transitions. When the process starts from say $m = \sum_{i=1}^{n} \delta_{r_i}$ with $r_1 \geq r_2 \geq \ldots \geq r_n > \varepsilon$, the first jump occurs at an exponential time with parameter $\sum_{i=1}^{n} r_i^{\alpha}$. The configuration immediately after the jump is independent of the exponential time and has the same distribution as the point measure obtained from m by picking an atom r_i at random with probability proportional to r_i^{α}, and replacing it by a random family of atoms, say $(r_i s_1, \ldots, r_i s_k)$ where $\mathbf{s} = (s_1, \ldots)$ has the law ν and $k = \max\{j \in \mathbb{N} : r_i s_j > \varepsilon\}$. It is then immediately checked from calculations of branching rates that for every $\varepsilon > \eta > 0$, the restriction of the point measure process $X^{(\eta)}$ to $]\varepsilon, \infty[$ is a version of $X^{(\varepsilon)}$. By Kolmogorov's extension theorem, this ensures the existence of a unique law of a process $X = (X(t), t \geq 0)$ with values in the space of Radon point measures on $]0, \infty[$, such that the restriction of X to $]\varepsilon, \infty[$ is a version of $X^{(\varepsilon)}$ for every $\varepsilon > 0$. The natural identification of a Radon point measure with the ranked sequence of its atoms enables us to view X as a self-similar fragmentation chain with the desired dynamics.

We stress that in general, despite of the terminology, X is *not* a continuous time Markov chain. Indeed, even though the lifetime of a fragment is exponentially distributed with a positive parameter, the process may create infinitely many fragments in a finite time (but of course, only finitely many of them have a size greater than ε for any fixed $\varepsilon > 0$), and the infimum of the lifetimes of an infinite family of fragments can be zero.

In the special case $\alpha = 0$, the lifetime of each fragment is a standard exponential variable, independently of the size of the fragment, and we say that the fragmentation chain is *homogeneous*. In this situation, there is a natural connection with branching random walks in continuous times (cf. Uchiyama [55]). Specifically, consider a homogeneous fragmentation chain X with a dislocation law ν that charges only the sub-space of sequences $\mathbf{s} = (s_1, \ldots)$ with $s_k = 0$ for k sufficiently large. Then the process

$$Z^{(t)}(dx) := \sum \delta_{-\ln X_i(t)}(dx),$$

where the sum is taken over the fragments with strictly positive size, is a branching random walk. More precisely, its branching measure is the image of ν by the map $x \to -\ln x$. This

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[1] Condition (1) is needed to prevent a possible explosion, as otherwise it could happen that the dynamics would create an infinite number of atoms of size $> \varepsilon$ in finite time.

elementary connection has a number of interesting consequences as it essentially reduces the study of the class of homogeneous fragmentations associated to a dislocation laws charging only finite configurations, to that of branching random walks on $]0, \infty[$, for which a great deal of results are known.

The infinitesimal generator $\mathbf{G}$ of a self-similar fragmentation chain has a simple and useful expression for so-called additive and multiplicative functionals, which is easily obtained from the construction described above in terms of the jump rates.

Proposition 1 (i) *Consider a measurable function $f : [0, \infty[\to \mathbb{R}$ with $f = 0$ on some neighborhood of 0, and define an additive functional $A : \mathcal{S} \to \mathbb{R}$ by*

$$A(\mathbf{s}) = \sum_{i=1}^{\infty} f(s_i), \qquad \mathbf{s} = (s_1, s_2, \ldots).$$

Then for every $\mathbf{x} = (x_1, \ldots) \in \mathcal{S}$, we have

$$\mathbf{G}A(\mathbf{x}) = \sum_{i=1}^{\infty} \mathbf{1}_{\{x_i > 0\}} x_i^{\alpha} \int_{\mathcal{S}} (A(x_i \mathbf{s}) - f(x_i)) \nu(d\mathbf{s}).$$

(ii) *Consider a measurable function $g : [0, \infty[\to]0, \infty[$ with $g = 1$ on some neighborhood of 0, and define a multiplicative functional $M : \mathcal{S} \to]0, \infty[$ by*

$$M(\mathbf{s}) = \prod_{i=1}^{\infty} g(s_i), \qquad \mathbf{s} = (s_1, s_2, \ldots).$$

Then for every $\mathbf{x} = (x_1, \ldots) \in \mathcal{S}$, we have

$$\mathbf{G}M(\mathbf{x}) = \sum_{i=1}^{\infty} \mathbf{1}_{\{x_i > 0\}} x_i^{\alpha} \frac{M(\mathbf{x})}{g(x_i)} \int_{\mathcal{S}} (M(x_i \mathbf{s}) - g(x_i)) \nu(d\mathbf{s}).$$

2.2 Some analytic expressions and Malthusian hypotheses

Throughout the rest of this section, we will be working with a self-similar fragmentation chain X with index $\alpha \in \mathbb{R}$ and dislocation law ν, where ν is a probability measure of $\mathcal{S}$ satisfying the conditions of the preceding section. To avoid uninteresting discussions, we shall not consider the situation when $s_i = 0$ or 1 for all $i \in \mathbb{N}$, $\nu(d\mathbf{s})$-a.s., since the fragmentation chain can then be identified as a branching process in continuous time.

For future purposes, it is convenient to introduce the notation

$$\underline{p} := \inf \left\{ p > 0 : \int_{\mathcal{S}} \sum_{i=1}^{\infty} s_i^p \nu(d\mathbf{s}) < \infty \right\}$$

(with the convention $\inf \emptyset = \infty$), and to assume from now on that $\underline{p} < \infty$. Then we define for every $p \geq \underline{p}$

$$\kappa(p) := \int_{\mathcal{S}} \left(1 - \sum_{i=1}^{\infty} s_i^p \right) \nu(d\mathbf{s}). \tag{2}$$

Note that our assumptions ensure that κ is always a continuous strictly increasing function on $]\underline{p}, \infty[$; $\kappa(\underline{p}-)$ may be finite or equal to $-\infty$.

We then make the fundamental :

Malthusian Hypotheses. *There exists a (unique) solution $p^* > \underline{p}$ to the equation $\kappa(p^*) = 0$, which is called the* Malthusian exponent. *Furthermore the integral*

$$\int_{\mathcal{S}} \left(\sum_{i=1}^{\infty} s_i^{p^*} \right)^p \nu(d\mathbf{s})$$

is finite for some $p > 1$.

We stress that the Malthusian hypotheses are quite weak. For instance, if $s_1 < 1$, $\nu(d\mathbf{s})$-a.s. (which means that the fragments resulting from the dislocation of a particle are always strictly smaller than the initial particle), then we have by dominated convergence that $\lim_{p \to \infty} \kappa(p) = 1$. If moreover we can find some $p > \underline{p}$ such that $\kappa(p) < 0$ (this occurs for instance whenever $\int_{\mathcal{S}} \#(\mathbf{s}) \nu(d\mathbf{s}) \in]1, \infty[$ where $\#(\mathbf{s}) := \mathtt{Card}\{i : s_i > 0\}$, then the fact that κ is s a continuous and strictly increasing ensures the existence of the Malthusian parameter.

Throughout the rest of this work, the Malthusian hypotheses will always be taken for granted. The function κ and the Malthusian exponent p^* play a crucial role in the study of the asymptotic behavior of fragmentation chains, as we shall see soon.

3 Fragmentation chains and multiplicative cascades

In this section, we point at a representation of a fragmentation chain as an infinite tree with random marks. This representation can be viewed as a different parametrization of the process, in which the natural time is replaced by the generation of the different particles.

