THE WIENER-HOPF FACTORIZATION

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Abstract. We give a description of the classical Wiener-Hopf factorization from the point of view of excursion theory concentrating mainly on the case of random walks as opposed to Lévy processes. The exposition relies primarily on the ideas of Greenwood and Pitman (1979, 1980).

Key words: Lévy process, Wiener-Hopf factorization, infinite divisibility.

1 Introduction

A fundamental part of the theory of random walks and Lévy processes is a set of conclusions which in modern times are loosely referred to as *the Wiener–Hopf factorization*. Historically the identities around which the Wiener–Hopf factorization is centred are the culmination of a number of works which include Spitzer (1956, 1957, 1964), Feller (1971), Borovkov (1976), Pecherskii and Rogozin (1969), Gusak (1969), Greenwood and Pitman (1979, 1980), Fristedt (1974) and many others; although the analytical roots of the so-called Wiener–Hopf method go much further back than these probabilistic references; see for example Hopf (1934) and Payley and Wiener (1934). The importance of the Wiener–Hopf factorization for either a random walk or a Lévy process is that it characterizes the range of the process's running maximum as well as the times at which new maxima are attained.

The overview we give here will first deal with the Wiener-Hopf factorization for random walks before moving to the case of Lévy processes. The discussion will follow very closely the ideas of Greenwood and Pitman (1979, 1980). Indeed for the case of random walks we shall not shy away from providing proofs as their penetrating and yet elementary nature reveals a simple path decomposition which is arguably more fundamental than the Wiener-Hopf factorization itself. The Wiener-Hopf factorization for Lévy processes is essentially a technical variant of the case for random walks and we only state it without proof.

2 Random walks and infinite divisibility

Let us start by reminding ourselves of some standard definitions and facts. Suppose that $\{\xi_i : i = 1, 2, ...\}$ are a sequence of $\mathbb{R}$-valued identically and independently distributed random variables defined on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with common distribution function F. Let

$$S_0 = 0 \text{ and } S_n = \sum_{i=1}^{n} \xi_i.$$

The process $S = \{S_n : n \geq 0\}$ is called a (real valued) random walk. For convenience we shall make a number of assumptions on F. First,

$$\min\{F(0,\infty), F(-\infty,0)\} > 0,$$

meaning that the random walk may experience both positive and negative jumps, and second, F has no atoms.

In the prevailing analysis we shall repeatedly refer to general and specific classes of infinitely divisible random variables. For this reason we need also to spend some time discussing the latter.

Suppose that X is an $\mathbb{R}^d$-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, then X is infinitely divisible if for each $n = 1, 2, 3, ...$

$$X \stackrel{d}{=} X_{(1,n)} + \cdots + X_{(n,n)}$$

where $\{X_{(i,n)} : i = 1, ..., n\}$ are independent and identically distributed and the equality is in distribution. Said another way, if μ is the characteristic function of X then for each $n = 1, 2, 3, ...$ we have that $\mu = (\mu_n)^n$ where μ_n is the the characteristic function of some $\mathbb{R}^d$-valued random variable.

In general, if X is any $\mathbb{R}^d$-valued random variable which is also infinitely divisible then for each $\theta \in \mathbb{R}^d$, $E(e^{i\theta \cdot X}) = e^{-\Psi(\theta)}$ where

$$\Psi(\theta) = ia \cdot \theta + \frac{1}{2}Q(\theta) + \int_{\mathbb{R}^d} (1 - e^{i\theta \cdot x} + i\theta \cdot x \mathbf{1}_{(|x|<1)})\Pi(dx), \quad (1)$$

$a \in \mathbb{R}^d$, Q is a positive semi-definite quadratic form on $\mathbb{R}^d$ and Π is a measure supported in $\mathbb{R}^d \backslash \{0\}$ such that

$$\int_{\mathbb{R}^d} 1 \wedge |x|^2 \Pi(dx) < \infty.$$

Here, $|\cdot|$ is Euclidian distance and, for $a, b \in \mathbb{R}^d$, $a \cdot b$ is the usual Euclidian inner product.

