Law of the time to absorption at zero of a (not-necessarily) symmetric stable Lévy process

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Stable processes

**Definition I**

A Lévy process $X$ is called (strictly) $\alpha$-stable if it satisfies the scaling property

$$(cX_{c^{-\alpha}t})_{t\geq 0}\bigg|_{P_x} \overset{d}{=} X|_{P_{cx}}, \quad c > 0.$$  

Necessarily $\alpha \in (0, 2]$. [$\alpha = 2 \rightarrow$ BM, exclude this.]

The quantity $\rho = P_0(X_t \geq 0)$ will frequently appear as will $\hat{\rho} = 1 - \rho$. 
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The quantity $\rho = P_0(X_t \geq 0)$ will frequently appear as will $\hat{\rho} = 1 - \rho$.

**Definition II**

Let $\alpha, \rho$ be admissible parameters, $X$ the Lévy process with Lévy density

$$
c_+ x^{-(\alpha+1)} \mathbb{1}_{x > 0} + c_- |x|^{-(\alpha+1)} \mathbb{1}_{x < 0}, \quad x \in \mathbb{R},
$$

no Gaussian part.
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**Stable processes**

Additional notes:

- $X$ does not have one-sided jumps,
- We assume that $\alpha \in (1, 2)$, in which case $X$ is point-recurrent.
The problem

Let

$$T_0 = \inf\{ t > 0 : X_t = 0 \}$$

be the first hitting time of \{0\}. Can we find an explicit expression for

$$p(t)dt := P_1(T_0 \in dt)?$$
### Problem: history


Positive, self-similar Markov processes

$\alpha$-pssMp

$[0, \infty)$-valued Markov process, equipped with initial measures $P_x$, $x > 0$, with 0 an absorbing state, satisfying the scaling property

$$(cX_{c^{-\alpha}t})_{t \geq 0} \bigg|_{P_x} \overset{d}{=} X|_{P_{cx}}, \quad x, c > 0$$
Lamperti transform

\[(X, P_x)_{x>0} \text{ pssMp} \quad \leftrightarrow \quad (\xi, P_y)_{y \in \mathbb{R}} \text{ killed Lévy} \]

\[X_t = \exp(\xi S(t)), \quad \xi_s = \log(X_{T(s)}), \]

\[S \text{ a random time-change} \quad T \text{ a random time-change} \]
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\[\begin{align*}
X \text{ never hits zero} & \quad \leftrightarrow \quad \xi \to \infty \text{ or } \xi \text{ oscillates} \\
X \text{ hits zero continuously} & \quad \leftrightarrow \quad \xi \to -\infty \\
X \text{ hits zero by a jump} & \quad \leftrightarrow \quad \xi \text{ is killed}
\end{align*} \]
Example 1

Let $X$ be a stable process, and define $X^* t = X_t 1(t < \tau - 0)$, $t \geq 0$, where $\tau - 0 = \inf\{t > 0 : X_t < 0\}$.

Then $X^*$ is a pssMp, with Lamperti transform $\xi^*$. $\xi^*$ has Lévy density $c + e^x (e^x - 1)^{\alpha + 1} 1(x > 0) + c - e^x (1 - e^x)^{\alpha + 1} 1(x < 0)$, and is killed at rate $c - \alpha/\alpha = \Gamma(\alpha) / \Gamma(\alpha \hat{\rho}) \Gamma(1 - \alpha \hat{\rho})$. 
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$$c_+ \frac{e^x}{(e^x - 1)^{\alpha+1}} \mathbb{1}_{(x>0)} + c_- \frac{e^x}{(1 - e^x)^{\alpha+1}} \mathbb{1}_{(x<0)},$$

and is killed at rate $c_- / \alpha = \frac{\Gamma(\alpha)}{\Gamma(\alpha \hat{\rho}) \Gamma(1 - \alpha \hat{\rho})}$. 
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Let $X$ be a symmetric $\alpha$-stable process with $\alpha \in (1, 2)$, and define

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Then $R$ is a pmM with Lamperti-transform $\xi = \xi^L \oplus \xi^C$, such that

(i) The Lévy process $\xi^L$ has characteristic exponent

$$\Psi^*(\theta) - k/\alpha, \quad \theta \in \mathbb{R},$$

where $\Psi^*$ is the characteristic exponent of the process $\xi^*$. 