3.1 Genealogical coding of fragmentation chains

We start by introducing some notation. We consider the infinite tree

$$\mathcal{U} := \bigcup_{n=0}^{\infty} \mathbb{N}^n,$$

with the convention $\mathbb{N}^0 = \{\emptyset\}$. In the sequel $\mathcal{U}$ will often be referred to as the *genealogical tree*; its elements are called *nodes* (or sometimes also individuals) and the distinguished node $\emptyset$ the *root*. Nodes will be used to label the particles produced by a fragmentation chain. For each $u = (u_1, \ldots, u_n) \in \mathcal{U}$, we call *generation* of u and write $|u| = n$, with the obvious convention $|\emptyset| = 0$. When $n \geq 1$ and $u = (u_1, \ldots, u_n)$, we call $u- := (u_1, \ldots, u_{n-1})$ the parent of u. Similarly, for every $i \in \mathbb{N}$ we write $ui = (u_1, \ldots, u_n, i) \in \mathbb{N}^{n+1}$ for the i-th child of u. Finally, we call *mark* any map from $\mathcal{U}$ to some (measurable) set.

Now, we associate to each trajectory of the fragmentation chain a mark on the infinite tree $\mathcal{U}$. The mark at a node u is the triple (ξ_u, a_u, ζ_u) where ξ_u is the size, a_u the birth-time and

ζ_u the lifetime of the particle with label u. More precisely, the initial particle corresponds to the ancestor $\emptyset$ of the tree $\mathcal{U}$, and the mark at $\emptyset$ is the triple $(1, 0, \zeta_\emptyset)$ where $\zeta_\emptyset$ is the lifetime of the initial particle; in particular $\zeta_\emptyset$ has the standard exponential law. The nodes at the first generation are used as the labels of the particles arising from the first split. Again, the mark associated to each node $i \in \mathbb{N}^1$ at the first generation, is the triple (ξ_i, a_i, ζ_i), where ξ_i is the size of the i-th child of the ancestor, $a_i = a_\emptyset + \zeta_\emptyset$ (the birth-time of a child particle coincides with the death-time of the parent), and ζ_i stands for the lifetime of the i-th child. And we iterate the same construction with each particle at each generation.

Clearly, the description of the dynamics of fragmentation entails that its genealogical coding also enjoys the branching property. Specifically, we have the following recursive description :

Proposition 2 *There exists two independent families of i.i.d. variables indexed by the nodes of the genealogical tree, $\left(\tilde{\xi}_{u\bullet}, u \in \mathcal{U} \right)$ and $(\mathbf{e}_{u,\bullet}, u \in \mathcal{U})$, where each $\tilde{\xi}_{u\bullet} := (\tilde{\xi}_{u_1,\ldots,u_n,i}, i \in \mathbb{N})$ is distributed according to the law ν, and each $\mathbf{e}_{u,\bullet} = (\mathbf{e}_{ui}, i \in \mathbb{N})$ is a sequence of i.i.d. standard exponential variables, and such that the following holds:*

Given the marks $((\xi_v, a_v, \zeta_v), |v| \le n)$ of the first n generations, the marks at generation $n + 1$ can be expressed in the form

$$(\xi_{ui}, a_{ui}, \zeta_{ui}) = (\tilde{\xi}_{ui}\xi_u, a_u + \zeta_u, \xi_{ui}^{-\alpha}\mathbf{e}_{ui}),$$

where $u = (u_1, \ldots, u_n)$ and $ui = (u_1, \ldots, u_n, i)$ is the i-th child of u.

Proposition 2 shows that the sizes at nodes $(\xi_u, u \in \mathcal{U})$ define a so-called *multiplicative cascade*; see the pioneer works of Mandelbrot [43], Kahane and Peyriere [36], Mauldin and Williams [44]; see also Liu [41] for further references. Although this multiplicative cascade alone does not enable us to recover the fragmentation chain (we also need the information on birth-times and lifetimes as it is amplified in Proposition 3 below), classical notions and results in this field have a key role in the study of fragmentation chains. It should be intuitively obvious that one can express the fragmentation chain at time t in terms of the particles which are alive at time t, i.e. which are born at or before t and die after t.

Proposition 3 *With probability one, for every $t \ge 0$ and every measurable function $f : [0, \infty[\to [0, \infty[$ with $f(0) = 0$, there is the identity*

$$\sum_{i=1}^{\infty} f(X_i(t)) = \sum_{u \in \mathcal{U}} \mathbf{1}_{\{a_u \le t < a_u + \zeta_u\}} f(\xi_u).$$

Proof: We have to check that all the fragments with positive size which are present at time t have a finite generation, i.e. result from finitely many dislocations of the initial particle. In this direction, let us fix some arbitrarily small $\varepsilon > 0$, and consider the threshold operator φ_ε which consists of removing all the fragments with size less than or equal to ε. Recall from Section 2.1 that $\varphi_\varepsilon(X)$ is a Markov chain, in particular the number of jumps accomplished by this chain before time t is finite a.s. This number obviously is an upperbound for the generation of all fragments with size greater than ε. $\qquad\square$

3.2 Intrinsic martingale

The purpose of this section is to introduce the so-called *intrinsic martingale* which is naturally induced by Malthusian hypotheses and the genealogical coding of fragmentations, and plays a crucial role in the asymptotic behavior of the latter.

Theorem 1 *The process*

$$\mathcal{M}_n := \sum_{|u|=n} \xi_u^{p^*}, \qquad n \in \mathbb{Z}_+$$

is a martingale which is bounded in $L^1(\mathbb{P})$, and in particular, uniformly integrable. Moreover, the terminal value $\mathcal{M}_\infty$ is strictly positive a.s. whenever $\nu(s_1 = 0) = 0$ (i.e. a fragment may never disappear entirely after a dislocation).

In the sequel, $(\mathcal{M}_n, n \in \mathbb{Z}_+)$ will be referred to as the intrinsic martingale. Observe that in the important case when dislocations are conservative, in the sense that $\sum_{i=1}^\infty s_i = 1$, $\nu(d\mathbf{s})$-a.s., then $p^* = 1$ and $\mathcal{M}_n = 1$ for all $n \in \mathbb{Z}_+$, and the statement is trivial.

In general the distribution of the terminal value of the intrinsic martingale is not known explicitly. However, it is straightforward from the branching property that there is the identity in law

$$\mathcal{M}_\infty \overset{(d)}{=} \sum_{j=1}^\infty \xi_j^{p^*} \mathcal{M}_\infty^{(j)} \tag{3}$$

where $\xi = (\xi_j, j \in \mathbb{N})$ has the law ν, and $\mathcal{M}_\infty^{(j)}$ are independent copies of $\mathcal{M}_\infty$, also independent of ξ. It is known that under fairly general conditions, such equation characterizes the law $\mathcal{M}_\infty$ uniquely, see e.g. [42, 52]. We also refer to Liu [41] for information of the tail behavior of the solution.

4 Some applications

In this section, we present some results on the behavior of self-similar fragmentation chains which can be derived by the combination of standard techniques from the theories of Markov chain in continuous time and multiplicative cascades.

4.1 Some strong limit theorems $(\alpha \geq 0)$

The Malthusian parameter p^* and the terminal value $\mathcal{M}_\infty$ of the intrinsic martingale have a crucial role in the study of the asymptotic behavior of additive functionals of the fragmentation, i.e. of the type

$$A(X(t)) = \sum_{i=1}^\infty X_i^{p^*}(t) f(X_i(t), t), \qquad t \geq 0$$

for some measurable function $f : \mathbb{R}_+^2 \to \mathbb{R}_+$. Indeed, let us denote by $\mathcal{M}(t)$ this quantity for $f \equiv 1$, so $\mathcal{M}(t)$ can be viewed as the analog of the intrinsic martingale when the parameter

is time instead of generation. It is easy to check that for $\alpha \geq 0$, $\mathcal{M}(t)$ converges a.s. and in $L^1(\mathbb{P})$ to the terminal value $\mathcal{M}_{(t)}\infty$ of the intrinsic martingale. More generally, the branching property and a variation of the law of large number enable to show that under appropriate hypotheses on the function f, it holds that

$$\lim_{t \to \infty} A(X(t)) = c(f)\mathcal{M}_\infty, \qquad \text{in } L^1(\mathbb{P}). \tag{4}$$

Here, $c(f)$ is deterministic factor which can be determined by first moment calculations. The terminal value of the intrinsic martingale $\mathcal{M}_\infty$ thus appears as a kind of universal random weight. We refer e.g. to [34, 49] and references therein for many results in this vein for certain branching processes. In order to avoid some technical discussion related to periodicity, we shall often make the assumption that the dislocation law ν is *non-geometric*, in the sense that there exists no real number $r \in]0, 1]$ such that $s_i \in \{r^n, n \in \mathbb{Z}_+\} \cup \{0\}$ for all $i \in \mathbb{N}$, $\nu(d\mathbf{s})$-a.s.