A special example of an infinitely divisible distribution is the Geometric distribution. The symbol $\mathbf{\Gamma}_p$ will always denote a geometric distribution with parameter $p \in (0, 1)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$. In particular,

$$P(\mathbf{\Gamma}_p = k) = pq^k, \;\; k = 0, 1, 2, ...$$

where $q = 1-p$. The geometric distribution has the the following nice properties which are worth recalling for the forthcoming discussion. Firstly $P(\mathbf{\Gamma}_p \geq k) =$

q^k and secondly the lack of memory property: $P(\mathbf{\Gamma}_p \geq n+m | \mathbf{\Gamma}_p \geq m) = P(\mathbf{\Gamma}_p \geq n)$ for all $n, m = 0, 1, 2, ...$

A more general class of infinitely divisible distributions than the latter, which will shortly be of use, are those which may be expressed as the distribution of a random walk sampled at an independent and geometrically distributed time; $S_{\mathbf{\Gamma}_p} = \sum_{i=1}^{\mathbf{\Gamma}_p} \xi_i$. (Note, we interpret $\sum_{i=1}^{0}$ as the empty sum). To justify the previous claim, a straightforward computation shows that for each $n = 1, 2, 3, ...$

$$\mathbb{E}(e^{i\theta S_{\mathbf{\Gamma}_p}}) = \left(\left(\frac{p}{1 - q\mathbb{E}(e^{i\theta \xi_1})}\right)^{\frac{1}{n}}\right)^n = \mathbb{E}(e^{i\theta S_{\mathbf{\Lambda}_{1/n,p}}})^n$$

where $\mathbf{\Lambda}_{1/n,p}$ is a negative Binomial random variable with parameters $1/n$ and p which is independent of S. The latter has distribution mass function

$$\mathbb{P}(\mathbf{\Lambda}_{1/n,p} = k) = \frac{1}{k!}\frac{\Gamma(k+1/n)}{\Gamma(1/n)} p^{1/n} q^k$$

for $k = 0, 1, 2, ...$

3 The Wiener-Hopf factorization for random walks

We now turn out attention to the Wiener-Hopf factorization. Fix $0 < p < 1$ and define

$$G = \inf\{k = 0, 1, ..., \mathbf{\Gamma}_p : S_k = \max_{j=0,1,...,\mathbf{\Gamma}_p} S_j\}$$

where $\mathbf{\Gamma}_p$ is a geometrically distributed random variable with parameter p which is independent of the random walk S. In words, G is the first visit of S to its maximum over the time period $\{0, 1, ..., \mathbf{\Gamma}_p\}$. Now define

$$N = \inf\{n > 0 : S_n > 0\}.$$

In words, the first visit of S to $(0, \infty)$ *after time* 0.

Theorem 1 (Wiener-Hopf factorization for random walks) *Assume all of the notation and conventions above.*

(i) (G, S_G) *is independent of* $(\mathbf{\Gamma}_p - G, S_{\mathbf{\Gamma}_p} - S_G)$ *and both pairs are infinitely divisible*

(ii) *For* $0 < s \leq 1$ *and* $\theta \in \mathbb{R}$

$$E(s^G e^{i\theta S_G}) = \exp\left\{-\int_{(0,\infty)} \sum_{n=1}^{\infty} (1 - s^n e^{i\theta x}) q^n \frac{1}{n} F^{*n}(dx)\right\}.$$

(iii) *For* $0 < s \leq 1$ *and* $\theta \in \mathbb{R}$

$$E(s^N e^{i\theta S_N}) = 1 - \exp\left\{-\int_{(0,\infty)} \sum_{n=1}^{\infty} s^n e^{i\theta x} \frac{1}{n} F^{*n}(dx)\right\}.$$

Note that the third part of the Wiener-Hopf factorization characterizes what is known as the *ladder height process* of the random walk S. The latter is the bivariate random walk $(T, H) := \{(T_n, H_n) : n = 0, 1, 2, ...\}$ where $(T_0, H_0) = (0, 0)$ and otherwise for $n = 1, 2, 3, ...$,

$$T_n = \begin{cases} \min\{k = 1, 2, ... : S_{T_{n-1}+k} > H_{n-1}\} & \text{if } T_{n-1} < \infty \\ \infty & \text{if } T_{n-1} = \infty \end{cases}$$

and

$$H_n = \begin{cases} S_{T_n} & \text{if } T_n < \infty \\ \infty & \text{if } T_n = \infty. \end{cases}$$

That is to say, the process (T, H), until becoming infinite in value, represents the times and positions of the running maxima of S; the so-called ladder times and ladder heights. It is not difficult to see that T_n is a stopping time for each $n = 0, 1, 2, ..$ and hence thanks to the i.i.d. increments of S, the increments of (T, H) are independent and identically distributed with the same law as the pair (N, S_N).