(ii) The process $\xi^C$ is a compound Poisson process whose jumps occur at rate $k/\alpha$, whose Lévy density is

$$\pi^C(y) = k \frac{e^y}{(1 + e^y)^{\alpha+1}}, \quad y \in \mathbb{R}.$$
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(i) 

$$\psi(\theta) = 2^{\alpha} \frac{\Gamma(\alpha/2 - i\theta/2)}{\Gamma(-i\theta/2)} \frac{\Gamma(1/2 + i\theta/2)}{\Gamma((1-\alpha)/2 + i\theta/2)}, \quad \theta \in \mathbb{R}.$$ 

(ii) For later convenience we also note $\psi(z) := \log \mathbb{E} e^{-z\alpha \xi_1}$ is given by

$$\psi(z) = -2^{\alpha} \frac{\Gamma(1/2 - \alpha z/2)}{\Gamma(1/2 - \alpha(1+z)/2)} \frac{\Gamma(\alpha(1+z)/2)}{\Gamma(\alpha z/2)}, \quad \text{Re } z \in (-1, 1/\alpha).$$
Standard theory for pssMp

(i) \((T_0, P_1)\) has the same law as \((I(\alpha \xi), \mathbb{P}_0)\), where

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\[ \mathcal{M}(s + 1) = -\frac{s}{\psi(-s)} \mathcal{M}(s), \]
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(iii) Because of the explicit form of $\psi$, we can guess (and then prove) that

$$\mathbb{E}_1[T_0^{s-1}] = \sin(\pi/\alpha) \frac{\cos\left(\frac{\pi \alpha}{2}(s - 1)\right)}{\sin\left(\pi(s - 1 + \frac{1}{\alpha})\right)} \frac{\Gamma(1 + \alpha - \alpha s)}{\Gamma(2 - s)},$$

for $\text{Re} \, s \in (-\frac{1}{\alpha}, 2 - \frac{1}{\alpha})$. 
Markov additive processes (MAPs)

Let $E$ be a finite state space and $(\mathcal{G}_t)_{t \geq 0}$ a standard filtration. A càdlàg process $(\xi, J)$ in $\mathbb{R} \times E$ with law $\mathbb{P}$ is called a **Markov additive process (MAP)** with respect to $(\mathcal{G}_t)_{t \geq 0}$ if $(J(t))_{t \geq 0}$ is a continuous-time, irreducible Markov chain in $E$, and the following property is satisfied, for any $i \in E$, $s, t \geq 0$:

Given $\{J(t) = i\}$, the pair $(\xi(t + s) - \xi(t), J(t + s))$ is independent of $\mathcal{G}_t$, and has the same distribution as $(\xi(s) - \xi(0), J(s))$ given $\{J(0) = i\}$. 

Pathwise description of a MAP

The pair \((\xi, J)\) is a Markov additive process if and only if, for each \(i, j \in E\), there exist a sequence of iid Lévy processes \((\xi^n_i)_{n \geq 0}\) and a sequence of iid random variables \((U^n_{ij})_{n \geq 0}\), independent of the chain \(J\), such that if \(T_0 = 0\) and \((T_n)_{n \geq 1}\) are the jump times of \(J\), the process \(\xi\) has the representation

\[
\xi(t) = \mathbb{1}_{(n>0)}(\xi(T_n^-) + U^n_{J(T_n^-), J(T_n)} + \xi^n_{J(T_n)}(t - T_n)),
\]

for \(t \in [T_n, T_{n+1})\), \(n \geq 0\).
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$$X_t = x \exp \{ \xi(\tau(t)) + i\pi(J(\tau(t)) + 1) \quad 0 \leq t < T_0, \}$$

where

$$\tau(t) = \inf \left\{ s > 0 : \int_0^s \exp(\alpha \xi(u)) du > t|x|^{-\alpha} \right\}$$

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Then $X_t$ is a real-valued self-similar Markov process in the sense that the law of $(cX_{tc^{-\alpha}} : t \geq 0)$ under $P_x$ is $P_{cx}$. 
rssMps, MAPs, Lamperti-Kiu (Chaumont, Panti, Rivero)