Let us first consider the case of homogeneous fragmentation chains, i.e. with index of self-similarity $\alpha = 0$. In this situation, each particle has a standard exponential lifetime, so informally when $n \in \mathbb{N}$ is large, particles at generation n are alive at times close to n (by the law of large numbers), and for the same reason, when t is large, the generation of particles alive at time t is close to $[t]$. In this direction, one naturally expects that the strong limit theorems for multiplicative cascades can be shifted without significant modifications to homogeneous fragmentation chains. This is indeed the case as shown in the following result which can be traced back (in a simpler setting) to Kolmogorov [38]; see also [5, 24, 55] for closely related statements.

Proposition 4 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous bounded function. In the homogenenous case $\alpha = 0$, the following limits hold in $L^1(\mathbb{P})$:*

$$\lim_{t \to \infty} \sum_{i=1}^{\infty} X_i^{p^*}(t) f(t^{-1} \ln X_i(t)) = \mathcal{M}_\infty f(-\kappa'(p^*))$$

and

$$\lim_{t \to \infty} \sum_{i=1}^{\infty} X_i^{p^*}(t) f(t^{-1/2}(\ln X_i(t) + \kappa'(p^*)t)) = \mathcal{M}_\infty \mathbb{E}(f(\mathcal{N}(0, -\kappa''(p^*)))),$$

where $\mathcal{N}(0, -\kappa''(p^*))$ denotes a centered Gaussian variable with variance $-\kappa''(p^*)$.

Roughly, the first part of Proposition 4 claims that in a homogenenous fragmentation, 'most' fragments decay exponentially fast with rate $\kappa'(p^*)$. The second part is sharper and specifies the pathwise fluctuations.

When the index of self-similarity α is positive, large fragments split faster than small ones, so the rate of fragmentation decays as time passes and one may expect a homogenisation phenomena. Theorem 2 below was first established by Filippov [29] in the special case when the dislocation measure ν is conservative, i.e. when $\sum_{i=1}^{\infty} s_i = 1$ for ν-a.e. $\mathbf{s}$ (see also Brennan and Durrett [27]). Recall that in this case, $p^* = 1$ and $\mathcal{M}_n \equiv 1$. The general case of non-conservative dislocation measures was recently proved by Bertoin and Gnedin [15].

Theorem 2 *Suppose that $\alpha > 0$ and that the dislocation law ν is non-geometric. Then for every bounded continuous function $f : \mathbb{R}_+ \to \mathbb{R}$*

$$\lim_{t \to \infty} \sum_{i=1}^{\infty} X_i^{p^*}(t) f(t^{1/\alpha} X_i(t)) = \mathcal{M}_\infty \int_0^\infty f(y) \rho(dy), \qquad in \ L^1(\mathbb{P}),$$

where $\mathcal{M}_\infty$ is the terminal value of the intrinsic martingale and ρ is a deterministic probability measure. More precisely, ρ is determined by the moments

$$\int_{]0,\infty[} y^{\alpha k} \rho(dy) = \frac{(k-1)!}{\alpha \kappa'(p^*) \, \kappa(p^* + \alpha) \cdots \kappa(p^* + (k-1)\alpha)} \quad for \ k \in \mathbb{N},$$

(with the usual convention that the right-hand side above equals $1/(\alpha \kappa'(p^*))$ for $k = 1$).

Comparison with the homogeneous case of Proposition 4 shows that the size of a typical fragment now decays as a power function of time. It is also interesting to observe that the limit is much more sensitive to the dislocation law than in the homogeneous situation: the function κ can be recovered from the limit measure ρ, whereas in the homogeneous case, the exponential rate of decay just depends on the derivative of κ at the Malthusian parameter.

Let us just sketch an argument for the proof of Theorem 2 based on moment calculations. Kolmogorov's backwards equation combined with Proposition 1 yields, in the special case of a power function, the equation

$$\frac{d}{dt} \mathbb{E} \left(\sum_{i=1}^{\infty} X_i^{p-\alpha}(t) \right) = -\kappa(p - \alpha) \mathbb{E} \left(\sum_{i=1}^{\infty} X_i^p(t) \right).$$

The solution is given in the form

$$\mathbb{E} \left(\sum_{i=1}^{\infty} X_i^p(t) \right) = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \gamma(n, p),$$

where $\gamma(0, p) = 1$ and for $n \geq 1$

$$\gamma(n, p) = \prod_{k=0}^{n-1} \kappa(p + \alpha k).$$

Asymptotics can be obtained using techniques from complex analysis, and one gets for $p = p^* + \alpha k$ with $k \in \mathbb{N}$, that

$$\mathbb{E} \left(\sum_{i=1}^{\infty} X_i^p(t) \right) \sim \frac{(k-1)!}{\alpha \kappa'(p^*) \, \kappa(p^* + \alpha) \cdots \kappa(p^* + (k-1)\alpha)} t^k, \qquad t \to \infty.$$

This shows that

$$\lim_{t \to \infty} \mathbb{E} \left(\sum_{i=1}^{\infty} X_i^{p^*}(t) f(t^{1/\alpha} X_i(t)) \right) = \int_0^\infty f(y) \rho(dy)$$

when $f(x) = x^{k\alpha}$, and then for any continuous function f bounded by a power function. Finally, L^1-convergence in Theorem 2 follows from the meta-limit theorem (4). We refer to [15] for details.

4.2 Extinction and formation of dust for $\alpha < 0$

Intuitively, when the index of self-similarity is negative, fragments with small sizes are subject to high splitting rates, and this makes them vanish entirely; see e.g. [30, 31, 35] for some works formalizing this intuition.

Proposition 5 *Suppose*

$$\int_{\mathcal{S}} \texttt{Card}\left\{i : s_i = 1\right\} \nu(d\mathbf{s}) < 1 \,.$$

Then the following assertions hold with probability one:

(i) For $\alpha < 0$, $X(t) = (0, \ldots)$ for all sufficiently large t.

(ii) When $\kappa(-\alpha) > 0$, for almost every $t > 0$

$$\texttt{Card}\left\{j \in \mathbb{N} : X_j(t) > 0\right\} < \infty \,.$$

We stress that in general, no matter what the value of α is, there may exist random instants t at which

$$\texttt{Card}\left\{j \in \mathbb{N} : X_j(t) > 0\right\} = \infty \,.$$

For instance in the case when the dislocation law fulfills

$$\nu\left(x_j > 0 \text{ for all } j \in \mathbb{N}\right) = 1 \,,$$

then with probability one, there occur infinitely many sudden dislocations in the fragmentation chain X, each of which produces infinitely many terms. This does not induce any contradiction with Proposition 5 (ii) when $\kappa(-\alpha) > 0$, because informally, as the index of self-similarity is negative, we know that fragments with small size vanish quickly.