Proof of Theorem 1. (i) The path of the random walk may be broken into $\nu \in \{0, 1, 2, ...\}$ finite (or completed) excursions from the maximum followed by an additional excursion which straddles the random time $\mathbf{\Gamma}_p$. Here we understand the use of the word straddle to mean that if ℓ is the index of the left end point of the straddling excursion then $\ell \leq \mathbf{\Gamma}_p$. By the Strong Markov Property for random walks and lack of memory, the completed excursions must have the same law; namely that of a random walk sampled on the time points $\{1, 2, ..., N\}$ conditioned on the event that $\{N \leq \mathbf{\Gamma}_p\}$ and hence ν is geometrically distributed with parameter $1 - P(N \leq \mathbf{\Gamma}_p)$. Mathematically we write

$$(G, S_G) = \sum_{i=1}^{\nu} (N^{(i)}, H^{(i)})$$

where the pairs $\{(N^{(i)}, H^{(i)}) : i = 1, 2, ...\}$ are independent having the same distribution as (N, S_N) conditioned on $\{N \leq \mathbf{\Gamma}_p\}$. Note also that G is the sum of the lengths of the latter conditioned excursions and S_G is the sum of the respective increment of the terminal value over the initial value of each excursion. In other words, (G, S_G) is the component-wise sum of ν independent copies of (N, S_N) (with $(G, S_G) = (0, 0)$ if $\nu = 0$). Infinite divisibility follows as a consequence of the fact that (G, S_G) is a geometric sum of i.i.d. random variables. The independence of (G, S_G) and $(\mathbf{\Gamma}_p - G, S_{\mathbf{\Gamma}_p} - S_G)$ is immediate from the decomposition described above.

Feller's classic Duality Lemma (cf. Feller (1971)) for random walks says that for any $n = 0, 1, 2...$ (which may later be randomized with an independent Geometric distribution), the independence and common distribution of increments implies that $\{S_{n-k} - S_n : k = 0, 1, ..., n\}$ has the same law as $\{-S_k : k = 0, 1, ..., n\}$. In the current context, the Duality Lemma also implies that the pair $(\mathbf{\Gamma}_p - G, S_{\mathbf{\Gamma}_p} - S_G)$ is equal in distribution to (D, S_D) where

$$D := \sup\{k = 0, 1, ..., \mathbf{\Gamma}_p : S_k = \min_{j=0,1,...,\mathbf{\Gamma}_p} S_j\}.$$

(ii) Note that, as a geometric sum of i.i.d. random variables, the pair $(\mathbf{\Gamma}_p, S_{\mathbf{\Gamma}_p})$ is infinitely divisible For $s \in (0,1)$ and $\theta \in \mathbb{R}$, let $q = 1-p$ and note that on the one hand,

$$\begin{aligned} E(s^{\mathbf{\Gamma}_p} e^{i\theta S_{\mathbf{\Gamma}_p}}) &= E(E(se^{i\theta S_1})^{\mathbf{\Gamma}_p}) \\ &= \sum_{k\geq 0} p(qsE(e^{i\theta S_1}))^k \\ &= \frac{p}{1-qsE(e^{i\theta S_1})}. \end{aligned}$$