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- The converse (within a special class of rssMps) is also true.
Characteristics of a MAP

- Denote the transition rate matrix of the chain $J$ by $Q = (q_{ij})_{i,j \in E}$. 
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- Let
  \[ F(z) = \text{diag}(\psi_1(z), \ldots, \psi_N(z)) + Q \circ G(z), \]
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  (when it exists), where $\circ$ indicates elementwise multiplication.
- The matrix exponent of the MAP $(\xi, J)$ is given by
  \[ \mathbb{E}_i(e^{z \xi(t)}; J(t) = j) = (e^{F(z)t})_{i,j}, \]
  \[ i, j \in E, \]
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- Denote the underlying MAP $(\xi, J)$, we prefer to give the matrix exponent of $(-\alpha \xi, J)$ as follows:

$$F(z) = \begin{pmatrix}
\frac{\Gamma(\alpha(1 + z))\Gamma(1 - \alpha z)}{\Gamma(\alpha\hat{\rho} + \alpha z)\Gamma(1 - \alpha\hat{\rho} - \alpha z)} & \frac{\Gamma(\alpha(1 + z))\Gamma(1 - \alpha z)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha\hat{\rho})} \\
-\frac{\Gamma(\alpha(1 + z))\Gamma(1 - \alpha z)}{\Gamma(\alpha\hat{\rho} + \alpha z)\Gamma(1 - \alpha\hat{\rho} - \alpha z)} & -\frac{\Gamma(\alpha(1 + z))\Gamma(1 - \alpha z)}{\Gamma(\alpha\rho + \alpha z)\Gamma(1 - \alpha\rho - \alpha z)}
\end{pmatrix}$$

for $\text{Re } z \in (-1, 1/\alpha)$. 
Cramér condition for a MAP

Proposition

(i) Suppose that $z \in \mathbb{C}$ is such that $F(z)$ is defined. Then, the matrix $F(z)$ has a real simple eigenvalue $\kappa(z)$, which is larger than the real part of all its other eigenvalues.

(ii) Suppose that $F$ is defined in some open interval $D$ of $\mathbb{R}$. Then, the leading eigenvalue $\kappa$ of $F$ is smooth and convex on $D$. 

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Assumption (Cramér condition for a MAP)

There exists $z_0 < 0$ such that $F(s)$ exists on $(z_0, 0)$, and some $\theta \in (0, -z_0)$, called the Cramér number, such that $\kappa(-\theta) = 0$. 
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Note that this dictates “$\kappa'(0) > 0$” which ensures that $\lim_{t \uparrow \infty} \frac{\xi_t}{t} = \kappa'(0) > 0$.
For a MAP \( \xi \), let

\[ I(-\xi) = \int_0^\infty \exp(-\xi(t)) \, dt. \]
Integrated exponential MAPs

For a MAP $\xi$, let

$$I(-\xi) = \int_0^\infty \exp(-\xi(t)) \, dt.$$ 

One way to characterise the law of $I(-\xi)$ is via its Mellin transform, which we write as $\mathcal{M}(s)$. This is the vector in $\mathbb{R}^N$ whose $i$th element is given by

$$\mathcal{M}_i(s) = \mathbb{E}_i[I(-\xi)^{s-1}], \quad i \in E.$$
Proposition

Suppose that $\xi$ satisfies the Cramér condition with Cramér number $\theta \in (0, 1)$. Then, $\mathcal{M}(s)$ is finite and analytic when $\Re s \in (0, 1 + \theta)$, and we have the following vector-valued functional equation:

$$\mathcal{M}(s + 1) = -s(F(-s))^{-1}\mathcal{M}(s), \text{ for } s \in (0, \theta).$$
Back to the case of an $\alpha$-stable process, $\alpha \in (1, 2)$

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- Recall that \( T_0 = \int_0^\infty \exp\{-(-\alpha \xi(u))\}du \) and that \( E = \{1, 2\} \).
- It is obvious (using asymmetry) that \( \mathbb{E}_1(T_0^{s-1}) \) is the same expression as \( \mathbb{E}_2(T_0^{s-1}) \) modulo interchanging the roles of \( \rho \) and \( \hat{\rho} \).
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- Easy to check that \( \kappa(1/\alpha - 1) = 0 \), i.e. \( \theta = 1 - 1/\alpha < 1 \).
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- It is obvious (using asymmetry) that $E_1(T_0^{s-1})$ is the same expression as $E_2(T_0^{s-1})$ modulo interchanging the roles of $\rho$ and $\hat{\rho}$.
- Easy to check that $\kappa(1/\alpha - 1) = 0$, i.e. $\theta = 1 - 1/\alpha < 1$.
- **Guess** a solution to the vector-valued functional equation and then **verify uniqueness**