It would be interesting to have information on the distribution of the extinction time

$$\zeta := \inf\left\{t \geq 0 : X(t) = (0, 0, \ldots)\right\} \,,$$

however it does not seem possible to express this law in a closed form. Nonetheless, we point that an application of the branching property at the first dislocation yields the identity in distribution

$$\zeta \overset{(d)}{=} \mathbf{e} + \max_{j \in \mathbb{N}} \xi_j^{-\alpha} \zeta_j \,, \tag{5}$$

where $\mathbf{e}$ is a standard exponential variable, $(\xi_j, j \in \mathbb{N})$ is distributed according to ν, $(\zeta_j, j \in \mathbb{N})$ is a sequence of independent copies of ζ, and finally, $\mathbf{e}$, $(\xi_j, j \in \mathbb{N})$ and $(\zeta_j, j \in \mathbb{N})$ are independent. We refer to [3] for a survey of this type of equations in distribution.

It is interesting to study in further details the extinction phenomenon in the case when the dislocation measure is conservative, i.e. when

$$\nu\left(\sum_{i=1}^{\infty} s_i \neq 1\right) = 0 \,. \tag{6}$$

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It is easy to deduce by iteration that for every $n \in \mathbb{N}$, the total mass of particles at the n-th generation is conserved, i.e.

$$\sum_{|u|=n} \xi_u = 1, \qquad \text{a.s.}$$

Turning our interest to the total mass of particles at time t, we introduce the quantity

$$D(t) := 1 - \sum_{n=1}^{\infty} X_i(t),$$

which can be viewed as the total mass of *dust*, that is of infinitesimal particles at time t. One could be tempted to believe that the assumption (6) would entail $D \equiv 0$; indeed it is easy to check that this holds when the index of self-similarity of the fragmentation is nonnegative. However Proposition 5 shows that for negative indices of self-similarity, D reaches 1 at a finite time a.s.

Proposition 6 *Suppose (6) holds. The following assertions hold with probability one:*

(i) *D is a continuous increasing process which reaches 1 in finite time.*

(ii) *If $\#(t) := \mathrm{Card}\{i : X_i(t) > 0\}$ denotes the number of fragments with positive mass at time t, then*

$$\int_0^{\infty} \mathbf{1}_{\{\#(t)<\infty\}} dD(t) = 0.$$

This statement again reflects the fact that, loosely speaking, dislocations occur faster and faster as time passes. Observe that it entails that almost-surely, there exists uncountably many times at which there are infinitely many fragments with positive size, which may be rather surprising (for instance in the case when dislocations are binary, i.e. produce exactly two fragments; see also Proposition 5).

Recall that when the dislocation law ν is conservative, $\kappa(p) > 0$ for every $p > 1$, and Proposition 5 shows that for $\alpha < -1$, at each fixed time t there is only a finite number of fragments with positive size in the system. Combining this observation with Proposition 6 entails that the random measure $dD(t)$ is singular with respect to Lebesgue measure on $\mathbb{R}_+$. Haas [32] established the following sharper result.

Proposition 7 *Suppose (6) holds and that there exists $k \in \mathbb{N}$ such that $\nu(s_{k+1} > 0) = 0$ (i.e. each dislocation splits a fragment into at most k pieces). The following assertions hold with probability one:*

(i) *If $\alpha \in]-1, 0[$, then D is absolutely continuous.*

(ii) *If $\alpha < -1$, then the random measure $dD(t)$ is supported by a set with Hausdorff dimension $1/|\alpha|$.*

5 Fragmentation chains and general branching processes

We now turn our attention to a slightly different aspect of fragmentation chains, by pointing at a connection with general (i.e. Crump-Mode-Jagers) branching processes; see [34, 49] and

references therein. Specifically, we use the genealogical coding of the fragmentation chain of Section 2, and for each node $u \in \mathcal{U}$, we think of $\sigma_u := -\ln \xi_u$ as the birth-time of an individual. Every individual which is born has an infinite lifetime, and gives birth to children according to a random point process η_i. More precisely, if $s_1 \geq s_2 \geq \ldots$ denote the sequence ranked in the decreasing order of the sizes of the fragments that result from the first dislocation of a fragment with size s (recall that we assume that $s \geq s_1$, i.e. dislocations always produce fragments of smaller sizes), the nth child of that individual is born at time $-\ln s_n$. In other words, each fragment with size s is viewed as an individual u born at time $\sigma_u = -\ln s$ and which has offspring described by the point process

$$\eta_u([0,t]) = \mathrm{Card}\left\{ n \in \mathbb{N} : s_n \geq \mathrm{e}^{-t} \right\}.$$

The particle system obtained in this way is then a general branching process with reproduction intensity

$$\mu(t) = \mathbb{E}(\eta_\emptyset([0,t])) = \int_\mathcal{S} \sum_{n=1}^\infty \mathbf{1}_{\{s_n \geq \mathrm{e}^{-t}\}} \nu(d\mathbf{s}), \qquad t \geq 0.$$

This point of view is useful to investigate problems related to situations when fragments with size less than a certain fixed parameter $\varepsilon > 0$ are instantaneously frozen. The fragmentation process then terminates when the system only constists in particles with size less than ε. This setting arises for instance in the mining industry where fragmentation is needed to reduce rocks into sufficiently small particles. For this purpose, rocks are broken in crushers and mills by a repetitive mechanism. Particles are screened so that when they become smaller than the diameter of the mesh of a thin grid, they are removed from the process.

Imagine the instantaneous dislocation of a block of size s into a set of smaller blocks of sizes $(s_1, s_2, \ldots)$ requires an energy of the form $s^\beta \varphi(s_1/s, s_2/s, \ldots)$, where $\varphi : \mathcal{S} \to \mathbb{R}$ is a cost function and $\beta > 0$ a fixed parameter. We are interested in the total energy-cost of the process that finishes when all particles have size less than $\varepsilon > 0$:

$$\mathcal{E}(\varepsilon) := \sum_{u \in \mathcal{U}} \mathbf{1}_{\{\xi_u > \varepsilon\}} \xi_u^\beta \varphi(\tilde{\xi}_{u\bullet}),$$

where $\tilde{\xi}_{u\bullet} = (\xi_{u,n}/\xi_u, n \in \mathbb{N})$ is the sequence of ratios of the sizes of the fragments resulting from the dislocation of the fragment labelled by the node u and the size of that fragment (see Proposition 2).

The following asymptotic result for the energy cost as $\varepsilon \to 0$ has been derived recently by Bertoin and Martinez [18] from the work of Nerman [49] on strong limit theorems for general branching processes.

Theorem 3 *Suppose that the dislocation law is non-geometric and fulfils $\int_\mathcal{S} |\varphi(\mathbf{s})| \nu(d\mathbf{s}) < \infty$. Then $\lim_{\varepsilon \to 0} \mathcal{E}(\varepsilon) := \mathcal{E}(0+)$ in $L^1(\mathbb{P})$ for $\beta > p^*$, whilst for $\beta < p^*$ it holds that*

$$\lim_{\varepsilon \to 0} \varepsilon^{p^* - \beta} \mathcal{E}(\varepsilon) = \frac{\mathcal{M}_\infty}{(p^* - \beta)\kappa'(p^*)} \int_\mathcal{S} \varphi(\mathbf{s}) \nu(d\mathbf{s}) \qquad in \ L^1(\mathbb{P}).$$

We may also consider the distribution of the small particles that can go across the grid. Specifically, we would like to get information about the random finite measure on $]0, 1[$

$$\varrho_\varepsilon(dx) := \sum_{u \in \mathcal{U}, u \neq \emptyset} \mathbf{1}_{\{\xi_{u-} \geq \varepsilon, \xi_u < \varepsilon\}} \xi_u^{p^*} \delta_{\xi_u/\varepsilon}(dx) \,,$$

which can be viewed as a weighted version of the empirical measure of the particles taken at the instant when they become smaller than ε and then renormalized. In the setting of the general branching process associated to the fragmentation chain, we can re-express ϱ_ε in the form

$$\langle \varrho_\varepsilon, f \rangle = \sum_{u \in \mathcal{U}} \sum_{n=1}^{\infty} \mathrm{e}^{-p^* \sigma_{un}} f(\mathrm{e}^{t - \sigma_{un}}) \mathbf{1}_{\{\sigma_u < t \leq \sigma_{un}\}} \,,$$

where $t = -\ln \varepsilon$ and $f :]0, 1[\to \mathbb{R}_+$ denotes a generic measurable function.