On the other hand, with the help of Fubini's Theorem,

$$\begin{aligned} &\exp\left\{-\int_{\mathbb{R}} \sum_{n=1}^{\infty} (1-s^n e^{i\theta x}) q^n \frac{1}{n} F^{*n}(dx)\right\} \\ &= \exp\left\{-\sum_{n=1}^{\infty} (1-s^n E(e^{i\theta S_n})) q^n \frac{1}{n}\right\} \\ &= \exp\left\{-\sum_{n=1}^{\infty} (1-s^n E(e^{i\theta S_1})^n) q^n \frac{1}{n}\right\} \\ &= \exp\left\{\log(1-q) - \log(1-sqE(e^{i\theta S_1}))\right\} \\ &= \frac{p}{1-qsE(e^{i\theta S_1})} \end{aligned}$$

where in the last equality we have appealed to the Mercator-Newton series expansion of the logarithm. Comparing the conclusions of the last two series of equalities, the required expression for $E(s^{\mathbf{\Gamma}_p} e^{i\theta S_{\mathbf{\Gamma}_p}})$ follows. The Lévy measure mentioned in (1) is thus identifiable as

$$\Pi(dy, dx) = \sum_{n=1}^{\infty} \delta_{\{n\}}(dy) F^{*n}(dx) \frac{1}{n} q^n$$

for $(y, x) \in \mathbb{R}^2$.

We know that $(\mathbf{\Gamma}_p, S_{\mathbf{\Gamma}_p})$ may be written as the independent sum of (G, S_G) and $(\mathbf{\Gamma}_p - G, S_{\mathbf{\Gamma}_p} - S_G)$ where both are infinitely divisible Further, the former has Lévy measure supported on $\{1, 2, ...\} \times (0, \infty)$ and the latter has Lévy measure supported on $\{1, 2, ...\} \times (-\infty, 0)$. Further, $E(s^G e^{i\theta S_G})$ extends to the upper half of the complex plane in θ (and is continuous to the real axis) and $E(s^{\mathbf{\Gamma}_p - G} e^{i\theta(S_{\mathbf{\Gamma}_p} - S_G)})$ extends to the lower half half of the complex plane in θ (and is continuous to the real axis)[1]. Taking account of (1), this forces the factorization of the expression for $E(s^{\mathbf{\Gamma}_p} e^{i\theta S_{\mathbf{\Gamma}_p}})$ in such a way that

$$E(s^G e^{i\theta S_G}) = \exp\left\{-\int_{(0,\infty)} \sum_{n=1}^{\infty} (1-s^n e^{i\theta x}) q^n \frac{1}{n} F^{*n}(dx)\right\}. \tag{2}$$

[1] It is this part of the proof which makes the connection with the general analytic technique of the Wiener-Hopf method of factorising operators. This also explains the origin of the terminology 'Weiner-Hopf factorization' for what is otherwise a path, and consequently distributional, decomposition.

(iii) Note that the path decomposition given in part (i) shows that

$$E(s^G e^{i\theta S_G}) = E(s^{\sum_{i=1}^{\nu} N^{(i)}} e^{i\theta \sum_{i=1}^{\nu} H^{(i)}})$$

where the pairs $\{(N^{(i)}, H^{(i)}) : i = 1, 2, ...\}$ are independent having the same distribution as (N, S_N) conditioned on $\{N \leq \mathbf{\Gamma}_p\}$. Hence we have

$$\begin{aligned}
E(s^G e^{i\theta S_G}) &= \sum_{k\geq 0} P(N > \mathbf{\Gamma}_p) P(N \leq \mathbf{\Gamma}_p)^k E(s^{\sum_{i=1}^{k} N^{(i)}} e^{i\theta \sum_{i=1}^{k} H^{(i)}}) \\
&= \sum_{k\geq 0} P(N > \mathbf{\Gamma}_p) P(N \leq \mathbf{\Gamma}_p)^k E(s^N e^{i\theta S_N} | N \leq \mathbf{\Gamma}_p)^k \\
&= \sum_{k\geq 0} P(N > \mathbf{\Gamma}_p) E(s^N e^{i\theta S_N} \mathbf{1}_{(N\leq \mathbf{\Gamma}_p)})^k \\
&= \sum_{k\geq 0} P(N > \mathbf{\Gamma}_p) E((qs)^N e^{i\theta S_N})^k \\
&= \frac{P(N > \mathbf{\Gamma}_p)}{1 - E((qs)^N e^{i\theta S_N})}. \qquad (3)
\end{aligned}$$

Note in the fourth equality we use the fact that $P(\mathbf{\Gamma}_p \geq n) = q^n$.