**Theorem**

For $-1/\alpha < \text{Re}(s) < 2 - 1/\alpha$ we have

$$E_1[T_0^{s-1}] = \frac{\sin\left(\frac{\pi}{\alpha}\right) \sin\left(\pi \hat{\rho}(1 - \alpha + \alpha s)\right)}{\sin(\pi \hat{\rho}) \sin\left(\frac{\pi}{\alpha}(1 - \alpha + \alpha s)\right)} \frac{\Gamma(1 + \alpha - \alpha s)}{\Gamma(2 - s)}.$$
Inversion (rational $\alpha \in (1, 2)$): $p(t) = dP_1(T_0 \leq t)/dt$

If $\alpha = m/n$ (where $m$ and $n$ are coprime natural numbers) then for all $t > 0$ we have

$$p(t) = \frac{\sin \left( \frac{\pi}{\alpha} \right)}{\pi \sin(\pi \hat{\alpha})} \sum_{k \geq 1, \atop k \neq -1 \mod m} \sin(\pi \hat{\alpha}(k + 1)) \frac{\sin \left( \frac{\pi k}{\alpha} \right)}{\sin \left( \frac{\pi}{\alpha} (k + 1) \right)} \frac{\Gamma \left( \frac{k}{\alpha} + 1 \right)}{k!} (-1)^{k-1} t^{1 - \frac{k}{\alpha}}$$

$$- \frac{\sin \left( \frac{\pi}{\alpha} \right)^2}{\pi \sin(\pi \hat{\alpha})} \sum_{k \geq 1, \atop k \neq 0 \mod n} \sin(\pi \alpha \hat{\alpha}k) \frac{\Gamma \left( k - \frac{1}{\alpha} \right)}{\sin(\pi \alpha k) \Gamma (\alpha k - 1)} t^{-k + 1 + \frac{1}{\alpha}}$$

$$- \frac{\sin \left( \frac{\pi}{\alpha} \right)^2}{\pi^2 \alpha \sin(\pi \hat{\alpha})} \sum_{k \geq 1} (-1)^{km} \frac{\Gamma \left( kn - \frac{1}{\alpha} \right)}{(km - 2)!} R_k(t) t^{-kn + 1 + \frac{1}{\alpha}},$$

where

$$R_k(t) := \pi \alpha \hat{\alpha} \cos(\pi \hat{\alpha} km)$$

$$- \sin(\pi \hat{\alpha} km) \left[ \pi \cot \left( \frac{\pi}{\alpha} \right) - \psi(kn - \frac{1}{\alpha}) + \alpha \psi(km - 1) + \ln(t) \right].$$

The three series converge uniformly for $t \in [\varepsilon, \infty)$ and any $\varepsilon > 0$. 
Inversion (almost every irrational $\alpha \in (1, 2)$)

Define $||x|| = \min_{n \in \mathbb{Z}} |x - n|$, and

$$\mathcal{L} = \mathbb{R} \setminus (\mathbb{Q} \cup \{ x \in \mathbb{R} : \lim_{n \to \infty} \frac{1}{n} \ln ||nx|| = 0 \}).$$

If $\alpha \notin \mathcal{L} \cup \mathbb{Q}$ then

$$p(t) = \frac{\sin \left( \frac{\pi}{\alpha} \right)}{\pi \sin(\pi \hat{\rho})} \sum_{k \geq 1} \sin(\pi \hat{\rho}(k + 1)) \frac{\sin \left( \frac{\pi}{\alpha} k \right)}{\sin \left( \frac{\pi}{\alpha} (k + 1) \right)} \frac{\Gamma \left( \frac{k}{\alpha} + 1 \right)}{k!} (-1)^{k-1} t^{-1 - \frac{k}{\alpha}}$$

$$- \frac{\sin \left( \frac{\pi}{\alpha} \right)^2}{\pi \sin(\pi \hat{\rho})} \sum_{k \geq 1} \frac{\sin(\pi \alpha \hat{\rho}k)}{\sin(\pi \alpha k)} \frac{\Gamma \left( \frac{k - 1}{\alpha} \right)}{\Gamma(\alpha k - 1)} t^{-k - 1 + \frac{1}{\alpha}}.$$