Proposition 8 *Suppose that the dislocation law ν is non-geometric. As $\varepsilon \to 0$, ϱ_ε converges in probability to $\mathcal{M}_\infty \mu$, where ϱ is a deterministic probability measure on $[0, 1]$ given by*

$$\varrho(dx) = \left(\int_{\mathcal{S}} \sum_{i=1}^{\infty} \mathbf{1}_{\{s_i < x\}} s_i^{p^*} \nu(d\mathbf{s}) \right) \frac{dx}{x \kappa'(p^*)} \,.$$

6 Fragmentations with instantaneous dislocations

In this section, we turn our attention to the situation when dislocations can occur instantaneously. Examples arise naturally e.g. in the study of Brownian motion and Continuum Random Trees; see [4, 10, 33, 46, 47]. Then one cannot consider the first dislocation of a particle, and the genealogical structure of the process is no longer discrete, which impedes the representation as a multiplicative cascade. Our purpose is two-fold; first we would like to characterize such fragmentation processes and their structure, and second we aim at extending properties which have been established for fragmentation chains to this more general setting.

We shall focus on the case when the process X takes values in the space of *mass-partitions*

$$\mathcal{P}_{\mathrm{m}} := \left\{ \mathbf{s} \in \mathcal{S} : \sum_{i=1}^{\infty} s_i \leq 1 \right\} \,.$$

A mass-partition $\mathbf{s} \in \mathcal{P}_{\mathrm{m}}$ can be thought of as the ranked sequence of the masses of the fragments of some object with unit total mass; the case when $\sum_{i=1}^{\infty} s_i < 1$ corresponds to the existence of dust, i.e. a part of the object with positive mass has been reduced to infinitesimal particles each with zero mass.

It might be natural to try to investigate fragmentations with instantaneous dislocations as limit of fragmentation chains when the parameter of the exponential lifetime of a particle with unit mass tends to ∞ (so the lifetime of that particle tends to 0). However this approach is far from being easy, and we shall rather follow a different route initiated by Kingman [37] and then further developed by Pitman [50] in the framework of coalescents (see also Schweinsberg [54]).

6.1 Exchangeable partitions and interval representations

Kingman's idea is to encode mass-partitions by random partitions of $\mathbb{N}$. Specifically, consider some space E endowed with a probability measure ϱ, and disjoint measurable subsets $E_1, \ldots$ with positive ϱ-measure. Set $E_0 := E \backslash (\cup_{k \geq 1} E_k)$, so $(E_0, E_1, \ldots)$ is a partition of E. The sets $E_1, \ldots$ are viewed as fragments of E and E_0 as the set of dust.

Then let $U_1, \ldots$ be a sequence of i.i.d. variables in E with law ϱ, and define a random partition π of $\mathbb{N}$ as follows. If $U_i \in E_0$, then $\{i\}$ is a singleton of the partition π, and the other blocks of π are of the type $\pi_k = \{\ell \in \mathbb{N} : U_\ell \in E_k\}$ for $k \geq 1$. The strong law of large numbers enables us to recover the sequence of the masses of the partition of E as the asymptotic frequences of the blocks of π; more precisely

$$\varrho(E_k) = \lim_{n \to \infty} \frac{1}{n} \mathrm{Card} \left\{ \ell \in \pi_k : \ell \leq n \right\}.$$

We stress that only a special class of partitions of $\mathbb{N}$ can arise in this framework, as blocks of such partitions always have asymptotic frequences. More importantly, the random partitions resulting from Kingman's construction are *exchangeable*, that is their distribution is invariant by the action of permutations. Indeed the action of a permutation σ amounts to a permutation of the indices in the sequence of i.i.d. variables, and since $U_{\sigma(1)}, U_{\sigma(2)}, \ldots$ has the same law as $U_1, U_2, \ldots$, we see that the image of π by the action of σ has the same law as π.

Kingman [37] (see also Aldous [1] for a simpler proof) has shown that for any random exchangeable partition γ of $\mathbb{N}$, the blocks of γ possess asymptotic frequences, and more precisely, γ has the same distribution as some partition π constructed as above for a certain random partition of a space E. Specifically, one can take $E =]0, 1[$ endowed with the Lebesgue measure, let $E_1, \ldots$ be the interval components of some random open set $G \subseteq]0, 1[$ and $E_0 =]0, 1[\backslash G$. Then $U_1, \ldots$ is a sequence of i.i.d. uniform variables which is independent of the random open set G.

Now consider a fragmentation process X in the sense of Definition 1, and assume that X takes values in the space of mass-partitions. For our purpose, it is convenient to think of X in terms of a fragmentation of the unit interval, in the sense that there is a Markov process $(G(t), t \geq 0)$ with values in the space of open sets in $]0, 1[$, such that $G(t) \subseteq G(s)$ when $s \leq t$ and for each $t \geq 0$, $X(t)$ is the ranked sequence of the interval components of $G(t)$. The fact that such a representation exists is explained in [7, 12].

Next, let $U_1, \ldots$ be a sequence of i.i.d. uniform variables which is independent of $(G(t), t \geq 0)$, and for each $t \geq 0$, write $\Pi(t)$ for the random exchangeable partition of $\mathbb{N}$ such that two distinct indices i, j belong to the same block of $\Pi(t)$ if and only if U_i and U_j belong to the same interval component of $G(t)$. We make the key observation that for every $t \geq 0$, given an interval component, say I of $G(t)$, if $B = \{i \in \mathbb{N} : U_i \in I\}$ denotes the block of $\Pi(t)$ corresponding to I, then conditionally on B, $(U_i, i \in B)$ is a sequence of i.i.d. uniform variables in I. Essentially, this observation implies that the partition-valued process $\Pi = (\Pi(t), t \geq 0)$ is Markovian. Thus to each fragmentation process X we can associate a Markov process Π with values in the space of partitions of $\mathbb{N}$ such that $\Pi(t)$ gets finer as t increases. More precisely, the law of Π is invariant by the action of permutations and the branching property of X is transferred to Π. Thus Π can be thought of as a fragmentation process with values in the space of partitions of

$\mathbb{N}$.

Conversally, given an arbitrary fragmentation process Π with values in the space of partitions of $\mathbb{N}$ and which is invariant by the action of permutations, Kingman's theorem enables us to recover a fragmentation process X with values in $\mathcal{P}_m$ by considering the asymptotic frequences of the blocs of $\Pi(t)$. We refer to [7] for a precise argument.

6.2 The structure of homogeneous fragmentation processes

In this section, we suppose that the index of self-similarity is $\alpha = 0$. In this situation, the fragmentation process Π with values in the space of partitions of $\mathbb{N}$ associated to X as above enjoys a crucial additional property : for every $n \in \mathbb{N}$, the restricted process $\Pi_{|[n]}$ is Markovian, where for a partition π of $\mathbb{N}$, $\pi_{|[n]}$ denotes the restriction of π to $[n] := \{1, \ldots, n\}$. Informally, this follows from exchangeability and the fact that the dislocation rates of blocks of Π do not depend on their sizes. Since the space of partitions of $[n]$ is finite, $\Pi_{|[n]}$ is a continuous time Markov chain, whose evolution is thus specified by its jump rates. Combining this with Kingman's theory of exchangeable random partitions enables us to reveal the structure of homogeneous fragmentation processes; see [7, 11] for details. Roughly, one gets that homogeneous fragmentations result from the combination of two different phenomena: a continuous erosion and sudden dislocations. The erosion is a deterministic mechanism, analogous to the drift for subordinators, whereas the dislocations occur randomly according to some Poisson random measure, and can be viewed as the jump-component of the fragmentation.