The required equality to be proved follows by setting $s = 0$ in (2) to recover

$$P(N > \mathbf{\Gamma}_p) = \exp\left\{ -\int_{(0,\infty)} \sum_{n=1}^{\infty} \frac{q^n}{n} F^{*n}(dx) \right\}$$

and then plugging this back into the right hand side of (3) and rearranging. ■

4 Lévy processes and infinite divisibility

We give a brief reminder of the definition of a Lévy process. The (one-dimensional) stochastic process $X = \{X_t : t \geq 0\}$ is called a Lévy process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if

(i) X has paths that are $\mathbb{P}$-almost surely right continuous with left limits.

(ii) given $0 \leq s \leq t < \infty$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$

(iii) given $0 \leq s \leq t < \infty$, $X_t - X_s$ is equal in distribution to X_{t-s} and

(iv) $\mathbb{P}(X_0 = 0) = 1$.

It is easy to deduce that if X is a Lévy process, then for each $t > 0$ the random variable X_t is infinitely divisible. Indeed one may also show via a straightforward computation that

$$\mathbb{E}(e^{i\theta X_t}) = e^{-\Psi(\theta)t} \text{ for all } \theta \in \mathbb{R},\, t \geq 0 \qquad (4)$$

where, in its most general form, Ψ takes the form given in (1). Conversely, it can also be shown that given a Lévy-Khintchine exponent (1) of an infinitely divisible random variable, there exists a Lévy process which satisfies (4). In the special case that the Lévy-Khintchine exponent Ψ belongs to that of a positive valued infinitely divisible distribution, it follows that the increments of the associated Lévy process must be positive and hence its paths are necessarily monotone increasing. In full generality, a Lévy process may be naively thought of as the independent sum of a linear Brownian motion plus an independent process with discontinuities in its path which in turn may be seen as the limit (in an appropriate sense) of the partial sums of a sequence compound poisson processes with drift. The book of Bertoin (1996) gives a comprehensive account of the above details.

The definition of a Lévy processes suggests that it may be thought of as a continuous-time analogue of a random walk. Let us introduce the exponential random variable with parameter p, denoted $\mathbf{e}_p$, which henceforth is assumed to be independent of all other random quantities under discussion and defined on the same probability space. Like the geometric distribution, the exponential distribution also has a lack of memory property in the sense that for all $0 \leq s, t < \infty$ we have $\mathbb{P}(\mathbf{e}_p > t + s | \mathbf{e}_p > t) = \mathbb{P}(\mathbf{e}_p > s) = e^{-ps}$. Moreover, $\mathbf{e}_p$, and more generally $X_{\mathbf{e}_p}$, is infinitely divisible. Indeed, straightforward computations show that for each $n = 1, 2, 3, \ldots$

$$\mathbb{E}(e^{i\theta X_{\mathbf{e}_p}}) = \left(\left(\frac{p}{p + \Psi(\theta)} \right)^{\frac{1}{n}} \right)^n = \mathbb{E}(e^{i\theta X_{\gamma_{1/n,p}}})^n$$

where $\gamma_{1/n,p}$ is a gamma distribution with parameters $1/n$ and p which is independent of X. The latter has distribution

$$\mathbb{P}(\gamma_{1/n,p} \in dx) = \frac{p^{1/n}}{\Gamma(1/n)} x^{-1+1/n} e^{-px} dx$$

for $x > 0$.

5 Wiener-Hopf factorization for Lévy processes

The Wiener-Hopf factorization for a one dimensional Lévy processes is a slightly more technical affair than for random walks but, in principle, appeals to essentially the same ideas that have been exhibited in the above exposition of the Wiener-Hopf factorization for random walks. We therefore only give in this section a statement of the Wiener-Hopf factorization. The reader interested in the full technical details is directed primarily to the article of Greenwood and Pitman (1979) for a natural and insightful probabilistic presentation (to the authors opinion). Alternative accounts based on the aforementioned article can be found in the books of Bertoin (1996) and Kyprianou (2006) and derivation of the Wiener-Hopf factorization for Lévy processes from the Wiener-Hopf factorization for random walks can be found in Sato (1999).