The two series in the right-hand side of the above formula converge uniformly for $t \in [\varepsilon, \infty)$ and any $\varepsilon > 0$. 
Conditioning to avoid zero (Chaumont, Panti, Rivero)

Let $X$ be an $\alpha$ stable process with $\alpha \in (1, 2)$ and let $h$ the function

$$h(x) = -\Gamma(1 - \alpha) \frac{\sin(\pi \alpha \hat{\rho})}{\pi} x^{\alpha - 1}, \quad x > 0,$$

and the same expression with $\hat{\rho}$ replaced by $\rho$ when $x < 0$.

- The function $h$ is invariant for the stable process killed on hitting 0, that is,

$$E_x[h(X_t), t < T_0] = h(x), \quad t > 0, \ x \neq 0. \quad (2)$$

Therefore, we may define a family of measures $P_X^\uparrow$ by

$$P_X^\uparrow(\Lambda) = \frac{1}{h(x)} E_x[h(X_t) \mathbb{1}_\Lambda, t < T_0], \quad x \neq 0, \ \Lambda \in \mathcal{F}_t,$$

for any $t \geq 0$. 
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- The function $h$ can be represented as

$$h(x) = \lim_{q \downarrow 0} \frac{P_x(T_0 > e_q)}{n(\zeta > e_q)}, \quad x \neq 0,$$

where $e_q$ is an independent exponentially distributed random variable with parameter $q$. Furthermore, for any stopping time $T$ and $\Lambda \in \mathcal{F}_T$, and any $x \neq 0$,

$$\lim_{q \downarrow 0} P_x(\Lambda, T < e_q | T_0 > e_q) = P_x^\uparrow(\Lambda).$$
Another representation of $P$

- $P_x(T_0 > t) = P_1(T_0 > x^{-\alpha} t), \text{ for } x > 0, t \geq 0.$
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- The density of $T_0$

$$p(t) = -\frac{\sin^2(\pi/\alpha) \sin(\pi\alpha \rho) \Gamma(1 - 1/\alpha)}{\pi \sin(\pi\bar{\rho}) \sin(\pi\alpha) \Gamma(\alpha - 1)} t^{1/\alpha - 2} + O(t^{-1/\alpha - 1}).$$
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- Stable (inverse) local time at zero:

$$n(\zeta \in dt) = \frac{\alpha - 1}{\Gamma(1/\alpha)} \frac{\sin(\pi / \alpha)}{\cos(\pi (\rho - 1/2))} t^{1/\alpha - 2} dt, \quad t \geq 0.$$
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\]
- Verify directly
  \[
h(x) = \lim_{s \to \infty} \frac{P_x(T_0 > s)}{n(\zeta > s)}.
\]
Another representation of $P^\uparrow$

- For any a.s. finite stopping time $T$ and $\Lambda \in \mathcal{F}_T$,

$$P_x(\Lambda | T_0 > T + s)$$

$$= E_x \left[ \frac{P_x(1_\Lambda, T_0 > T + s | \mathcal{F}_T)}{P_x(T_0 > T + s)} \right]$$

$$= E_x \left[ 1_\Lambda 1(T_0 > T) \frac{P_{X_T}(T_0 > s)}{P_x(T_0 > T + s)} \right]$$

$$= E_x \left[ 1_\Lambda 1(T_0 > T) \frac{h(X_T)}{h(x)} \frac{P_{X_T}(T_0 > s)}{h(X_T)n(\zeta > s)} \frac{n(\zeta > s)}{n(\zeta > T + s)} \frac{h(x)n(\zeta > T + s)}{P_x(T_0 > T + s)} \right].$$

- For any a.s. stopping time $T$, $\Lambda \in \mathcal{F}_T$,

$$P_x^\uparrow(\Lambda) = \lim_{s \to \infty} P_x(\Lambda | T_0 > T + s).$$