More precisely, let us first focus on dislocations. We call *dislocation measure* a measure ν on $\mathcal{P}_m$, which gives no mass to $(1, 0, \ldots)$ and fulfills

$$\int_{\mathcal{P}_m} (1 - x_1)\, \nu(dx) \, < \, \infty\,. \tag{7}$$

Then, consider a Poisson random measure on $\mathcal{P}_m \times \mathbb{N} \times \mathbb{R}_+$,

$$\sum_{i=1}^{\infty} \delta_{(\Delta(i), k(i), t(i))}\,,$$

with intensity $\nu \otimes \# \otimes dt$, where $\#$ denotes the counting measure on $\mathbb{N}$. One can construct a pure jump process $(Y(t), t \geq 0)$ in $\mathcal{P}_m$ which jumps only at times $t(i)$ at which there is an atom $(\Delta(i), k(i), t(i))$ of the Poisson measure. More precisely, the jump (i.e. the dislocation) induced by such an atom can be described as follows.

The mass-partition $Y(t(i))$ at time $t(i)$ is obtained from that immediately before $t(i)$, i.e. $Y(t(i)-)$, by replacing its $k(i)$-th term, viz. $Y_{k(i)}(t(i)-)$, by the sequence $Y_{k(i)}(t(i)-)\Delta(i)$, and ranking all the terms in the decreasing order. For instance, if

$$Y(t(i)-) \, = \, \left(\frac{2}{3}, \frac{1}{4}, \frac{1}{12}, 0, \ldots\right) \quad, \quad k(i) = 2 \quad \text{and} \quad \Delta(i) \, = \, \left(\frac{3}{4}, \frac{1}{4}, 0, \ldots\right)$$

then we look at the 2-nd largest term in the sequence $Y(t(i)-)$, which is $\frac{1}{4}$, and split it according to $\Delta(i)$. This produces two fragments of size $\frac{3}{16}$ and $\frac{1}{16}$, and thus

$$Y(t(i)) \, = \, \left(\frac{2}{3}, \frac{3}{16}, \frac{1}{12}, \frac{1}{16}, 0, \ldots\right)\,.$$

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Next, call *erosion coefficient* an arbitrary real number $c \geq 0$, and set $X(t) = \mathrm{e}^{-ct} Y(t)$. So X is obtained from Y by letting the fragments of the latter be eroded at constant rate c. The process $(X(t), t \geq 0)$ is again a homogeneous fragmentation, and conversely any homogeneous fragmentation process can be constructed like this. In conclusion, the distribution of a homogeneous fragmentation is entirely specified by its erosion rate and its dislocation measure.

In the special case when $c = 0$ and ν is a finite measure (and, in particular, a probability), it is easy to check that X is then a homogeneous fragmentation chain with dislocation law ν in the sense of Section 2. Note however that the condition (7) allows ν to be infinite, which corresponds to the situation when particles dislocate instantaneously. Informally, mass-partitions **s** for which $1 - s_1$ is small should be thought of as small, in the sense that a small mass-partition produces one large fragment and all the remaining ones are small (in particular, the mass-partition $(1, 0, \ldots)$ has to be viewed as a neutral element for dislocations). So, roughly speaking, condition (7) allows infinitely many small dislocations, but guaranties that the accumulation of these small dislocations does not reduce instantaneously the initial mass into dust. This bears obvious similarities with subordinators, which are constructed by the Itô-Lévy decomposition from atoms of a certain Poisson random measure on $\mathbb{R}_+$, see Chapter 1 in [9]. The intensity of this Poisson random measure is given by the so-called Lévy measure Λ of the subordinator, and the integrability condition $\int_{\mathbb{R}_+} (1 \wedge x) \Lambda(dx) < \infty$ for a measure on $\mathbb{R}_+$ to be the Lévy measure of some subordinator is the necessary and sufficient condition for the summability of the atoms.

The Poissonian structure of homogeneous fragmentation is a fundamental tool which enables to circumvent difficulties related to the absence of a discrete genealogical structure. However, although the law of a homogeneous fragmentation process is characterized by its erosion rate and dislocation measure, in general we do not know how to describe explicitly e.g. the distribution of the process at a fixed time. The next crucial tool for the study lies in the fact that partial but most useful information can be derived from a so-called size-biased sampling, and it turns out that the law of the latter is simple to formulate.

Recall that we may represent X in terms of fragmentation of the unit interval. So we consider a Markovian family $(G(t), t \geq 0)$ of nested open subsets of the unit interval, in particular for every $t \geq 0$, $X(t)$ is the ranked sequence of the lengths of the interval components of $G(t)$. Now suppose that U is a uniform random variable on $]0, 1[$, which is independent of $(G(t), t \geq 0)$, and for every $t \geq 0$, denote by $\chi(t)$ the length of the interval component of $G(t)$ which contains U (with the convention that $\chi(t) = 0$ if $U \notin G(t)$). In other words, the process $\chi = (\chi(t), t \geq 0)$ gives the size of the fragment containing a point which has been tagged independently of the fragmentation process; one refers to χ as the process of the tagged fragment. Note that $\chi(t)$ is a size-biased pick from the sequence $X(t) = (X_1(t), \ldots)$, i.e. there is the identity in law

$$\chi(t) \overset{\mathcal{L}}{=} X_K(t),$$

where K is an integer valued variable whose conditional distribution given $X(t)$ is

$$\mathbb{P}(K = k \mid X(t)) = X_k(t), \quad k = 1, \ldots .$$

It turns out that the process of the tagged fragment is closely related to a subordinator (i.e. an increasing process with independent and stationary increments; see [9] for background) that can be characterized explicitly in terms of the dislocation measure and the erosion coefficient.

Theorem 4 *The process*

$$\sigma(t) = -\ln \chi(t), \qquad t \geq 0$$

is a subordinator, whose law is determined by its Laplace transform of its one-dimensional distributions. We have

$$\mathbb{E}\left(\chi(t)^q\right) = \mathbb{E}(\exp(-q\sigma(t))) = \exp\left(-t\kappa(q+1)\right), \qquad t, q > 0,$$

where the function κ is given in terms of the erosion rate c and the dislocation measure ν by the identity

$$\kappa(q) := cq + \int_{\mathcal{P}_{\mathrm{m}}} \left(1 - \sum_{i=1}^{\infty} s_i^q\right) \nu(d\mathbf{s}), \qquad q \geq 1. \tag{8}$$

Many features (like asymptotic behavior) of fragmentation process can be read on properties of tagged fragments, the combination of Theorem 4 and the theory of subordinators provides the key to many results on homogeneous fragmentations.

6.3 Additive martingales and applications

The independence and stationarity of the increments of subordinators entail that for every $q \geq 0$, the process

$$\exp(-q\sigma(t) + \kappa(q+1)t) = \exp(t\kappa(q+1))\chi^q(t), \qquad t \geq 0$$

is a martingale. As the tagged fragment $\chi(t)$ is picked at random from the mass-partition $X(t)$ by size-biased sampling, it follows (take $q = p + 1$) that

$$M(p, t) := \exp(t\kappa(p)) \sum_{i=1}^{\infty} X_i^p(t), \qquad t \geq 0$$

is a nonnegative martingale for every $p > \underline{p}$ (this can also be checked directly from the branching and scaling properties).

One refers to $M(p, t)$ as an *additive martingale*. Plainly $M(p, t)$ converges a.s. as $t \to \infty$, and in order to investigate the asymptotic behavior of homogeneous fragmentation processes, it is important to know if the limit of this martingale is strictly positive or zero. We shall investigate this question in the special case when the dislocation measure is conservative and there is no erosion, i.e. we assume from now on that

$$c = 0 \text{ and } \sum_{i=1}^{\infty} s_i = 1 \quad \nu(d\mathbf{s})\text{-a.e.} \tag{9}$$

Observe that the Malthusian hypotheses are then automatically fulfilled and the Malthusian parameter is $p^* = 1$. A first step in the analysis is the following elementary lemma.