Before proceeding to the statement of the Wiener-Hopf factorization, we need first to introduce the *ladder process* associated with any Lévy process X. Here we encounter more subtleties than for the random walk. Consider the range of the times and positions at which the process X attains new maxima. That is to say the random set $\{(t, \overline{X}_t) : \overline{X}_t = X_t\}$ where $\overline{X}_t = \sup_{s\leq t} X_s$ is the running maximum. It turns out that this range is equal in law to the range of a killed bivariate subordinator $(\tau, H) = \{(\tau_t, H_t) : t < \zeta\}$ where the killing time ζ is an independent and exponentially distributed random variable with some rate $\lambda \geq 0$. In the case that $\lim_{t\uparrow\infty} \overline{X}_t = \infty$ there should be no killing in the process (τ, H) and hence $\lambda = 0$ and we interpret $\mathbb{P}(\zeta = \infty) = 1$. Note that we may readily define the Laplace exponent of the killed process (τ, H) by

$$\mathbb{E}(e^{-\alpha\tau_t-\beta H_t}\mathbf{1}_{(t<\zeta)}) = e^{-\kappa(\alpha,\beta)t}$$

for all $\alpha, \beta \geq 0$ where necessarily $\kappa(\alpha, \beta) = \lambda + \phi(\alpha, \beta)$ is the rate of ζ and ϕ is the bivariate Laplace exponent of the unkilled process $\{(\tau_t, H_t) : t \geq 0\}$. Analogously to the role played by joint probability generating and characteristic exponent of the pair (N, S_N) in Theorem 1 (iii) the quantitiy $\kappa(\alpha, \beta)$ also is prominent in the Wiener-Hopf factorization for Lévy processes which we state below. In order to do so we make one final definition. For each $t > 0$, let

$$\overline{G}_{\mathbf{e}_p} = \sup\{s < \mathbf{e}_p : X_s = \overline{X}_s\}.$$

Theorem 2 (The Wiener–Hopf factorization for Lévy processes) *Suppose that X is any Lévy process other than a compound Poisson process. As usual, denote by $\mathbf{e}_p$ an independent and exponentially distributed random variable.*

(i) *The pairs*

$$(\overline{G}_{\mathbf{e}_p}, \overline{X}_{\mathbf{e}_p}) \text{ and } (\mathbf{e}_p - \overline{G}_{\mathbf{e}_p}, \overline{X}_{\mathbf{e}_p} - X_{\mathbf{e}_p})$$

are independent and infinitely divisible.

(ii) *For $\alpha, \beta \geq 0$*

$$\mathbb{E}\left(\mathrm{e}^{-\alpha\overline{G}_{\mathbf{e}_p}-\beta\overline{X}_{\mathbf{e}_p}}\right) = \frac{\kappa(p,0)}{\kappa(p+\alpha,\beta)}.$$

(iii) *The Laplace exponent $\kappa(\alpha, \beta)$ may be identified in terms of the law of X in the following way,*

$$\kappa(\alpha,\beta) = k\exp\left(\int_0^\infty\int_{(0,\infty)}\left(\mathrm{e}^{-t}-\mathrm{e}^{-\alpha t-\beta x}\right)\frac{1}{t}\mathbb{P}(X_t\in dx)dt\right)$$

where $\alpha, \beta \geq 0$ and k is a dimensionless strictly positive constant.

6 The first passage problem and mathematical finance

There are many applications of the Wiener-Hopf factorization in applied probability and mathematical finance is no exception in this respect. One of the most prolific links is the relationship between the information contained in the Wiener-Hopf factorization and the distributions of the first passage times

$$\tau_x^+ := \inf\{t > 0 : X_t > x\} \text{ and } \tau_x^- := \inf\{t > 0 : X_t < x\}$$

together with the overshoots $X_{\tau_x^+} - x$ and $x - X_{\tau_x^-}$, where $x \in \mathbb{R}$. In turn this is helpful for the pricing of certain types exotic options.

For example in a simple market model for which there is one risky asset modelled by an exponential Lévy process and one riskless asset with a fixed rate of return, say $r > 0$, the value of a perpetual American put, or indeed a perpetual down-and-in put, boils down to the computation of the following quantity

$$v_y(x) := \mathbb{E}\left(e^{-r\tau_y^-}(K - e^{X_{\tau_y^-}})^+ | X_0 = x\right), \tag{5}$$

where $y \in \mathbb{R}$ and $z^+ = \max\{0, z\}$ and the expectation is taken with respect to an appropriate risk neutral measure which keeps X in the class of Lévy processes (for example the measure which occurs as a result of the Escher transform). To see the connection with the Wiener-Hopf factorization consider the following lemma and its corollary.

Lemma 3 *For all $\alpha > 0$, $\beta \geq 0$ and $x \geq 0$ we have*

$$\mathbb{E}\left(e^{-\alpha\tau_x^+ - \beta X_{\tau_x^+}} \mathbf{1}_{(\tau_x^+ < \infty)}\right) = \frac{\mathbb{E}\left(e^{-\beta \overline{X}_{\mathbf{e}_\alpha}} \mathbf{1}_{(\overline{X}_{\mathbf{e}_\alpha} > x)}\right)}{\mathbb{E}\left(e^{-\beta \overline{X}_{\mathbf{e}_\alpha}}\right)}. \tag{6}$$

Proof. First, assume that α, β, $x > 0$ and note that

$$\begin{aligned}
&\mathbb{E}\left(e^{-\beta \overline{X}_{\mathbf{e}_\alpha}} \mathbf{1}_{(\overline{X}_{\mathbf{e}_\alpha} > x)}\right) \\
&= \mathbb{E}\left(e^{-\beta \overline{X}_{\mathbf{e}_\alpha}} \mathbf{1}_{(\tau_x^+ < \mathbf{e}_\alpha)}\right) \\
&= \mathbb{E}\left(\mathbf{1}_{(\tau_x^+ < \mathbf{e}_\alpha)} e^{-\beta X_{\tau_x^+}} \mathbb{E}\left(e^{-\beta(\overline{X}_{\mathbf{e}_\alpha} - X_{\tau_x^+})} \middle| \mathcal{F}_{\tau_x^+}\right)\right).
\end{aligned}$$

Now, conditionally on $\mathcal{F}_{\tau_x^+}$ and on the event $\tau_x^+ < \mathbf{e}_\alpha$ the random variables $\overline{X}_{\mathbf{e}_\alpha} - X_{\tau_x^+}$ and $\overline{X}_{\mathbf{e}_\alpha}$ have the same distribution thanks to the lack of memory property of $\mathbf{e}_\alpha$ and the strong Markov property. Hence, we have the factorization

$$\mathbb{E}\left(e^{-\beta \overline{X}_{\mathbf{e}_\alpha}} \mathbf{1}_{(\overline{X}_{\mathbf{e}_\alpha} > x)}\right) = \mathbb{E}\left(e^{-\alpha\tau_x^+ - \beta X_{\tau_x^+}}\right) \mathbb{E}\left(e^{-\beta \overline{X}_{\mathbf{e}_\alpha}}\right).$$

The case that β or x are equal to zero can be achieved by taking limits on both sides of the above equality. ■

By replacing X by $-X$ in the previous lemma, we get the following analogous result for first passage into the negative half line.

Corollary 4 *For all $\alpha, \beta \geq 0$ and $x \geq 0$ we have*

$$\mathbb{E}\left(e^{-\alpha\tau^-_{-x}+\beta X_{\tau^-_{-x}}}\mathbf{1}_{(\tau^-_{-x}<\infty)}\right) = \frac{\mathbb{E}\left(e^{\beta \underline{X}_{\mathbf{e}_\alpha}}\mathbf{1}_{(-\underline{X}_{\mathbf{e}_\alpha}>x)}\right)}{\mathbb{E}\left(e^{\beta \underline{X}_{\mathbf{e}_\alpha}}\right)}. \tag{7}$$

In that case, we may develop the expression in (5) by using Corollary 4 and get that

$$v_y(x) = \frac{\mathbb{E}\left((K\mathbb{E}[e^{\underline{X}_{\mathbf{e}_r}}] - e^{x+\underline{X}_{\mathbf{e}_r}})\mathbf{1}_{(-\underline{X}_{\mathbf{e}_r}>x-y)}\right)}{\mathbb{E}(e^{\underline{X}_{\mathbf{e}_r}})}.$$

Ultimately, further development of the expression on the right hand side above requires knowledge of the distribution of $\underline{X}_{\mathbf{e}_r}$. This is information which, in principle, can be extracted from the Wiener-Hopf factorization.

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