Lemma 1 *Assume (9). The function $p \to \kappa(p)/p$ reaches its maximum at a unique location $\bar{p} > 1$, which is the unique solution to the equation*

$$p\kappa'(p) = \kappa(p).$$

More precisely, the function $p \to \kappa(p)/p$ increases on $]\underline{p}, \bar{p}[$ and decreases on $]\bar{p}, \infty[$, and the value of its maximum is $\kappa'(\bar{p}) = \kappa(\bar{p})/\bar{p}$.

Proof: We first point out that the function κ is concave and increasing. It follows that

$$\text{the function } p \to p\kappa'(p) - \kappa(p) \text{ decreases on }]\underline{p}, \infty[. \tag{10}$$

Indeed, this function has derivative $p\kappa''(p)$, which is negative since κ is concave. Recall from (8) and (9) that $\kappa(1) = 0$; on the other hand, it is obvious that $\lim_{q \to \infty} \kappa(q)/q = 0$, hence the function $p \to \kappa(p)/p$ has the same limit at 1 and at ∞, so it reaches its overall maximum at a unique point $\bar{p} > 1$. In particular, we deduce from (10) that the derivative of $p \to \kappa(p)/p$ is positive on $]\underline{p}, \bar{p}[$ and negative on $]\bar{p}, \infty[$. Finally, the derivative must be zero at $\bar{p}$, which entails that the overall maximum is given by $\kappa'(\bar{p}) = \kappa(\bar{p})/\bar{p}$. $\qquad\square$

We may now state the main result of this section which can be proved using the Poissonian structure of homogeneous fragmentations and stochastic calculus, see [13] for details.

Theorem 5 *Assume (9). For every $p \in]\underline{p}, \bar{p}[$, the martingale $M(p, \cdot)$ is bounded in $L^1(\mathbb{P})$ and its terminal value is strictly positive.*

Just as Theorem 1 for the intrinsic martingale, Theorem 5 has crucial role in the study of the asymptotic behavior of homogeneous fragmentation. First, we specify the rate of decay of the largest fragment, refering to Biggins [23] for a similar result in the framework of branching random walks.

Corollary 1 *Assume (9). It holds with probability one that*

$$\lim_{t \to \infty} \frac{1}{t} \ln X_1(t) = -\kappa'(\bar{p}) = -\frac{\kappa(\bar{p})}{\bar{p}}.$$

Proof: For every $p > \underline{p}$, we have

$$\exp(t\kappa(p))X_1^p(t) \le \exp(t\kappa(p)) \sum_{i=1}^{\infty} X_i^p(t)$$

and the right-hand side remains bounded as t tends to infinity. Hence

$$\limsup_{t \to \infty} \frac{1}{t} \ln X_1(t) \le -\frac{\kappa(p)}{p},$$

and optimizing over p yields

$$\limsup_{t \to \infty} \frac{1}{t} \ln X_1(t) \le -\frac{\kappa(\bar{p})}{\bar{p}}.$$

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On the other hand, for every $p \in]\underline{p}, \bar{p}[$ and $\varepsilon > 0$ sufficiently small, we have the lower bound

$$\exp(t\kappa(p)) \sum_{i=1}^{\infty} X_i^p(t) \leq X_1^\varepsilon(t) \exp(t\kappa(p)) \sum_{i=1}^{\infty} X_i^{p-\varepsilon}(t) .$$

We know that both limits

$$\lim_{t \to \infty} \exp(t\kappa(p)) \sum_{i=1}^{\infty} X_i^p(t) \quad \text{and} \quad \lim_{t \to \infty} \exp(t\kappa(p-\varepsilon)) \sum_{i=1}^{\infty} X_i^{p-\varepsilon}(t)$$

are finite and strictly positive a.s., and we deduce that

$$\liminf_{t \to \infty} \frac{1}{t} \ln X_1(t) \geq -\frac{\kappa(p) - \kappa(p-\varepsilon)}{\varepsilon} .$$

We take the limit of the right-hand side as $\varepsilon \to 0+$ and then as p tends to $\bar{p}$ to conclude that

$$\liminf_{t \to \infty} \frac{1}{t} \ln X_1(t) \geq -\kappa'(\bar{p}) .$$

Now, this quantity coincides with $-\kappa(\bar{p})/\bar{p}$, as we know from Lemma 1. $\qquad \square$

It is interesting to compare Corollary 1 with Proposition 2(i), which claims that the size of most fragments decays exponentially fast with rate $\kappa'(p^*) = \kappa'(1)$. The size of largest fragment thus also decays exponentially fast, but with a slower rate $\kappa'(\bar{p}) < \kappa'(1)$. We refer to Berestycki [8] for the multi-fractal analysis of the exponential rates of decay of fragments in homogeneous fragmentations.

Alternatively, one can also establish Theorem 5 by discretization, using the following connection with branching random walks and classical results on the latter; see [19]. If we consider the point measure $Z^{(t)}$ with atoms at the logarithms of the fragments

$$Z^{(t)} := \sum_{i=1}^{\infty} \delta_{\ln X_i(t)} , \qquad t \geq 0 , \tag{11}$$

we can think of the discrete skeleton $(Z^{(n)}, n \in \mathbb{N})$ as a non-interacting particle system. Specifically, particles evolve independently one of the other, and at each step, each particle, say y, is replaced birth to a random cloud of particles $y + \mathcal{Z}$, where the law of $\mathcal{Z}$ is that of $(\ln X_i(1), i \in \mathbb{N})$. This means that $(Z^{(n)}, n \geq 0)$ is a branching random walk in the sense of [5], [22], ... The latter have been throughoutly studied in the literature, and many of their properties can be translated to homogeneous fragmentation processes. For instance, we can derive precise information on almost sure large deviations for the empirical measure; using a genuine result of Biggins [24] for branching random walks.

Corollary 2 *Assume (9) and that the dislocation measure ν is non-geometric, and for $p \in]\underline{p}, \bar{p}[$, let $M(p, \infty)$ denote the terminal value of the uniformly integrable martingale $M(p, \cdot)$. If $\bar{f} : \mathbb{R} \to \mathbb{R}$ is a function with compact support which is directly Riemann integrable, then*

$$\lim_{t \to \infty} \sqrt{t}\, \mathrm{e}^{-t(p\kappa'(p) - \kappa(p))} \int_{\mathbb{R}} f(t\kappa'(p) + y) Z^{(t)}(dy) = \frac{M(p, \infty)}{\sqrt{2\pi |\kappa''(p)|}} \int_{-\infty}^{\infty} f(y) \mathrm{e}^{-py} dy .$$

uniformly for p in compact subsets of $]\underline{p}, \bar{p}[$, almost surely.

In a different direction, one can also use the discretization techniques to estimate the probability of presence of abnormally large fragments as time goes to infinity; see [19] for details.

Corollary 3 *Assume (9) and that the dislocation measure ν is non-geometric, fix two real numbers $a < b$ and take $p > \bar{p}$. Then as $t \to \infty$*

$$\mathbb{P}\left(\exists i \in \mathbb{N} : X_i(t) \in [a\mathrm{e}^{-t\kappa'(p)}, b\mathrm{e}^{-t\kappa'(p)}]\right) \sim Kt^{-1/2}\mathrm{e}^{t(p\kappa'(p)-\kappa(p))},$$

where K is some positive and finite constant depending on a, b and the characteristics of the fragmentation.

6.4   Changing the index of self-similarity

So far, we have only been able to study fragmentations with instantaneous dislocations in the homogeneous case, i.e. when the index of self-similarity is $\alpha = 0$. In this section, we present a simple transformation that changes a homogeneous fragmentation X into a self-similar one with an arbitrary index of self-similarity, $X^{(\alpha)}$, completing the construction of general self-similar fragmentation processes. We refer to [12] for details.

In this direction, it is convenient to start from an interval-representation $(G(t), t \geq 0)$ of some homogeneous fragmentation X as in section 6.1. For every $y \in]0, 1[$, let $I_y(t)$ denote the interval component of $G(t)$ that contains y if $y \in G(t)$, and $I_y(t) = \emptyset$ otherwise. We write $|I|$ for the length of an interval $I \subseteq]0, 1[$, and for every $y \in]0, 1[$ we consider the time-substitution

$$T_y^{(\alpha)}(t) := \inf\left\{u \geq 0 : \int_0^u |I_y(v)|^{-\alpha}dv > t\right\}.$$

Because the open sets $G(t)$ are nested, we see that for every $y, z \in]0, 1[$, the intervals $I_y(T_y^{(\alpha)}(t))$ and $I_z(T_z^{(\alpha)}(t))$ are either identical or disjoint, so the family $\left\{I_y(T_y^{(\alpha)}(t)), 0 < y < 1\right\}$ can be viewed as the interval components of an open set $G^{(\alpha)}(t)$. It is straightforward that the family $(G^{(\alpha)}(t), t \geq 0)$ is nested. More precisely, if we write $X^{(\alpha)}(t)$ for the ordered sequence of the lengths of the interval components of $G^{(\alpha)}(t)$, then $(X^{(\alpha)}(t), t \geq 0)$ is a self-similar fragmentation with index α.

Any self-similar fragmentation $X^{(\alpha)}$ can be constructed from some homogeneous one X as above, and this construction can be inverted. In particular, the distribution of $X^{(\alpha)}$ is entirely determined by the index of self-similarity α, and the erosion coefficient $c \geq 0$ and the dislocation measure ν of the homogeneous fragmentation X.

The key tool which is needed to extend the results of Sections 3-5 to self-similar fragmentation with instantaneous dislocations is provided by the stochastic structure of the process of the tagged fragment, $(\chi^{(\alpha)}(t), t \geq 0)$. Recall that in the homogeneous case, $(\chi(t), t \geq 0)$ can be described as the exponential of a subordinator. The construction above of a self-similar fragmentation from a homogeneous one by time substitution enables us to derive the law of the tagged fragment $\chi^{(\alpha)}(t)$ for a self-similar fragmentation.

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Specifically, let $\sigma = (\sigma(t), t \geq 0)$ be a subordinator with Laplace exponent $\kappa(1 + \cdot)$ given by (8). Introduce the time-change

$$\tau(t) = \inf\left\{u : \int_0^u \exp(\alpha\sigma(r))dr > t\right\}, \qquad t \geq 0,$$

and set $\zeta_t = \exp(-\sigma(\tau(t)))$ (with the convention that $\zeta_t = 0$ if $\tau(t) = \infty$). Then the processes $(\zeta_t, t \geq 0)$ and $\left(\chi^{(\alpha)}(t), t \geq 0\right)$ have the same law. In particular, this shows that the process of the tagged fragment is a decreasing self-similar Markov process as introduced by Lamperti [40], see the survey [20] and the references therein.

Results stated for self-similar fragmentation chains in Sections 4-5 can be extended verbatim to self-similar fragmentation processes; see [13, 15, 18, 32].

7   Duality with certain coalescent processes

Coagulation processes are used as models to describe the evolution of particle systems in which pairs (or, more generally, families) of particles merge as time passes. At this level of generality, a simple time-reversal provides an obvious connection with fragmentation.

However in practice, just as for fragmentation, one has to make some restrictive assumptions on the dynamics of coagulation in order to deal with processes that can be studied mathematically. First, one assumes that particles are determined by their masses (i.e. a positive real number such that the total mass is a preserved quantity when merging occurs). So there is no geometry involved in the system; physicists call such models *mean-field*. Second, one assumes that the evolution is Markovian, and third, that the rate at which a family of particles merges only depends on the particles in this family, and not on the other particles in the system. We shall call *stochastic coalescent* a coagulation process that fulfils these requirements, and refer to the survey by Aldous [2] and the references therein for much on this notion.

The first two requirements (mean-field and Markov properties) are clearly compatible with time-reversal. However, even though the third one bears some vague resemblance with the branching property, there is in general no reason why it should yield the latter after time-reversal. Despite the absence of a general result on duality by time-reversal between fragmentation and coalescent processes, there are nontheless several important examples for which duality holds. It would be very interesting to establish general critera for duality; we leave this question open as a challenge, and now conclude this survey by discussing some examples.

The first example is the simplest, it can be constructed from a sequence $U_1, \ldots$ of independent uniform variables on $[0, 1]$ and an independent Poisson process $N = (N_t, t \geq 0)$. Specifically, for each time $t \geq 0$, write $F(t)$ for the ranked sequence of the lenghts of the interval components of the random open set $]0, 1[\backslash\{U_i : 1 \leq i \leq N_t\}$. It is easy to see that $F = (F(t), t \geq 0)$ is a self-similar fragmentation chain with index $\alpha = 1$. More precisely, its dislocation law is that of the random mass partition $(1 - U/2, U/2, 0, \ldots)$ where U is uniformly distributed on $[0, 1]$. Then it can be checked (see [14] for details) that the exponential time reversal $C(t) := F(\mathrm{e}^{-t})$ transforms the fragmentation process F into a coalescent process C. Specifically, when there are $n \geq 2$ particles in the coalescent C, the first coagulation occurs after an exponential time

with parameter n, and the pair of particles involved is uniformly distributed amongst the $n(n-1)/2$ possible pairs. One reckognizes a variation of the celebrated coalescent of Kingman [37] (which is used to describe the genealogy of large populations and has many applications in Biology). More generally, a similar duality holds for self-similar fragmentation chains with index $\alpha = 1$ and dislocation law given in terms of certain n-dimensional Dirichlet distributions. These fragmentation and coalescent processes appear in the genealogy of Yule processes; see [16].

The second example concerns the *additive coalescent*, a coagulation process which arises e.g. as a model for the formation of drops of rain in clouds. Roughly speaking, in an additive coalescent, any pair of particles, say (x, y), merges at rate $x+y$, independently of the other pairs in the system. Evans and Pitman [28] have observed that if $C^{+,n} = (C^{+,n}(t), t \geq 0)$ denotes the process started from the monodisperse initial condition which consists in n particles each with mass $1/n$, then as $n \to \infty$, the process $(C^{+,n}(t + \frac{1}{2} \ln n), t \geq -\frac{1}{2} \ln n)$ converges in the sense of finite dimensional distributions to $C^+ := (C^+(t), t \in \mathbb{R})$. The latter is known as the *standard* additive coalescent. Again the exponential time-change, $F(t) = C^+(-\ln t)$, transforms the coalescent into a fragmentation process which is self-similar with index $\alpha = 1/2$. We refer to [4, 10] for more on this topic. In this direction, we also mention that Miermont [45] has shown that the exponential time-change also transforms some (but not all) non-standard additive coalescent into fragmentation processes; however the latter are not self-similar in general.

The final example is due to Pitman [50] who established a remarkable duality between certain coagulation and fragmentation operators based on adequat Poisson-Dirichlet distributions. An important special case involves Ruelle cascades and the Bolthausen-Sznitman coalescent. The former have been introduced in [53] as a tool for studying Derrida's Generalized Random Energy Model of spin glass; and the latter in [26] to describe the time-reversed dynamics in Ruelle's cascades. The Bolthausen-Sznitman coalescent also appears naturally in the genealogy of Neveu's branching process, see [17]. Pitman's duality result enables us to view Ruelle's cascades as a time-inhomogeneous fragmentation process, see also Basdevant [6] for a recent development in this direction.

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